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STUDIA MATHEMATICA BULGARICA

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БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. Bulgar. **21** (2012), 257–266

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ON MODIFIED METHOD OF SIMPLEST EQUATION FOR OBTAINING EXACT SOLUTIONS OF NONLINEAR PDES: CASE OF ELLIPTIC SIMPLEST EQUATION

Nikolay K. Vitanov

Devoted to the 65th Anniversary of the leading Bulgarian researcher in the area of partial differential equations acad. Petar Popivanov. Let the next generations know that in the dark times of misery of the Bulgarian science the society of researchers on nonlinear partial differential equations worked hard for new achievements thus paving the way for a better future of Bulgarian mathematics.

Abstract. The modified method of simplest equation is useful tool for obtaining exact and approximate solutions of nonlinear PDEs. These solutions are constructed on the basis of solutions of more simple equations called simplest equations. In this paper we study the role of the simplest equation for the application of the modified method of simplest equation. As simplest equation we discuss the elliptic equation.

2010 *Mathematics Subject Classification:* 74J30, 34L30.

Key words: nonlinear partial differential equations, method of simplest equation, exact travelling-wave solutions, elliptic equation.

1. Introduction. The nonlinear PDEs are widely used for modelling natural and social phenomena [1, 2, 3]. Many of these model systems are large and are accompanied by complex boundary conditions. For such systems one can obtain numerical solutions. But for more simple model NPDEs it is of great interest to obtain exact analytic solutions. Such exact solutions describe important classes of waves and processes in the investigated systems. Moreover the exact solutions can be used to test the computer programs for obtaining numerical solutions of the corresponding nonlinear PDEs. Finally the exact solutions can be useful as initial conditions in the process of obtaining of numerical solutions.

Because of all above an important research area is connected to obtaining exact analytic or approximate numerical solutions of nonlinear PDEs. The inverse scattering transform [4] and the method of Hirota [5] are famous methods for obtaining exact soliton solutions of various NPDEs. In addition in the last several years several approaches for obtaining exact special solutions of nonlinear PDE have been developed (see for examples [6, 7, 8]). By means of such methods numerous exact solutions of many equations have been obtained such as for an example the Kuramoto-Shivasinsky equation, etc. [7, 9, 10, 11]. The discussion below will be concentrated around the modified method of simplest equation for obtaining exact and approximate solutions of nonlinear PDEs. The method of simplest equation has been developed by Kudryashov [12]–[15] on the basis of a procedure analogous to the first step of the test for the Painleve property. In the modified method of the simplest equation [7, 8] this procedure is substituted by the concept for the balance equation. Modified method of simplest equation is already successfully applied for obtaining exact travelling wave solutions of numerous nonlinear PDEs such as versions of generalised Kuramoto-Sivashinsky equation, reaction–diffusion equation, reaction–telegraph equation [7], [11] generalised Swift-Hohenberg equation and generalised Rayleigh equation [8], generalised Fisher equation, generalised Huxley equation, generalised Degasperis-Processi equation and b-equation[16].

In 2004 Kudryashov [17] used the equation for the Weierstrass elliptic function as building block to find a number of differential equations with exact solutions. Below we follow this idea and use the elliptic equation as building block to find classes of equations with exact solutions.

2. The modified method of simplest equation and the role of the simplest equation. Let us have a partial differential equation and let by means of an appropriate ansatz this equation be reduced to a nonlinear ordinary differential equation

$$P\left(F(\xi), \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \dots\right) = 0 \tag{2.1}$$

For large class of equations from the kind (2.1) exact solution can be constructed as finite series

$$F(\xi) = \sum_{\mu=-\nu}^{\nu_1} p_\mu [\Phi(\xi)]^\mu \tag{2.2}$$

where $\nu > 0$, $\mu > 0$, p_μ are parameters and $\Phi(\xi)$ is a solution of some ordinary differential equation referred to as the simplest equation. The simplest equation is of lesser order than (2.1) and we know the general solution of the simplest equation or we know at least exact analytic particular solution(s) of the simplest equation [12, 13].

The modified method of simplest equation can be applied to nonlinear partial differential equations of the kind

$$E\left(\frac{\partial^{\omega_1} F}{\partial x^{\omega_1}}, \frac{\partial^{\omega_2} F}{\partial t^{\omega_2}}, \frac{\partial^{\omega_3} F}{\partial x^{\omega_4} \partial t^{\omega_5}}\right) = G(F) \tag{2.3}$$

where $\omega_3 = \omega_4 + \omega_5$ and we use the following short notations: $\frac{\partial^{\omega_1} F}{\partial x^{\omega_1}}$ denotes the set of derivatives $\frac{\partial^{\omega_1} F}{\partial x^{\omega_1}} = \left(\frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial F^3}{\partial x^3}, \dots\right)$; $\frac{\partial^{\omega_2} F}{\partial t^{\omega_2}}$ denotes the set of derivatives $\frac{\partial^{\omega_2} F}{\partial t^{\omega_2}} = \left(\frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t^2}, \frac{\partial F^3}{\partial t^3}, \dots\right)$ and $\frac{\partial^{\omega_3} F}{\partial x^{\omega_4} \partial t^{\omega_5}}$ denotes the set of derivatives $\frac{\partial^{\omega_3} F}{\partial x^{\omega_4} \partial t^{\omega_5}} = \left(\frac{\partial^2 F}{\partial x \partial t}, \frac{\partial^3 F}{\partial x^2 \partial t}, \frac{\partial F^3}{\partial x \partial t^2}, \dots\right)$. $G(F)$ is a polynomial of F. E can be an arbitrary sum of products of arbitrary number of its arguments. Each argument in each product can have arbitrary power. Each of the products can be multiplied by a function of F which can be polynomial of F. The application of the modified method of simplest equation is based on the following steps. By means of an appropriate ansatz (for an example the travelling-wave ansatz) the solved class of NPDE of kind (2.3) is reduced to a class of nonlinear ODEs of the kind (2.1). The finite-series solution (2.2) is substituted in (2.1) and as a result a polynomial of $\Phi(\xi)$ is obtained. Eq. (2.2) is a solution of (2.1) if all coefficients of the obtained polynomial of $\Phi(\xi)$ are equal to 0. By means of a balance equation one ensures that there are at least two terms in the coefficient of the highest power of $\Phi(\xi)$. The balance equation gives a relationship between the parameters of the solved class of equations and the parameters of the solution. The application of the balance equation and the equalising the coefficients of the polynomial of $\Phi(\xi)$ to 0 leads to a system of nonlinear relationships among the parameters of the solution and the parameters of the solved class of equation. Each solution of the obtained system of nonlinear algebraic equations leads to a solution a nonlinear PDE from the investigated class of nonlinear PDEs.

Below in order to investigate the role of simplest equation we shall be interested in exact solutions $F(\xi)$ of nonlinear ODEs which can be obtained on the basis of given simplest equation. After this we shall determine what class of nonlinear PDEs can be reduced to the corresponding nonlinear ODE by means of the travelling-wave ansatz $F(x,t) = F(\xi) = F(x - vt)$. In more detail we start from a simplest equation of the kind $Q\left(\Phi, \frac{d\Phi}{d\xi}, \frac{d^2\Phi}{d\xi^2}, \dots, \frac{d^n\Phi}{d\xi^n}\right) = 0$ and on the basis of a solution of this simplest equation we construct the function $F = F(\Phi)$ which has to be a solution of the more complicated equation

$$P\left(F, \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \dots, \frac{d^nF}{d\xi^n}\right) = 0 \tag{2.4}$$

Now the problem can be defined as follows. We choose Q and $F(\Phi(\xi))$. The question is what are the nonlinear ODEs which have this function $F(\xi)$ as a solution? Let us assume that we have obtained a class of such nonlinear ODEs. Then we easily can restore the class of nonlinear PDEs that are reduced by means of the travelling-wave ansatz to this class of ODEs. In such a way we can find the class of nonlinear PDEs that have corresponding function $F(\xi)$ as travelling-wave solution.

In this paper we shall investigate a sub-problem of the general problem. First of all we shall discuss simplest equations of the kind

$$\left(\frac{d\Phi}{d\xi}\right)^{\epsilon} = \sum_{\pi=0}^{\sigma} \gamma_\pi [\Phi(\xi)]^\pi \tag{2.5}$$

and second the function $F(\Phi)$ will be assumed polynomial of Φ

$$F(\Phi) = \sum_{\mu=0}^{\nu} p_\mu [\Phi(\xi)]^\mu \tag{2.6}$$

As it can be seen from Eq. (2.4) a very important role is played by the derivatives of F with respect to ξ. The first several of these derivatives are

$$\frac{dF}{d\xi} = \frac{dF}{d\Phi}\frac{d\Phi}{d\xi}; \qquad \frac{d^2F}{d\xi^2} = \frac{d^2F}{d\Phi^2}\left(\frac{d\Phi}{d\xi}\right)^2 + \frac{dF}{d\Phi}\frac{d^2\Phi}{d\xi^2} \tag{2.7}$$

$$\frac{d^3F}{d\xi^3} = \frac{d^3F}{d\Phi^3}\left(\frac{d\Phi}{d\xi}\right)^3 + 3\frac{d^2F}{d\Phi^2}\frac{d\Phi}{d\xi}\frac{d^2\Phi}{d\xi^2} + \frac{dF}{d\Phi}\frac{d^3\Phi}{d\xi^3} \tag{2.8}$$

$$\dots\dots\dots$$

For different versions of the simplest equation (2.5) and for different forms of (2.6) the relationships for the derivatives above will have different forms. If we choose a simplest equation then we can construct a class of nonlinear ODEs which have corresponding $F(\xi)$ as solution. If we choose another simplest equation then we obtain another class of nonlinear ODEs that have as solution the function $F(\xi)$ corresponding to the second simplest equation. Thus the choice of the simplest equation determines the class of nonlinear ODEs that have $F(\xi)$ as solution. And the class of nonlinear ODE determines the class of nonlinear PDEs that can be reduced to the corresponding class of nonlinear ODE by means of an appropriate ansatz (the travelling-wave ansatz in our case).

3. Elliptic equation as simplest equation. The elliptic equation is

$$\left(\frac{d\Phi}{d\xi}\right)^2 = a\Phi^4 + b\Phi^2 + c \tag{3.1}$$

where a, b and c are parameters. The elliptic functions of Jacobi [18] are among the solutions of the elliptic equation.

For the case of the elliptic equation the derivatives of $F(\xi)$ are as follows

$$\frac{dF}{d\xi} = \sqrt{a\Phi^4 + b\Phi^2 + c}\frac{dF}{d\Phi} \tag{3.2}$$

$$\frac{d^2F}{d\xi^2} = a\Phi^4\frac{d^2F}{d\Phi^2} + b\Phi^2\frac{d^2F}{d\Phi^2} + c\frac{d^2F}{d\Phi^2} + 2a\Phi^3\frac{dF}{d\Phi} + b\Phi\frac{dF}{d\Phi} \tag{3.3}$$

$$\frac{d^3F}{d\xi^3} = \frac{d^3F}{d\Phi^3}(a\Phi^4 + b\Phi^2 + c)\sqrt{a\Phi^4 + b\Phi^2 + c} + \frac{d^2F}{d\Phi^2}(6a\Phi^3 + 3b\Phi)\sqrt{a\Phi^4 + b\phi^2 + c} + \frac{dF}{d\Phi}(6a\Phi^2 + b)\sqrt{a\Phi^4 + b\Phi^2 + c}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{3.4}$$

For illustrative purposes we shall discuss the following simple equation

$$A(F)\left(\frac{dF}{d\xi}\right)^m = B(F) \tag{3.5}$$

where $A(F)$ and $B(F)$ are polynomial of F as follows

$$A(F) = \sum_{q=0}^{Q} \alpha_q F^q; \quad B(F) = \sum_{r=0}^{R} \beta_r F^r \tag{3.6}$$

The substitution of (3.6), (2.6), and (3.1) in (3.5) leads to an equation containing polynomial of Φ and $(\sqrt{a\Phi^4 + b\Phi^2 + c})^m$. Because of this m must be even, i.e., $m = 2n$. Following the modified method of simplest equation we have to balance the largest powers of the polynomial from the left-hand side and from the right-hand side of the equation 3.5. As a result we obtain the following balance equation

$$n = \frac{\nu(R - Q)}{2(\nu + 1)} \tag{3.7}$$

Now we have many cases (we note that n, R, Q must be integers). For example let $\nu = 1$. Then $F = p_0 + p_1\Phi$; $n = \dfrac{R-Q}{4}$. Let $\nu = 2$. Then $F = p_0 + p_1\Phi + p_2\Phi^2$; $n = \dfrac{R-Q}{3}$. Let $\nu = 3$. Then $F = p_0 + p_1\Phi + p_2\Phi^2 + p_3\Phi^3$; $n = \dfrac{3(R-Q)}{8}$ etc.

Let us now discuss the case $\nu = 2$. $F(\xi)$ and the balance equation have been mentioned just above. We must have $R = 4Q$ and then $n = Q$. The simplest possibility is $Q = 1$. Then $R = 4$, $n = 1$ and $m = 2$. This corresponds to the equation

$$(\alpha_0 + \alpha_1 F)\left(\frac{dF}{d\xi}\right)^2 = \beta_0 + \beta_1 F + \beta_2 F^2 + \beta_3 F^3 + \beta_4 F^4 \tag{3.8}$$

where $\alpha_{0,1}$, $\beta_{0,1,2,3,4}$ are parameters. The substitution of Eq.(3.2) in Eq. (3.8)) leads to the a system of 9 nonlinear relationships among the parameters of the solution and the parameters of Eq. (3.8). The solution of this system can be obtained for all values of the parameters $\alpha_{0,1}, \beta_{0,1,2,3,4}$ but it is too large. In order to illustrate the solution let us set $\alpha_0 = \alpha_1 = 1$; $\beta_3 = \beta_4 = 1$. One solution of the system of nonlinear relationships for this case is

$$p_0 = \left[18\beta_2 - 18\beta_1 + 2\left(-162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\right)^{1/2}\right]^{1/3} - \frac{2\beta_2}{\left[18\beta_2 - 18\beta_1 + 2\left(-162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\right)^{1/2}\right]^{1/3}}$$

$$
\begin{aligned}
p_1 &= 0, \;\; p_2 = 4a \\
b &= \frac{1}{8}\left[108\beta_2 - 108\beta_1 + 12\left(-162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\right)^{1/2}\right]^{1/3} - \\
&\quad \frac{3}{2}\frac{\beta_2}{\left[108\beta_2 - 108\beta_1 + 12\left(-162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\right)^{1/2}\right]^{1/3}}
\end{aligned}
$$

$$
\begin{aligned}
c = \frac{1}{48a}\Bigg\{3\beta_2+16\Bigg\{\frac{1}{8}\left[108\beta_2-108\beta_1+12\left(-162\beta_1\beta_2+12\beta_2^3+81\beta_1^2+81\beta_2^2\right)^{1/2}\right]^{1/3} - \\
\frac{3}{2}\frac{\beta_2}{\left[108\beta_2 - 108\beta_1 + 12\left(-162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\right)^{1/2}\right]^{1/3}}\Bigg\}\Bigg\}
\end{aligned}
$$

$$
\beta_0 = \beta_1 - \beta_2 \tag{3.9}
$$

and then the solution of the equation

$$
(1+F)\left(\frac{dF}{d\xi}\right)^2 = \beta_1 - \beta_2 + \beta_1 F + \beta_2 F^2 + F^3 + F^4 \tag{3.10}
$$

is

$$
\begin{aligned}
F(\xi) &= \left[18\beta_2 - 18\beta_1 + 2\left(-162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\right)^{1/2}\right]^{1/3} - \\
&\quad \frac{2\beta_2}{\left[18\beta_2 - 18\beta_1 + 2\left(-162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\right)^{1/2}\right]^{1/3}} + \\
&\quad 4a\Phi(\xi)^2
\end{aligned} \tag{3.11}
$$

where $\Phi(\xi)$ is solution of the elliptic equation

$$
\left(\frac{d\Phi}{d\xi}\right)^2 =
$$

$$
a\Phi^4 + \Bigg\{ \frac{1}{8}\Bigg[108\beta_2 - 108\beta_1 + 12\Bigg(- 162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\Bigg)^{1/2}\Bigg]^{1/3} -
\frac{3}{2}\frac{\beta_2}{\Bigg[108\beta_2 - 108\beta_1 + 12\Bigg(- 162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\Bigg)^{1/2}\Bigg]^{1/3}}\Bigg\}\Phi^2 +
\frac{1}{48a}\Bigg\{3\beta_2 + 16\Bigg\{\frac{1}{8}\Bigg[108\beta_2 - 108\beta_1 + 12\Bigg(-162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\Bigg)^{1/2}\Bigg]^{1/3} -
\frac{3}{2}\frac{\beta_2}{\Bigg[108\beta_2 - 108\beta_1 + 12\Bigg(- 162\beta_1\beta_2 + 12\beta_2^3 + 81\beta_1^2 + 81\beta_2^2\Bigg)^{1/2}\Bigg]^{1/3}}\Bigg\}\Bigg\}
\tag{3.12}
$$

The following notes are in order here. First of all comparing (3.12) to the differential equation for the elliptic functions of Jacobi [18] we can easily write a solution of Eq. (3.10) by means of one of these functions (for an example cn(x; k)). Taking into account that $F(\xi)$ describes a travelling wave we have obtained exact solutions of several nonlinear ODEs that can be reduced to (3.10). One example for such equation is

$$
(1+F)\left(\frac{\partial F}{\partial x}\right)^2 = \beta_1 - \beta_2 + \beta_1 F + \beta_2 F^2 + F^3 + F^4 \tag{3.13}
$$

4. Concluding remarks. In this paper we have discussed the role of the simplest equation for the application of the modified method of simplest equation for obtaining exact and approximate travelling-wave solutions of nonlinear PDEs. The main idea of the study was that when we fix the simplest equation then for each of the functions $p_0 + p_1\Phi(\xi)$; $p_0 + p_1\Phi(\xi) + p_2[\Phi(\xi)]^2; \dots$, constructed by solution $\Phi(\xi)$ of the simplest equation there exists a class of NPDEs for which the so constructed function is a travelling-wave solution. In this paper we have studied parts of the corresponding classes of NPDEs by means of the following algorithm: (1) Choose the simplest equation; (2) Construct a polynomial function on the basis of a solution of the simplest equation; (3) Find the class of nonlinear ODEs for which the mentioned above polynomial function is a solution; (4) Find the class of nonlinear PDEs that can be reduced to the above class of nonlinear

ODEs. In this paper we have discussed just one simplest equation: the elliptic equation. The formulated research topic is promising as one can use numerous ODEs as simplest equations and one can use different forms of the polynomial in order to construct functions: $p_0 + p_1\Phi$; $p_0 + p_1\Phi + p_2\Phi^2$; $p_0 + p_1\Phi + p_2\Phi^2 + p_3\Phi^3$, ... And for each of these function one can obtain classes of nonlinear ODEs and PDEs which have the corresponding function as a solution.

REFERENCES

[1] J. D. Murray. Lectures on nonlinear differential equation models in biology. Oxford, England, Oxford University Press, 1977.

[2] M. Ablowitz, P. A. Clarkson. Solitons, nonlinear evolution equations and inverse scattering. Cambridge, England, Cambridge University Press, 1991.

[3] N. K. Vitanov, Z. I. Dimitrova, M. Ausloos. Verhulst-Lotka-Volterra (VLV) model of ideological struggle. *Physica A* **389** (2010), 4970–4980.

[4] M. J.Ablowitz, D. J. Kaup, A. C. Newell, H. Segur. Inverse scattering transform – Fourier analysis for nonlinear problems. *Studies in Applied Mathematics* **53** (1974), 249–315.

[5] R. Hirota. Exact solution of Korteweg-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **27** (1971), 1192–1194.

[6] N. A. Kudryashov. Exact solutions of the generalized Kuramoto- Sivashinsky equation. *Phys. Lett. A* **147** (1990), 287–291.

[7] N. K. Vitanov, Z. I. Dimitrova, H. Kantz. Modified method of simplest equation and its application to nonlinear PDEs. *Applied Mathematics and Computation* **216** (2010), 2587–2595.

[8] N. K. Vitanov. Modified method of simplest equation: Powerful tool for obtaining exact and approximate traveling-wave solutions of nonlinear PDEs. *Commun. Nonlinear Sci. Numer. Simulat.* **16** (2011), 1176–1185.

[9] N. A. Kudryashov. Simplest equation method to look for exact solutionsof nonlinear differential equations. *Chaos Solitons and Fractals* **24** (2005), 1217–1231.

[10] N. K.Vitanov, I. P. Jordanov, Z. I. Dimitrova. On nonlinear dynamics of interacting populations: Coupled kink waves in a system of two populations. *Commun. Nonlinear Sci. Numer. Simulat.* **14** (2009), 2379–2388.

[11] N. K. Vitanov,Z. I. Dimitrova. Application of the method of simplest equation for obtaining exact traveling-wave solutions for two classes of model PDEs from ecology and population dynamics. *Commun. Nonlinear Sci. Numer. Simulat.* **15** (2010), 2836–2845.

[12] N. A. Kudryashov, N. B. Loguinova. Extended simplest equation method for nonlinear differential equations. *Applied Mathematics and Computation* **205** (2008), 396–402.

[13] N. A. Kudryashov, M. V. Demina. Polygons of differential equations for finding exact solutions. *Chaos Solitons and Fractals* **33** (2007), 480–496.

[14] N. A. Kudryashov, N. B. Loguinova. Be careful with the exp-function method. *Commun. Nonlinear Sci. Numer. Simulat.* **14** (2009), 1881–1890.

[15] N. A. Kudryashov. Seven common errors in finding exact solutions of nonlinear differential equations. *Commun. Nonlinear Sci. Numer. Simulat.* **14** (2009), 3507–3529.

[16] N. K. Vitanov, Z. I. Dimitrova, K. N. Vitanov. On the class of nonlinear PDEs that can be treated by the modified method of simplest equation. Application to generalized Degasperis-Processi equation and b-equation. *Commun. Nonlinear Sci. Numer. Simulat.* **16** (2011), 3033–3044.

[17] N. A. Kudryashov. Nonlinear differential equations with exact solutions via the Weierstrass function. *Zeitschrift für Naturforschung A* **59** (2004), 443–454.

[18] E. Janke, F. Emde. Tables of functions with formulae and curves. New York, Dover, 1945.

Institute of Mechanics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 4
1113 Sofia, Bulgaria
e-mail: vitanov@imbm.bas.bg