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HYPERBOLIC FIBRATIONS AND PDE

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Dedicated to the 65 years jubilee
of Professor Petar Popivanov full member of Bulg. Acad. Sci.

Abstract. In this note we try to distinguish the hyperbolic fibrations from the Euclidean one with the help of the invariant action of partial differential operators on the fibration. Two examples are given.

1. Introduction. In these notes one considers some examples of multiplicative geometric structures over algebras with singular elements. The description of the singular elements is difficult in general case. The simplest example is the one of the hyperbolic complex plane $\widetilde{\mathbf{R}}^2(1,j)$,

$$\widetilde{\mathbf{R}}^2(1,j) = \{z = x + jy,\ x, y \in \mathbf{R},\ j^2 = +1, j \notin \mathbf{C}\},$$

which admits a fibration defined by the 2-dimensional Minkowski metric

$$\mu(z) = x^2 - y^2.$$

In some larger context the problem of the interconnection between more general fibrations and the corresponding function theory and PDE concerns the fibre bundles over some matrix algebras.

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2. Multiplicative geometric structures. One can consider manifolds modeled on the algebra of real matrices. Here we consider only globally defined modeling over vector spaces of dimensions 2 or 4. As it was remarked above the simplest case is the hyperbolic complex plane $\widetilde{\mathbf{R}}^2(1,j)$, $j^2 = +1$, $j \notin \mathbf{C}$. If $z = x + jy$, the conjugate of z is $z_* = x - jy$. Then the product zz_* defines a 2-dimensional Minkowski metric

$$zz_* = x^2 - y^2. \tag{2.1}$$

Clearly, $\widetilde{\mathbf{R}}^2(1,j)$ is a multiplicative geometric structure. If $z, w \in \widetilde{\mathbf{R}}^2(1,j)$ then

$$zw = (x + jy)(u + jv) = xu + yv + j(xv + yu), \tag{2.2}$$

or $\widetilde{\mathbf{R}}^2(1,j)$ is two-dimensional vector space over the algebra of matrices

$$z = \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \ x, y \in \mathbf{R}. \tag{2.3}$$

The singular elements are

$$\{x + jx, \ x - jx\}, x \in \mathbf{R}. \tag{2.4}$$

The set of point $\{x + jx, \ x \in \mathbf{R}\}$ determines the bisectrix of the first and the third Descartes quadrants, and the set $\{x - jx, \ x \in \mathbf{R}\}$ determines the bisectrix of the second and the fourth Descartes quadrants. The metric $zz_* = x^2 - y^2$ annihilates on the both of these bisectrices.

We shall distinguish the so-called hyperbolic type quadrants from the Descartes ones. These hyperbolic quadrants are defined by the pair of the bisectrices. Two of the hyperbolic quadrants are of nonnegative metric zz_*, i.e. $x^2 - y^2 \geq 0$, and other two – of non-positive metric $x^2 - y^2 \leq 0$. The corresponding interiors are open subsets defined by the strong inequality $x^2 - y^2 > 0$ and $x^2 - y^2 < 0$. Every of these interiors is a connected open set in $\mathbf{R}^2$.

Let us remark that we distinguish between the notation $\mathbf{R}^2(1,j)$, $j^2 = -1$, and $\widetilde{\mathbf{R}}^2(1,j)$, $j^2 = +1$, $j \notin \mathbf{C}$. The first notation concerns the field of complex numbers with the Euclidean metric $\mu(z) = x^2 + y^2$, $z = x + iy$, and the second – the hyperbolic complex numbers with the Minkowski metric $\mu(z) = x^2 - y^2$. In the first case the Euclidean metric defines the fibration which consists of concentrated at the origin circles $x^2 + y^2 = r^2$. In the second case we have a fibration of hyperbolas of the type $x^2 - y^2 = \text{const}$. Both of these fibrations are 2-dimensional.

3. Examples of functions and PDE in dimension two. One shall consider functions defined on hyperbolic complex plane which are coherent with the fibration determined by the metric $zz_* = x^2 - y^2$ on the axis of abscissae ($y = 0$) in the positive quadrants and, respectively, to $zz_* = -y^2$ on the axis of ordinates ($x = 0$) on the negative quadrants. The non-degenerate hyperbola's fibration is defined by the both inequalities $x \neq 0$ and $y \neq 0$.

Proposition 1. *If the function $\varphi(t)$ is differentiable on $\mathbf{R}$ of class $C^{(1)}(\mathbf{R})$ then the function*

$$f(x,y) = \varphi(x^2 - y^2) \tag{3.1}$$

satisfies the following partial differential equation of first order

$$y\frac{\partial f(x,y)}{\partial x} + x\frac{\partial f(x,y)}{\partial y} = 0, \tag{3.2}$$

for both $x \neq 0$ and $y \neq 0$.

P r o o f. One calculates that

$$\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= \varphi'(x^2 - y^2)2x, \\ \frac{\partial f(x,y)}{\partial y} &= \varphi'(x^2 - y^2)(-2y). \end{aligned}$$

After the elimination of $\varphi'(x^2 - y^2) \neq 0$, one receives the equation (3.2).

The following theorem concerns the hyperbolic operator denoted here Δ_-, namely

$$\Delta_- f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} - \frac{\partial^2 f(x,y)}{\partial y^2}. \tag{3.3}$$

Proposition 2. *If the function φ is of class $C^{(2)}(\mathbf{R})$ then*

$$\Delta_- f(x,y) = 4\varphi'(x^2 - y^2) + 4(x^2 - y^2)\varphi''(x^2 - y^2). \tag{3.4}$$

P r o o f. One calculates that

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 2\varphi'(x^2 - y^2) + 4x^2\varphi''(x^2 - y^2),$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = -2\varphi'(x^2 - y^2) + 4y^2\varphi''(x^2 - y^2).$$

The difference of the above two formulas gives just the formula (3.4).

Consequence. According to the formula (3.4) the restriction of the hyperbolic operator Δ_- on each non-degenerate hyperbola $x^2 - y^2 = const$ is of constant value. This fact is expressed here by the phrase that the Hyperbolic operator Δ_- is coherent with the non-degenerate part of the fibration by hyperbolas.

Remark. The above described coherence of Δ_- with the considered fibration of the hyperbolic complex plane is not valid for the Euclidean (elliptic) Laplace operator

$$\Delta_+ f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}. \tag{3.5}$$

Indeed, we have

$$\Delta_+ f(x,y) = 4(x^2 + y^2)\varphi''(x^2 - y^2),$$

where $x^2 - y^2 = const$ does not implies $x^2 + y^2 = \text{const}$.

4. Remarks on the algebraic properties of the biplanar geometric structures (dimension 4). In the case of 4 dimensions we consider the biplanar geometric structure

$$\widetilde{\mathbf{R}}^2_{02} \oplus j\widetilde{\mathbf{R}}^2_{13}, \tag{4.1}$$

where

$$\widetilde{\mathbf{R}}^2_{02} = \{z = x_0 + j^2 x_2;\ x_0, x_2 \in \mathbf{R},\ j^4 = +1\},$$

$$\widetilde{\mathbf{R}}^2_{13} = \{w = x_1 + j^2 x_3;\ x_1, x_3 \in \mathbf{R},\ j^4 = +1\}.$$

This geometric structure corresponds to the algebra $\mathfrak{A}^*(4)$ [3], i.e. $X \in \mathfrak{A}^*(4)$ says that

$$X = x_0 + jx_1 + j^2 x_2 + j^3 x_3 = z + jw. \tag{4.2}$$

According to the Minkowski 4-dimensional signature (1,3) one considers the possible four conjugate X_*

$$(4.3)\qquad \left| \begin{array}{l} X_* = x_0 - j(x_1 + jx_2 + j^2x_3), \\ X_* = x_1 - j^3(x_0 + j^2x_2 + j^3x_3), \\ X_* = x_2 - j^2(x_0 + jx_1 + j^3x_3), \\ X_* = x_3 - j(x_0 + jx_1 + j^2x_2). \end{array} \right.$$

Calculating XX_* in the above written four cases for X_* on obtains the following two equivalence relations.

Theorem 1. *The equality $XX_* = 0$ is equivalent to the next two real number systems*

$$(4.4)\qquad x_0^2 - x_2^2 = 0,\ x_1 = x_3 = 0,\ x_0 \neq 0,\ x_2 \neq 0,$$

or

$$(4.5)\qquad x_0 = x_2 = 0,\ x_1^2 - x_3^2 = 0, x_1 \neq 0, x_3 \neq 0.$$

Proof. First, one calculates that

$$(4.6)\qquad \left| \begin{array}{l} XX_* = x_0^2 - x_2^2 - 2x_1x_3 - 2jx_2x_3 - j^2x_3^2 - 2j^3x_1x_2, \\ XX_* = x_0^2 + x_2^2 + 2jx_2x_3 + j^2(x_3^2 + 2x_0x_2 - x_1^2) + 2j^3x_1x_2, \\ XX_* = x_0^2 - x_2^2 + 2x_1x_3 + 2jx_0x_1 + j^2x_3^2 + 2j^3x_0x_3, \\ XX_* = x_0^2 + x_2^2 + 2jx_0x_2 + j^2(x_1^2 - x_3^2 + 2x_0x_2) + 2j^3x_1x_2. \end{array} \right.$$

Second, if $XX_* = 0$ for the first case in the system (4.6) we obtain the real number system

$$(4.7)\qquad \left| \begin{array}{l} x_0^2 - x_2^2 - 2x_1x_3 = 0, \\ x_2x_3 = 0, \\ x_3^2 = 0, \\ x_1x_2 = 0, \end{array} \right.$$

which implies $x_3 = 0$, $x_0^2 - x_2^2 = 0$. In the case $x_1 \neq 0$ one obtains $X = jx_1$ and $X_* = -jx_1$. So $XX_* = -j^2x_1^2 \neq 0$ which contradicts $XX_* = 0$. The equivalence relation (4.5) holds.

In the second case of (4.6) we have the real number system

$$\left|\begin{array}{l} x_0^2 + x_2^2 = 0, \\ x_2 x_3 = 0, \\ x_3^2 - 2x_0 x_2 - x_1^2 = 0, \\ x_1 x_2 = 0. \end{array}\right.$$

If $x_3 \neq 0$, then $x_2 = x_0 = 0$ and $x_3^2 - x_1^2 = 0$, and $XX_* = 2j^3 x_1 x_3 = 0$, or $x_1 x_3 = 0$, which implies $x_1 = 0$ or $x_3 = 0$ which is contradiction. We obtain (4.4).

Repeating the same reasoning we prove the statement for the other 2 cases of (4.6).

The system $x_0^2 - x_2^2 = x_1 = x_3 = 0$ determines the zero-divisors in $\widetilde{\mathbf{R}}^2_{02}$, and respectively $x_1^2 - x_3^2 = x_0 = x_2 = 0$ – the zero-divisors in $\widetilde{\mathbf{R}}^2_{13}$.

5. Examples of a fibration and PDE in dimension four. On the biplanar geometric structure we shall consider the fibration which coincides with the Cartesian product of the fibration considered in the section 2, i.e. we shall consider pairs of hyperbolas.

Now one takes a pair of Minkowski metrics and the function

$$f(x_0, x_1, x_2, x_3) = \varphi(x_0^2 - x_2^2) + \psi(x_1^2 - x_3^2),$$

where $\varphi(t)$ and $\psi(s)$ are two differentiable functions of class C^2 of one real variable.

One calculates

$$(5.1) \qquad \left|\begin{array}{l} \dfrac{\partial f}{\partial x_0} = 2x_0 \varphi'(x_0^2 - x_2^2), \\ \dfrac{\partial f}{\partial x_1} = 2x_1 \psi'(x_1^2 - x_3^2), \\ \dfrac{\partial f}{\partial x_2} = -2x_2 \varphi'(x_0^2 - x_2^2), \\ \dfrac{\partial f}{\partial x_3} = -2x_3 \psi'(x_1^2 - x_3^2) \end{array}\right.$$

and also

$$
(5.2)\qquad \left|\begin{aligned}
\frac{\partial^2 f}{\partial x_0^2} &= 4x_0^2\varphi''(x_0^2 - x_2^2) + 2\varphi'(x_0^2 - x_2^2),\\
\frac{\partial^2 f}{\partial x_1^2} &= 4x_1^2\psi''(x_1^2 - x_3^2) + 2\psi'(x_1^2 - x_3^2),\\
\frac{\partial^2 f}{\partial x_2^2} &= 4x_2^2\varphi''(x_0^2 - x_2^2) - 2\varphi'(x_0^2 - x_2^2),\\
\frac{\partial^2 f}{\partial x_3^2} &= 4x_3^2\psi''(x_1^2 - x_3^2) - 2\psi'(x_1^2 - x_3^2).
\end{aligned}\right.
$$

In the next we shall use the following ultrahyperbolic PDE of second order

$$
\Delta_-^{(4)} f = \frac{\partial^2 f}{\partial x_0^2} - \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_3^2} = 0.
$$

It is easy to calculate that for the special function (5.2) we have

$$
(5.3)\qquad \begin{aligned}
\Delta_-^{(4)} f &= 4(x_0^2 - x_2^2)\varphi''(x_0^2 - x_2^2) + 4(x_1^2 - x_3^2)\psi''(x_1^2 - x_3^2)\\
&\quad + 4\varphi'(x_0^2 - x_2^2) + 4\psi'(x_1^2 - x_3^2).
\end{aligned}
$$

In such a way we obtain the following theorem

Theorem 2. *If on the considered fibration we have $x_0^2 - x_2^2 = const$, $x_1^2 - x_3^2 = const$, then $\Delta_-^{(4)} f = const$. In other words the ultra-hyperbolic operator in R^4 (5.3) is coherent (see section 2) with the considered here hyperbolic fibration.*

Remark. The ordinary Laplace equation of dimension 4 is not coherent with the considered here hyperbolic fibration. In the paper [1] is introduced a kind of Laplace operator, called hyperbolic double-complex Laplace operator. In real form it has four real variables. This operator is not sensitive to the fibration considered here, i.e. it is not coherent with them.

6. Generalities and questions. (6.1) The theorem for the coherency of biplanar hyperbolic structures can be generalized for fourplanar hyperbolic structures defined with the help of 8-dimensional hyperbolic matrix numbers [2]. The same is valid for eightplanar hyperbolic structures etc.

(6.2) The following question can be formulated: may we describe a kind of hyperbolic fibrations on spacelike surfaces coherent with an appropriate PDO?

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