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OSCILLATION PROPERTIES OF SOME FUNCTIONAL FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper are considered oscillation properties of some classes of functional ordinary differential equations, namely equations of the type

$$z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) + \sum_{i=1}^{n} \beta_i(t) z(t - \gamma_i) = f(t),$$

where $m > 0$ is constant, $f(t) \in C([T, \infty); \mathbf{R})$, $T \geq 0$ is a large enough constant, $g(z, \xi, \eta, \zeta) \in C(\mathbf{R}^4; \mathbf{R})$, $\beta_i(t) \in C([0, \infty); [0, \infty))$, $\forall\ i = \overline{1, n}$, $n \in \mathbf{N}$ and $\{\gamma_i\}_{i=1}^{n}$ are nonnegative constants.

As a main result of this work we derive a sufficient condition for the distribution of the zeros of the above equations.

Furthermore we discuss the complexity of the oscillation behavior of such equations and its relation to some properties of the corresponding solutions. Finally, we comment the oscillation behavior of a neutral fourth order ordinary differential equation, which appears in two papers of Ladas and Stavroulakis, as well as in a paper of Grammatikopoulos et al.

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1. Introduction. As it is well known, the oscillation theory of ordinary differential equations is an important branch of the qualitative theory of ODE's. In this paper is given a sufficient condition for oscillation of all sufficiently smooth solutions of the equation

$$z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) + \sum_{i=1}^{n} \beta_i(t) z(t - \gamma_i) = f(t), \tag{1}$$

where

$$m > 0 \ \text{ and } \ \gamma_i \geq 0 \ \text{ are constants}, \ \ \forall\, i = \overline{1, n}, \ n \in \mathbf{N}; \ \ \gamma = \max_{1 \leq i \leq n} \gamma_i. \tag{2}$$

We suppose that $g(z, \xi, \eta, \zeta) \in C(\mathbf{R}^4; \mathbf{R})$ and there is a positive constant q such that

$$\begin{aligned} g(z, \xi, \eta, \zeta) &\geq qz, \ \ \forall\, (z, \xi, \eta, \zeta) \in \mathbf{R}_+ \times \mathbf{R}^3; \\ g(z, \xi, \eta, \zeta) &\leq qz, \ \ \forall\, (z, \xi, \eta, \zeta) \in \mathbf{R}_- \times \mathbf{R}^3. \end{aligned} \tag{3}$$

Furthermore, we assume $f(t) \in C([T, \infty); \mathbf{R})$, where $T \geq 0$ is a large enough constant, and

$$\beta_i(t) \in C([0, \infty); [0, \infty)), \ \forall\, i = \overline{1, n}. \tag{4}$$

Some remarks on the oscillation behavior of two particular fourth order ordinary differential equations that are not covered by our main result, are given in Section 4.

This paper generalizes works of Kusano and Yoshida [2], Yoshida [8], and Petrova [5], where is studied the inequality

$$z^{iv}(t) + mz''(t) + qz(t) \leq f(t). \tag{5}$$

Here $m > 0$ and $q > 0$ are proper constants, $f(t) \in C([T, \infty); \mathbf{R})$ for any large enough constant T in [2] and [5] as well as $f(t) \in C(\mathbf{R}_+; \mathbf{R})$ in [8].

Remark 1. In fact, in [5] we stated the assumption (3), which guarantees that every positive solution of the inequality

$$z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) \leq f(t) \tag{6}$$

satisfies the inequality (5) also. Further, in [5] we applied (6) to the equation

$$z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) = f(t), \tag{7}$$

since every solution of (7) is a solution of (6).

Note that in [5] are considered

$$z^{iv}(t) + mz''(t) + qz(t) + \lambda z(t-\tau) = f(t) \quad \text{and} \tag{8}$$

$$z^{iv}(t) + mz''(t) + qz(t) + \lambda z(t-\tau) \le f(t) \tag{9}$$

where $m > 0$, $q > 0$, $\lambda \in \mathbf{R}$ are constants, $f(t) \in C([T,\infty);\mathbf{R})$,$T \ge 0$ is a large enough constant. Further in this work we mention that the most interesting particular case of (8) and (9) is for $\lambda \ge 0$. This fact explains the assumption (4).

The original equation (1) is a generalization of (7), (8), and the following one

$$z^{iv}(t) + mz''(t) + qz(t) = f(t). \tag{10}$$

2. Preliminary results. In the text below we follow the assumptions of the famous lemma of Yoshida [7], where $L > 0$ and $\rho \ge 0$ are constants:

Lemma 1 ([7]). *If there is a number $s \ge \rho$ such that*

$$\int_s^{s+\pi/L} F(t)\sin L(t-s)dt \le 0, \tag{11}$$

then the ordinary differential inequality

$$z''(t) + L^2 z(t) \le F(t) \tag{12}$$

has no positive solution in $(s, s+\pi/L]$.

Lemma 1 is a key point in [5] and [8] for instance, and it is generalized many times. For more details see [6] and especially to the cited literature there.

Let us remind the definitions, where we suppose that c is a real constant.

Definition 1. *Function $\varphi(t) \in C([c,\infty);\mathbf{R})$ is* **oscillating** *when $t \to \infty$, if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} t_n = \infty \qquad \textit{and} \qquad \varphi(t_n) = 0.$$

Definition 2. *Function $\varphi(t) \in C([c,\infty);\mathbf{R})$ is* **eventually positive (eventually negative)***, if there exists a constant $\tilde{c} \ge c$ such that*

$$\varphi(t) > 0 \;\; (\varphi(t) < 0), \qquad \forall\; t \in [\tilde{c},\infty).$$

Further we apply the following function

(13)
$$\Phi_4(\theta) = \int\limits_{\theta}^{\theta+\pi/\tilde{L}} \int\limits_{s}^{s+\pi/L} f(t)\sin L(t-s)\,\sin \tilde{L}(s-\theta)dtds, \quad \tilde{L} = \sqrt{m-L^2},$$

which is defined in [5]. In fact, we found a family of such functions $\Phi_4(\theta)$, which depend on the constant L, unlike Kusano and Yoshida [2] and Yoshida [8], which obtained an unique function of a similar type. The results in [5] are qualitatively different from the other two papers. Below we give an example of application of the results from [5], in which it is impossible to apply the results from [2] and [8]. The idea is simple. We omit more details about the results for the inequality (5) given in [2] and in [8], just mention that the constants

(14)
$$\omega_{\pm} = \left(\frac{1}{2}(m\pm(m^2-4q)^{1/2})\right)^{1/2}$$

play essential roles there. Hence, the condition

(15)
$$m^2 > 4q$$

is a necessary one in both publications. So we construct an example (15) is not fulfilled for.

Theorem 1 ([5]). *Assume that* (3) *holds and the positive constants* L *and* q *satisfy the conditions:*

(16)
$$L \in (0, \sqrt{m}) \quad \textit{and} \quad q + (L^2 - m)L^2 > 0.$$

If there is a number $s \geq \rho$ *such that*

(17)
$$\Phi_4(s + \pi/L) \leq 0,$$

then the ordinary differential inequalities (6) *and* (5) *have no positive solution in* $(s, s+\pi/L+\pi/\tilde{L}]$. *Furthermore, if the function* $\Phi_4(\theta)$ *is eventually nonpositive, then the ordinary differential inequalities* (6) *and* (5) *have no eventually positive solution.*

Theorem 2 ([5]). *Let* (3) *and* (16) *be satisfied. If the function* $\Phi_4(s)$ *is oscillating, then every solution of the ordinary differential equations* (7) *and* (10) *oscillates.*

Theorem 3 ([5]). *Let* (3) *and* (16) *be fulfilled and there is a number* $s \geq \rho$ *such that* (17) *holds. If* $\lambda \geq 0$ *then* (9) *has no positive solution in* $(s-\tau, s+\pi/L+\pi/\tilde{L}]$. *If* $\lambda \in [-\left(q+(L^2-m)L^2\right), 0]$ *then* (9) *has no positive and monotonically increasing solution in* $(s-\tau, s+\pi/L+\pi/\tilde{L}]$.

Furthermore, let the function $\Phi_4(\theta)$ *be eventually nonpositive. If* $\lambda \geq 0$ *then* (9) *has no eventually positive solution. If* $\lambda \in [-\left(q+(L^2-m)L^2\right), 0]$ *then the ordinary differential inequality* (9) *has no eventually positive and monotonically increasing solution.*

Theorem 4 ([5]). *Let* (3) *and* (16) *be fulfilled and let the function* $\Phi_4(s)$ *be oscillating. If* $\lambda \geq 0$ *then every solution of the equation* (8) *oscillates. If* $\lambda \in [-\left(q+(L^2-m)L^2\right), 0]$ *then the equation* (8) *has nor eventually positive and monotonically increasing solution neither eventually negative and monotonically decreasing solution.*

Example 1. *Let us consider*

$$z^{iv}(t) + z''(t) + z(t) \leq \sin t \quad \text{and} \tag{18}$$

$$z^{iv}(t) + z''(t) + z(t) = \sin t, \tag{19}$$

which are particular cases of (5) *and* (10), *where*

$$m = 1 \quad \text{and} \quad q = 1. \tag{20}$$

Obviously, the function $\sin t$ is solution of both (18) and (19).

Example 1 is an illustration of Theorem 2, since the assumption (16) is meaningful for $L = 1/2$. Hence, $\tilde{L} = \sqrt{3}/2$ and

$$F_4(\theta) = \int_{\theta}^{\theta+2\pi/\sqrt{3}} \int_{s}^{s+2\pi} \sin t . \sin \frac{t-s}{2} . \sin \frac{\sqrt{3}(s-\theta)}{2} dtds. \tag{21}$$

We point out that the function $F_4(s)$, defined by (21), is oscillating.

On one hand, the proof of Theorem 2 is based on Theorem 1. On the other hand, Theorem 1 differs from the respective results in [2] and [8], since (15) is not satisfied because of (20). In fact Theorem 2 is not comparable with any other results from [2] and [8], since there are not considered any ordinary differential equations at all.

3. Main results. Taking into account Remark 1, the inequality

$$z^{iv}(t) + mz''(t) + g(z(t), z'(t), z''(t), z'''(t)) + \sum_{i=1}^{n} \beta_i(t) z(t-\gamma_i) \le f(t) \tag{22}$$

plays the key role in the following results. As before, we suppose that $\rho \ge 0$ is a constant.

Theorem 5. *Let* (3) *and* (16) *be fulfilled. If there is a number* $s \ge \rho$ *such that* (17) *holds, then the inequality* (22) *has no positive solution in* $(s-\gamma, s+\pi/L+\pi/\tilde{L}]$. *Furthermore, if the function* $\Phi_4(\theta)$ *is eventually nonpositive, then the inequality* (22) *has no eventually positive solution.*

Proof. The statement follows from Theorem 1. The assumption (16) guarantees that every positive solution of (22) in $(s-\gamma, s+\pi/L+\pi/\tilde{L}]$ satisfies (6) in $(s, s+\pi/L+\pi/\tilde{L}]$. □

Theorem 6. *Let* (3) *and* (16) *be satisfied. If the function* $\Phi_4(\theta)$ *is oscillating, then every solution of the equation* (1) *oscillates.*

Proof. Definition 1 yields that there is a sequence $\{\theta_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \theta_n = \infty \qquad \text{and} \qquad \Phi_4(\theta_n) = 0. \tag{23}$$

In fact, from (23) follows that $\Phi_4(\theta_n) \le 0$ and $\Phi_4(\theta_n) \ge 0$ simultaneously. In this way, we suppose that $\theta_n = s_n + \pi/L$ in order to apply Theorem 5 in $(s_n - \gamma, s_n + \pi/L + \pi/\tilde{L}]$ for $\forall\, n \in \mathbf{N}$. The inequality (22) has no positive solution in every such interval. Similarly, we could prove that the inequality with just the opposite direction has no negative solution there. Hence, the equation (1) has at least one zero in $(s_n - \gamma, s_n + \pi/L + \pi/\tilde{L}]$, $\forall\, n \in \mathbf{N}$. □

Remark 2. We point out that we proved more than it has been formulated in the above theorem, since we explained the impact of the delays in the original equation (1).

According to Remark 2, here we generalize some of our previous results from [5]. We would not generalize the other results from there.

4. Oscillation properties of some other fourth order equations. Constructing equations of the type $l(\varphi(t)) = f(t)$, such that their oscillation behavior is investigated via the respective inequalities $l(\varphi(t)) \leq f(t)$ and $l(\varphi(t)) \geq f(t)$ is not an easy task. Moreover, this comment holds for an arbitrary order of the operator $l(\varphi(t))$. Let us give two examples in this direction. The first one is from [3].

In [3] are considered the respective inequalities

$$y^{(\eta)}(t) + (-1)^{\eta+1}\left[p^{\eta} + q(t)y(t-\eta\tau)\right] \leq 0; \tag{24}$$

$$y^{(\eta)}(t) + (-1)^{\eta+1}\left[p^{\eta} + q(t)y(t-\eta\tau)\right] \geq 0 \tag{25}$$

and the equation

$$y^{(\eta)}(t) + (-1)^{\eta+1}\left[p^{\eta} + q(t)y(t-\eta\tau)\right] = 0. \tag{26}$$

It is supposed that $q(t) \in C([0,\infty);[0,\infty))$ and that $\eta \in \mathbf{N}$, $p > 0$, and $\tau > 0$ are constants. We recall only Corollary 3 from [3], which is the final result in the paper.

Corollary 1 ([3]). *The condition*

$$p\tau e > 1. \tag{27}$$

is necessary and sufficient so that

(*i*) *when* η *is odd:* (24) *has no eventually positive solutions,* (25) *has no eventually negative solutions, and* (26) *has only oscillatory solutions*

(*ii*) *when* η *is even:* (24) *has no eventually negative bounded solutions,* (25) *has no eventually positive bounded solutions, and every bounded solution of* (26) *is oscillatory.*

Traditionally the ordinary differential inequalities have been applied only for sufficient conditions for oscillation and [3] and [5] are exceptions in the sense that both mentioned above papers treat the questions if the oscillatory properties could be considered together with some other properties of the solutions of the respective ordinary differential equations.

The second example is from Grammatikopoulos et al [1]. More exactly, in [1] have been investigated the equation

$$\frac{d^{\eta}}{dt^{\eta}}\left[y(t) + P(t)y(t-\tau)\right] + Q(t)y(t-\sigma) = 0, \quad t \geq t_0, \tag{28}$$

where $P, Q \in C([t_0, \infty); \mathbf{R})$, $\tau, \sigma \in \mathbf{R}^+$ and $\eta \geq 1$. There have been obtained two types of sufficient conditions for oscillation of all the solutions of (28). The first result is in the particular case, where η is an odd number and the second one — where η is an even number.

REFERENCES

[1] M. Grammatikopoulos, G. Ladas, A. Meimaridou. Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients. *Chin. Ann. of Math.* **9B**, *3* (1988), 322–338.

[2] T. Kusano, N. Yoshida. Forced oscillations of Timoshenko beams. *Quart. Appl. Math.* **43** (1985), 167–177.

[3] G. Ladas, I. Stavroulakis. On delay differential inequalities of higher order. *Canad. Math. Bull.* **25**, *3* (1988), 348–354.

[4] G. Ladas, I. P. Stavroulakis, Oscillations of differential equations of the mixed type. *J. Math. Phys. Sciences*, **18** (1984), 245–262.

[5] Z. Petrova. Oscillations of a class of equations and inequalities of fourth order, *Mathematics and Education in Mathematics* **35**, (2006), 195–200.

[6] Z. Petrova. Oscillations of some equations and inequalities of second order and applications in mechanics. In: Applications of Mathematics in Engineering & Economics (Ed. M. D. Todorov), Proc. 33rd International Conference AMEE, AIP CP946, Melville, New York, 2007, 224–234.

[7] N. Yoshida. On the zeros of solutions to nonlinear hyperbolic equations. *Proc. R. Soc. Edinb., Sect. A* **106** (1987), 121–129.

[8] N. Yoshida., On the zeros of solutions of beam equations. *Ann. Mat. Pura Appl., IV. Ser.* **151** (1988), 389–398.

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