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EXISTENCE THEOREMS FOR NON-COOPERATIVE ELLIPTIC SYSTEMS

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Abstract. Existence of classical $C^2(\Omega)\bigcap C(\overline{\Omega})$ solutions of non-cooperative weakly coupled systems of elliptic second-order PDE is proved via the method of sub- and super-solutions.

1. Introduction. Let $\Omega \in R^n$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper are considered weakly coupled linear elliptic systems of the form

(1) $$L_M u = f(x) \text{ in } \Omega$$

and boundary data

(2) $$u(x) = g(x) \text{ on } \partial\Omega,$$

where $L_M = L + M$, L is a matrix operator with null off-diagonal elements $L = \operatorname{diag}(L_1, L_2, \dots, L_N)$, and matrix $M = \{m_{ki}(x)\}_{k,i=1}^{N}$. Scalar operators

$$L_k u^k = -\sum_{i,j=1}^{n} D_j\left(a_{ij}^k(x) D_i u^k\right) + \sum_{i=1}^{n} b_i^k(x) D_i u^k + c^k u^k \ \texttt{in}\ \Omega$$

are supposed uniformly elliptic ones for $k = 1, 2, \dots, N$, i.e. there are constants $\lambda, \Lambda > 0$ such that

$$\lambda\,|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}^k(x)\xi_i\xi_j \le \Lambda\,|\xi|^2$$

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for every k and any $\xi = (\xi_1, \dots, \xi_n) \in R^n$.

Right-hand side $f(x)$ is supposed a bounded vector-function, that is

$$(*) \qquad |f^l(x)| \le C \text{ in } \Omega$$

for every $l = 1, \dots, N$, where C is a positive constant.

Coefficients c^k and m_{ik} in (1) are supposed continuous in $\overline{\Omega}$, and $a^k_{ij}(x)$, $b^k_i(x) \in C^1(\Omega) \cap C(\overline{\Omega})$. Assume in addition that for every $k = 1, \dots, N$

$$(3) \qquad \left\{ \sum_{i=1}^{n} \left(\sum_{j=1}^{n} D_j a^k_{ij}(x) + b^k_i(x) \right)^2, |c^k| \right\} \le b$$

holds for $x \in \overline{\Omega}$, where b is a positive constant.

Hereafter by $f^-(x) = \min(f(x), 0)$ and $f^+(x) = \max(f(x), 0)$ are denoted the non-negative and, respectively, the non-positive part of the function f. The same convention is valid for matrixes as well. For instance, we denote by M^+ the non-negative part of M, i.e. $M^+ = \{m^+_{ij}(x)\}^N_{i,j=1}$.

In this paper is employed the method of sub- and super-solutions in order to prove the existence of a classical $C^2(\Omega) \bigcap C(\overline{\Omega})$ solution of problem (1). A key-point of the method is the validity of the comparison principle. Unlike the cooperative systems, for non-cooperative ones there is no complete theory for the validity of the comparison principle. In [1] are given some sufficient conditions such that the comparison principle holds, which are recalled in section "Comparison principle for non-cooperative linear elliptic systems" below.

We consider linear systems only for the sake of simplicity. The results hold as well for quasi-linear weakly coupled elliptic systems

$$Q^l(u) = -diva^l(x, u^l, Du^l) + F^l(x, u^1, \dots, u^N, Du^l) = f^l(x) \text{ in } \Omega$$
$$u^l(x) = g^l(x) \text{ on } \partial\Omega$$

for $l = 1, \dots, N$, where the coefficients $a^l(x, u, p)$, $F^l(x, u, p)$, $f^l(x)$, $g^l(x)$ are supposed to be at least measurable functions with respect to the x variable and locally Lipschitz continuous on u and p.

2. Comparison principle for non-cooperative linear elliptic systems. Let us recall the following Theorem (Theorem 3 in [1]):

Theorem 1. *Let* (1) *be a weakly coupled elliptic system with irreducible cooperative part of* $L^*_{M^-}$. *Then the comparison principle holds for the classical solutions of system* (1) *if there is* $x_0 \in \Omega$ *such that*

$$\lambda + \sum_{k=1}^{N} m^+_{kj}(x_0) > 0 \quad for \quad j = 1\dots, N \tag{4}$$

and

$$\lambda + m^+_{jj}(x) \geq 0 \quad for\ every \quad x \in \Omega \quad and \quad j = 1\dots, N \tag{5}$$

where λ *is the principal eigenvalue of the operator* L_{M^-} *in* Ω.

The same result holds if the cooperative part of $L^*_{M^-}$ has structure with Jordan cells on the main diagonal and zeroes otherwise (Theorem 4 in [1]).

Theorem 2. *Assume* $m^-_{ij} \equiv 0$ *for* $i \neq j$ *and* (2) *is satisfied. Then the comparison principle holds for the classical* $C^2(\Omega)\bigcap C(\overline{\Omega})$ *solutions of system* (1) *if there is* $x_0 \in \Omega$ *such that*

$$\lambda_j + \sum_{k=1}^{N} m^+_{kj}(x_0) > 0 \quad for\ every \quad j = 1\dots, N, \quad and \tag{6}$$

$$\lambda_j + m^+_{jj}(x) \geq 0 \quad for\ every \quad x \in \Omega \quad and \quad j = 1\dots, N, \tag{7}$$

where λ_j *is the principal eigenvalue of* $\widetilde{L}_j = L_j + m^-_{jj}$ *in* Ω.

Theorem 2 is formulated for diagonal matrix M^-, but the statement is valid with obvious modification if M^- has Jordan cells on the main diagonal.

Finally (Theorem 5 in [1]), in case that the cooperative part M^- is triangular, we have

Theorem 3. *Assume the cooperative part* M^- *of system* (1) *is triangular, i.e.* $m^-_{ij} = 0$ *for* $i = 1, \dots, N$, $j > i$. *Then the comparison principle holds for the classical* $C^2(\Omega)\bigcap C(\overline{\Omega})$ *solutions of system (1), if there is* $\varepsilon > 0$ *such that*

$$\lambda_j - (1 - \delta_{1j})\varepsilon + \sum_{k=1}^{N} m^+_{kj}(x_0) > 0 \tag{8}$$

for $j = 1\dots, N$ *and some* $x_0 \in \Omega$ *and*

$$\lambda_j - (1 - \delta_{1j})\varepsilon + m^+_{jj}(x) \geq 0 \quad for\ every \quad x \in \Omega \quad and \quad j = 1\dots, N, \tag{9}$$

where λ_j *is the principal eigenvalue of the operator* $L_j + m^-_{jj}$.

3. Existence of classical solution. The first step of the method is existence of super- and sub-solution of system (1), (2). It is easy to check that constant-vector $(M,\dots,M)$ is a super-solution for any constant M such that

$$\sum_{i=1}^{n} m_{ki}(x) \geq \frac{C}{M}, \tag{10}$$

where C is the upper bound $|f^l(x)|$ (see $(*)$).

Theorem 4. *Suppose conditions* (4), (5); (6), (7) *or* (8), (9) *hold for system* (1), (2), *according to the structure of matrix M, as well as* (10). *Assume $v(x)$ is a classical super-solution and $w(x)$ is a a classical sub-solution of* (1), (2). *Then there exists a classical solution $u(x)$ of the problem* (1), (2) *with null boundary data.*

Since the system (1) is a linear one, we assume in the following proof without loss of generality that $g(x) = 0$.

S k e t c h o f t h e p r o o f. Let denote

$$F^k(x,u^1,\dots,u^N) = \sum_{i=1}^{n} m_{ki}(x)u^i + c^k u^k$$

1. Consider the sequence of vector - functions $u_0, u_1, \dots, u_l, \dots$, where $u_0 = w(x)$ and $u_l \in H_0^1(\Omega)$ defines u_{l+1} by induction as a solution of the problem

$$- \sum_{i,j=1}^{N} D_i(a_{ij}^k(x) D_j u_{l+1}^k) + \sum_{i=1}^{N} b_i^k(x) D u_{l+1}^k + \sigma u_{l+1}^k = \\ = f^k(x) - F^k(x, u_k^1, \dots, u_k^N) + \sigma u_l^k \quad \text{in} \quad \Omega \tag{11}$$

with null boundary conditions

$$u_{l+1}^k(x) = 0 \quad \text{on} \quad \partial\Omega \tag{12}$$

for every $k = 1,\dots,N$.

Let denote the left-hand side of (11) by $A^k(x,u,\sigma)$, and the right-hand side – by $B^k(x,u,\sigma)$, $k = 1,\dots,N$.

The problem (11), (12) is reducible system and in fact decomposes to N independent equations. Then Theorem 8.3 in [3] (page 348) is applicable, hence these equations are solvable in $C^{2,\alpha}(\overline{\Omega})$ and

$$\|u_l^k\|_{C^\beta(\overline{\Omega})} < c, \tag{13}$$

$$(14) \qquad \left\| \frac{\partial u_l^k}{\partial x_i} \right\|_{C^\beta(\overline{\Omega}} < c_1 \text{ for every } \; i = 1, \dots, n, \; \gamma = 1, \dots, m.$$

Furthermore $u_0^l \leq u_1^l \leq \cdots \leq u_{l+1}^k \leq \cdots$ by the comparison principle.

The proof of $u_0^l \leq u_1^l$ is trivial since u_0^l is a sub-solution of (1), (2).

3. Obviously the inequality $u_{l+1}(x) \leq v(x)$ holds for every u_{l+1}, since $v(x)$ is a super-solution of the same system (1), (2).

4. The sequence of vector-functions $\{u^k\}$ is monotonously increasing and bounded from above in Ω. Therefore there is a function u such that $u^k(x) \to u(x)$ point-wise in Ω. Furthermore, (13) yields $\{u^k\}$ is uniformly equicontinuous in $\overline{\Omega}$ and $\{u^k\} <$ const, since $u_l^k(x)$ is Holder continuous and therefore $|u_l^k(x) - u_l^k(x_0)| \leq c(|x - x_0|^\beta)$ for every $l = 1, \dots, N$. By Arzela–Ascoli compactness criterion there is a sub-sequence $\{u_{k_j}\}$ that converges uniformly to $u \in C(\overline{\Omega})$. For convenience we denote $\{u_{k_j}\}$ by $\{u^k\}$.

Since $u \in C(\overline{\Omega})$ and all functions $\{u_{k_j}\}$ satisfy the null boundary conditions, then u satisfies the boundary conditions as well.

The functions u^k are Holder continuous with the same Holder constant, therefore u is Holder continuous as well with the same Holder constant, i.e. $u \in C^\beta(\overline{\Omega})$.

Since $u_{l+1}(x)$ is monotone and $u(x)$ is continuous, then $\{(u^k)^2\} \to u^2$ in Ω. Then the Dominated Convergence Theorem (Theorem 5 at p.648 in [2]) yields $u^k \to u(x)$ in $(L^2(\Omega))^N$.

5. Analogously to the previous step, (14) yields $\{D_i u^k\}$ is uniformly equicontinuous in $\overline{\Omega}$ and $\{D_i u^k\} <$ const. According to Arzela–Ascoli compactness criterion there is sub-sequence $\{D_i u_{k_j}\}$ that converges uniformly to $D_i u \in C(\overline{\Omega})$. For convenience we denote $\{u_{k_j}\}$ by $\{u^k\}$.

6. For every $0 < \eta(x) = (\eta^1(x), \dots, \eta^N(x)) \in (H_0^1(\Omega))^N$

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij}^k(x) D_j u_{l+1}^k D_i \eta^k(x) + \sum_{i=1}^N b_i^k(x) D u_{l+1}^k \eta^k(x) + \sigma u_{l+1}^k \eta^k(x) \right) dx =$$

$$= \int_\Omega (f^k(x) - F^k(x, u_k^1, \dots, u_k^N) + \sigma u_l^k) \eta^k(x) dx$$

holds and for $k \to \infty$ we obtain

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij}^k(x) D_j u^k D_i \eta^k(x) + \sum_{i=1}^N b_i^k(x) D u^k \eta^k(x) \right) dx =$$
$$= \int_\Omega (f^k(x) - F^k(x, u^1, \dots, u^N)) \eta^k(x) dx$$

that is $u(x)$ is solution of (1), (2). $\square$

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