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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. Bulgar. **13** (2000), 79-90

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ASYMPTOTIC BEHAVIOUR OF A SUPERCRITICAL GALTON-WATSON PROCESS WITH CONTROLLED BINOMIAL MIGRATION

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This paper considers a branching process generated by an offspring distribution F with mean $m < \infty$ and variance $\sigma^2 < \infty$ with δ-migration controlled by the native population N_n^{bef} according to a binomial law with parameter $p_{N_n^{bef}}$. The δ-migration is an emigration if $\delta = 1$, an immigration if $\delta = -1$, and a partial observation of the population if $\delta = 0$; δ does not depend on n. We assume $\lim_n p_n = p$, $p_n = O(m_*^{-nx})$ with $0 \le x \le 1$ and $m_* = m(1-\delta p)$, $p \in [0,1]$. Moreover when $p = 0$, $\{p_n\}_n$ is either a deterministic sequence or a stochastic one. Under the assumption $m_* > 1$, we study the asymptotic behaviour of the different processes. For each $0 \le x \le 1$, $N_n \overset{a.s,L^2}{=} O(m_*^n)$ and $N_n^{bef} \overset{a.s,L^2}{=} O(m_*^n)$. In the case $x < 1$, $N_n^{obs} \overset{a.s,L^2}{=} O(m_*^{n(1-x)})$ whereas in the case $x = 1$, N_n^{obs} converges in distribution to a Poisson variable with a deterministic or random parameter depending on whether $\{p_n\}_n$ is stochastic or deterministic.

Keywords: Galton-Watson, supercritical, migration, binomial, size-dependent.

AMS subject classification: 60J80, 62F12, 62P10.

1 Introduction

Consider a native population in which each individual can mutate with the same probability ([5]) or consider the general epidemiologic problem where each individual of the population can catch a disease with the same probability. At last, consider a population which is only partially observed at each generation: for example, the population is in a volume V_n at generation n and the observation is done by means of an aliquot v_n, this aliquot being removed after observation. In this case each individual can be observed with the probability $p_n = v_n V_n^{-1}$.

In these examples, the population of individuals who change (by mutation or disease or observation) can be considered as an emigrating population. Models of systematic emigration are rare in the litterature ([10], [11],[7]). The reason is clear: systematic emigration can easily lead to the extinction of the population excepted when the emigration is controlled and the native process is supercritical.

We deal more generally with a Galton-Watson process generated by an offspring distribution F with mean $m < \infty$ and variance $\sigma^2 < \infty$ with, at each generation n, an observed δ-migration N_n^{obs} controlled by the native population N_n^{bef} according to a binomial law $B_{p_{N_n^{bef}}}^{*N_n^{bef}}$. The δ-migration is defined as an emigration if $\delta = 1$, an immigration if $\delta = -1$ and corresponds to a partial and non removed observation of the native population if $\delta = 0$. The parameter δ is assumed constant throughout the different generations.

The population size after migration, at the nth generation, N_n, is given, for $n \geq 1$, by the model (M):

$$N_n = N_n^{bef} - \delta N_n^{obs}, \tag{1}$$

where

$$N_n^{bef} = \sum_{i=1}^{N_{n-1}} Y_{n,i} \tag{2}$$

is the population size at the nth generation before migration and

$$N_n^{obs} = \sum_{j=1}^{N_n^{bef}} N_{n,j}^{obs} \tag{3}$$

is the migrating population size at the nth generation. Assume

(A1): The $\{Y_{n,i}\}_{n,i}$ are i.i.d. according to $F(m, \sigma^2)$ with mean $m < \infty$ and variance $\sigma^2 < \infty$;

(A2): Given N_n^{bef}, the $\{N_{n,j}^{obs}\}_j$ are i.i.d. according to a Bernoulli distribution $B_{p_{N_n^{bef}}}$ on $\{0, 1\}$ with parameter $P(N_{n,j}^{obs} = 1 | N_n^{bef}) = p_{N_n^{bef}}$;

(A3): $\lim_n p_n = p$ and $m_* > 1$ (where $m_* = m(1 - \delta p)$). Consider the following particular cases :

1. $p > 0$ and $\{p_{N_n^{bef}}\}_n$ is a deterministic sequence denoted $\{p_n\}_n$ and such that $m(1 - \delta p_n) > 1$, for all n, and $0 < \Pi_{n=1}^{\infty}[(1 - \delta p_n)(1 - \delta p)^{-1}] < \infty$ (or equivalently $\delta | \sum |p_n - p| < \infty$);

2. $p = 0$. Let $0 < \lambda \leq 1$, $0 < x \leq 1$.

   - $\{p_{N_n^{bef}}\}_n$ is the following controlled stochastic sequence : $p_{N_n^{bef}} = \lambda (N_n^{bef})^{-x}$ on $\{N_n^{bef} > 0\}$ and $p_{N_n^{bef}} = 0$ when $N_n^{bef} = 0$. Assume $N_0 [E(W^{1-x} | N_0)]^{-1} (m^x - 1) - \delta\lambda > 0$, where $W \overset{a.s.}{=} \lim_n N_n m^{-n}$.

- $\{p_{N_n^{bef}}\}_n$ is the following deterministic sequence denoted $\{p_n\}_n$: $p_n = \lambda(E(N_n^{bef}))^{-x}$, *i.e.* $p_n = \lambda(N_0 m\Pi_1^{n-1} m(1-\delta p_k))^{-x}$, $n \geq 2$, $p_1 = \lambda(N_0 m)^{-x}$. Assume $N_0^x(m^x - m^{-(1-x)}) - \delta\lambda > 0$.

By convention we set $x = 0$ when $p > 0$; m_*^{-nx} is the convergence rate to 0 of $\{p_n\}$.

In Dion and Yanev [2], the branching process with immigration independent of reproduction is viewed as a BGW (Bienaymé-Galton-Watson) defined according to "diagonal stopping lines", and starting from a random number of ancestors $Z_0(n)$ which is the number of immigrants up to generation $n-1$. But here, since the migration is controlled by the native population, we can show that the branching processes $\{N_n\}_n$ and $\{N_n^{bef}\}_n$ are non homogeneous BGW branching processes starting from the initial population size N_0 itself. $\{N_n\}_n$ corresponds to the individual δ-migration whereas $\{N_n^{bef}\}_n$ corresponds to the familial δ-migration. But $\{N_n^{obs}\}_n$ is generally not a martingale. The extinction time is the same one for the three processes to within one generation. We show that the asymptotic behaviour of $\{N_n\}_n$ and $\{N_n^{bef}\}_n$ does not depend on x, which is not the case of $\{N_n^{obs}\}_n$, the convergence rate of which depends on whether $x < 1$ or $x = 1$; $N_n m_*^{-n}$ and $N_n^{bef}[mm_*^{n-1}]^{-1}$ converge a.s. and in L^2 to a non degenerate variable W, $0 \leq W < \infty$, $E(W|N_0) > 0$ (for a sufficiently large N_0, when $\delta = 1$ and $\{p_n\}_n$ is stochastic). These results are a consequence of Klebaner'result concerning size-dependent processes when $\{p_{N_n^{bef}}\}_n$ is stochastic ([8]). For $x < 1$, $N_n^{obs}[m\tilde{p}_n m_*^{n-1}]^{-1}$ converges also a.s. and in L^2 to $W^{1-\tilde{x}}$, where $\tilde{p}_n = p_n$ and $\tilde{x} = 0$ when $p_{N_n^{bef}}$ is deterministic and $\tilde{p}_n = \lambda m^{-nx}$ and $\tilde{x} = x$ when $p_n = \lambda(N_n^{bef})^{-x}$. These results concerning a deterministic and homogeneous normalization of the processes are robust results with respect to the non homogeneity of the processes. Next using the normalization associated with the martingale deduced from $\{N_n\}_n$, and denoted Π_1^n for simplification, $N_n[\Pi_1^n]^{-1}$, $N_n^{bef}[m\Pi_1^{n-1}]^{-1}$, converges a.s. and in L^2 to W_{N_0}, $E(W_{N_0}) = N_0$. And for $x < 1$, $N_n^{obs}[mp_{N_{n-1}}\Pi_1^{n-1}]^{-1}$ converges a.s. and in L^2 to W_{N_0}, where $mp_{N_{n-1}} = E(\sum_1^{Y_{n,1}} N_{n,1,j}^{obs}|N_{n-1})$. And the same with the normalization associated with the martingale $\{N_n^{bef}\}_n$, the convergence occuring to $W_{N_0}^{bef}$, $E(W_{N_0}^{bef}) = N_0$.

In all the cases, the convergence in L^2 is obtained with an additional assumption when the normalization is stochastic, that is $x > \delta - \ln(\lambda^{-1}(m-1))(\ln m)^{-1}$.

In the case $x = 1$, N_n^{obs} converges in distribution to a Poisson variable with a deterministic or random parameter depending on whether $\{p_{N_n^{bef}}\}_n$ is stochastic or deterministic and $N_n^{obs}[mp_{N_{n-1}}\Pi_1^{n-1}]^{-1}$ converges in distribution to the previous Poisson distribution multiplied either by a random variable or a constant. Moreover when $\delta = -1$ (immigration), the model corresponds asymptotically to the model already described in the litterature as a branching process with a Poisson immigration independent of the native population.

By convention, $\sum_1^0 = 0$.

2 Asymptotic behaviour of $\{N_n\}_n$, $\{N_n^{bef}\}_n$ and $\{N_n^{obs}\}_n$

2.1 Asymptotic behaviour of $\{N_n\}_n$ and $\{N_n^{bef}\}_n$

Let $Y_{*n,i} = \sum_{j=1}^{Y_{n,i}}(1-\delta N_{n,i,j}^{obs})$, $i = 1,\ldots,N_{n-1}$. Denote $m_{*N_{n-1}} = E(Y_{*n,1}|\mathcal{F}_{n-1})$ and $\sigma^2_{*N_{n-1}} = Var(Y_{*n,1}|\mathcal{F}_{n-1})$, where $\mathcal{F}_{n-1}$ is the σ-algebra generated by $N_0, N_1, \ldots, N_{n-1}$.

Let $Y_{*n,i}^{bef} = \sum_{j=1}^{1-\delta N_{n-1,i}^{obs}} Y_{n,i,j}$. Denote $m^{bef}_{*N^{bef}_{n-1}} = E(Y^{bef}_{*n,1}|\mathcal{F}^{bef}_{n-1})$ and $\sigma^{2bef}_{*N^{bef}_{n-1}} = Var(Y^{bef}_{*n,1}|\mathcal{F}^{bef}_{n-1})$, where $\mathcal{F}^{bef}_{n-1}$ is the σ-algebra generated by $N_0, N_1^{bef}, \ldots, N^{bef}_{n-1}$.

Denote $Y^{obs}_{n,1} = \sum_1^{Y_{n,1}} N^{obs}_{n,1,j}$, $m^{obs}_{N_{n-1}} = E(Y^{obs}_{n,1}|\mathcal{F}_{n-1})$ and $\sigma^{2obs}_{N_{n-1}} = Var(Y^{obs}_{n,1}|\mathcal{F}_{n-1})$.

When $\{p_{N_n^{bef}}\}_n$ is a deterministic sequence, $m_{*N_{n-1}}$, $\sigma^2_{*N_{n-1}}$ depend only on n and F and will be also denoted respectively m_{*n}, σ^2_{*n}. And the same concerning $m^{bef}_{*N_{n-1}}$, $\sigma^{2bef}_{*N_{n-1}}$, $m^{obs}_{N_{n-1}}$ and $\sigma^{2obs}_{N_{n-1}}$. Denote $p_{N_{n-1}} = \lambda m^{-1} N_{n-1}^{-x} m_{n,1-x}$ when $N_{n-1} > 0$, where $m_{n,1-x} = E(\overline{Y}_n^{1-x}|\mathcal{F}_{n-1}, N_{n-1} > 0)$, $\overline{Y}_n = \frac{\sum_1^{N_{n-1}} Y_{n,i}}{N_{n-1}}$. Denote also $\sigma^2_{n,1-x} = Var(\overline{Y}_n^{1-x}|\mathcal{F}_{n-1})$. We set $p_{N_{n-1}} = 0$, if $N_{n-1} = 0$.

Lemma 1 *1.* $m_{n,1-x} \le m^{1-x}$*;*

2. On the non-extinction set, we have $\lim_n m_{n,1-x} = m^{1-x}$ *and* $\lim_n N_{n-1}^{1-(1+\varepsilon)x}\sigma^2_{n,1-x} = 0$*, for each* $\varepsilon > 0$*.*

PROOF.

1. Use $E(\overline{Y}_n|\mathcal{F}_{n-1}) = m$ and the Lyapunov inequality $[E(|X|^s)]^{1/s} \le [E(|X|^r)]^{1/r}$, $0 < s < r$, with $r = 1$ and $s = 1-x$.

2. First according to ([4]), $N_n \to \infty$ on the non-extinction set. Next use the standard result (R): if X_n and X are $\mathcal{L}^r$ r.v.s and $\lim_n X_n \overset{\mathcal{L}^r}{=} X$ then $\lim_n E(|X_n|^s) = E(|X|^s)$ for each $0 < s \le r$ ([1]). For the first result, apply to $X_n = \overline{Y}_n$, $X = m$, $r = 2$ and $s = 1-x$, and for the second result, use $Var(X_n) = E(X_n^2) - [E(X_n)]^2$ and apply to $X_n = 1_{\{N \le N_{n-1}} N^{\frac{1-(1+\varepsilon)x}{2(1-x)}} \overline{Y}_n$, $r = 2$ and $s = 2(1-x)$ for the first term and $s = 1-x$ for the second term.

□

Proposition 1 *1.* $\{N_n\}_n$ *is an inhomogeneous branching process generated by* $\{\mathcal{L}(Y_{*n,1})\}_n$. *When* $\{p_{N_n^{bef}}\}_n$ *is deterministic,* m_{*n} *and* σ^2_{*n} *are given by*

$$m_{*n} = m(1-\delta p_n); \sigma^2_{*n} = \sigma^2(1-\delta p_n)^2 + \delta^2 m p_n (1-p_n). \tag{4}$$

When $p_{N_n^{bef}} = \lambda(N_n^{bef})^{-x}$, $m_{*N_{n-1}}$ *and* $\sigma^2_{*N_{n-1}}$ *satisfy*

$$m_{*N_{n-1}} = m(1-\delta p_{N_{n-1}}) \tag{5}$$

$$\sigma^2_{*N_{n-1}} \leq (\sigma + |\delta| C_1 N_{n-1}^{-x/2})^2, \tag{6}$$

with equality when $\delta = 0$, and where $0 < C_1 < \infty$ is function of m, σ^2.

2. *$\{N_n^{bef}\}_n$ is an inhomogeneous branching process generated by $\{\mathcal{L}(Y^{bef}_{*n,1})\}_{n\geq 2}$, and by $\mathcal{L}(Y_{1,1})$, $n = 1$. $m^{bef}_{*N^{bef}_{n-1}}$ and $\sigma^{2bef}_{*N^{bef}_{n-1}}$ satisfy*

$$m^{bef}_{*N^{bef}_{n-1}} = m(1-\delta p_{N^{bef}_{n-1}});\ \sigma^{bef}_{*N^{bef}_{n-1}} = \sigma^2(1-\delta p_{N^{bef}_{n-1}}) + m^2\delta^2 p_{N^{bef}_{n-1}}(1-p_{N^{bef}_{n-1}}). \tag{7}$$

Moreover when $p_{N_n^{bef}} = \lambda(N_n^{bef})^{-x}$, then, $\{N_n\}_n$ and $\{N_n^{bef}\}_n$ are size dependent branching processes ([8]).

Proof.

1. The branching property of $\{N_n\}_n$ is deduced directly from model (M):

$$N_n = \sum_{i=1}^{N_{n-1}} Y_{*n,i} \text{ and } N_n^{obs} = \sum_{i=1}^{N_{n-1}} Y_{n,i}^{obs}, \tag{8}$$

where $Y_{*n,i} = \sum_{j=1}^{Y_{n,i}}(1-\delta N^{obs}_{n,i,j})$ and $Y^{obs}_{n,i} = \sum_{j=1}^{Y_{n,i}} N^{obs}_{n,i,j}$, the $\{N^{obs}_{n,i,j}\}_{i,j}$ being i.i.d. according to $B_{p_{N_n^{bef}}}$, given N_n^{bef}. Therefore $\{N_n\}_n$ is an inhomogeneous BGW branching process generated by the conditional distribution of $Y_{*n,i}$ given $\mathcal{F}_{n-1}$.

When $\{p_{N_n^{bef}}\}_n$ is a deterministic sequence $\{p_n\}_n$, m_{*n} can be calculated directly from the definition of $Y_{*n,1}$, and σ^2_{*n} from

$$Y_{*n,1} - m_{*n} = \delta \sum_{j=1}^{Y_{n,1}} (p_n - N^{obs}_{n,1,j}) + (1-\delta p_n)(Y_{n,1} - m).$$

Assume now that $p_{N_n^{bef}} = \lambda(N_n^{bef})^{-x}$. To calculate $m_{*N_{n-1}}$, use first on one hand the relationship deduced from (2) and (3):

$$\begin{aligned} E(N_n|\mathcal{F}_{n-1}) &= E(N_n^{bef}|\mathcal{F}_{n-1}) - \delta E(E(N_n^{obs}|N_n^{bef}, \mathcal{F}_{n-1})|\mathcal{F}_{n-1}) \\ &= mN_{n-1} - \delta\lambda E((N_n^{bef})^{1-x}|\mathcal{F}_{n-1}), \end{aligned} \tag{9}$$

and on the other hand, the branching property $N_n = \sum_1^{N_{n-1}} Y_{*n,i}$

$$E(N_n|\mathcal{F}_{n-1}) = N_{n-1} m_{*N_{n-1}}. \tag{10}$$

Comparing (9) and (10) leads to

$$m_{*N_{n-1}} = m(1 - \delta\lambda m^{-1} N_{n-1}^{-x} m_{n,1-x}). \tag{11}$$

Next, from $Y_{*n,1} - m_{*N_{n-1}} = (Y_{n,1} - m) - \delta(Y_{n,1}^{obs} - m_{N_{n-1}}^{obs})$,

(12) $\quad \sigma_{*N_{n-1}}^2 = \sigma^2 + \delta^2 \sigma_{N_{n-1}}^{2obs} - 2\delta E[(Y_{n,1} - m)(Y_{n,1}^{obs} - m_{N_{n-1}}^{obs})|\mathcal{F}_{n-1}]$.

But

$$|E[(Y_{n,1} - m)(Y_{n,1}^{obs} - m_{N_{n-1}}^{obs})|\mathcal{F}_{n-1}]| \leq \sigma\sigma_{N_{n-1}}^{obs}$$

implying by lemma 2.1.2 the bounding of $\sigma_{*N_{n-1}}^2$.

2. N_n^{bef} can be written

$$\begin{aligned} N_n^{bef} &= \sum_{i=1}^{N_{n-1}^{bef}} \sum_{j=1}^{1-\delta N_{n-1,i}^{obs}} Y_{n,i,j} \\ &\overset{not.}{=} \sum_{i=1}^{N_{n-1}^{bef}} Y_{*n,i}^{bef}. \end{aligned}$$

Then as for $\{N_n\}_n$, we obtain $m_{*N_{n-1}^{bef}}^{bef} = E(1-\delta N_{n-1,1}^{obs}|\mathcal{F}_{n-1}^{bef})E(Y_{n,1})$ and $\sigma_{*N_{n-1}^{bef}}^{2bef} = \sigma^2 E(1 - \delta N_{n-1,1}^{obs}|\mathcal{F}_{n-1}^{bef}) + m^2 Var(\delta N_{n-1,1}^{obs}|\mathcal{F}_{n-1}^{bef})$.

□

Lemma 2 *1. Assume $\{p_{N_n^{bef}}\}_n$ is a deterministic sequence. then*

$$m_n^{obs} = mp_n \text{ and } \sigma_n^{2obs} = \sigma^2 p_n^2 + mp_n(1 - p_n)$$

2. Assume $p_{N_n^{bef}} = \lambda(N_n^{bef})^{-x}$. Then

(a) $m_{N_{n-1}}^{obs} = mp_{N_{n-1}}$ and $m_{N_{n-1}}^{obs} \leq \lambda N_{n-1}^{-x} m^{1-x}$;

(b) There exists $0 < C < \infty$ function of m and σ^2 such that $\sigma_{N_{n-1}}^{2obs} \leq CN_{n-1}^{-x}$.

Proof.

1. The proof follows directly from the definition of $Y_{n,1}^{obs}$.

2. (a) From the relationships $m_{*N_{n-1}} = m - \delta m_{N_{n-1}}^{obs}$ obtained from the definition of $Y_{*n,1}$, and $m_{*N_{n-1}} = m(1 - \delta\lambda m^{-1} N_{n-1}^{-x} m_{n,1-x})$ (cf (11)), deduce

(13) $$m_{N_{n-1}}^{obs} = \lambda N_{n-1}^{-x} m_{n,1-x}.$$

Finally use item 1 of lemma 2.1.1.

(b) From $N_n^{obs} = \sum_1^{N_{n-1}}(Y_{n,i}^{obs} - m_{N_{n-1}}^{obs}) + m_{N_{n-1}}^{obs} N_{n-1}$ deduce

$$E((N_n^{obs})^2|\mathcal{F}_{n-1}) = N_{n-1}\sigma_{N_{n-1}}^{2obs} + (m_{N_{n-1}}^{obs})^2 N_{n-1}^2. \tag{14}$$

Next using (3), $N_n^{obs} = \sum_1^{N_n^{bef}}(N_{n,j}^{obs} - p_{N_n^{bef}}) + p_{N_n^{bef}} N_n^{bef}$ which implies

$$E((N_n^{obs})^2|N_n^{bef}, \mathcal{F}_{n-1}) = \lambda(N_n^{bef})^{1-x}(1 - p_{N_n^{bef}}) + \lambda^2(N_n^{bef})^{2-2x},$$

obtain

$$\begin{aligned} &E((N_n^{obs})^2|\mathcal{F}_{n-1}) \\ = \; &\lambda N_{n-1}^{1-x} E(\overline{Y}_n^{1-x}(1 - p_{N_n^{bef}})|\mathcal{F}_{n-1}) + \lambda^2 N_{n-1}^{2-2x} E(\overline{Y}_n^{2-2x}|\mathcal{F}_{n-1}). \end{aligned} \tag{15}$$

Comparing (14) and (15) and using (13) yields

$$\sigma_{N_{n-1}}^{2obs} = \lambda N_{n-1}^{-x} E(\overline{Y}_n^{1-x}(1-p_{N_n^{bef}})|\mathcal{F}_{n-1}) + \lambda^2 N_{n-1}^{1-2x} Var(\overline{Y}_n^{1-x}|\mathcal{F}_{n-1}). \tag{16}$$

from which we deduce $\sigma_{N_{n-1}}^{2obs} \leq \lambda N_{n-1}^{-x} m_{n,1-x} + \lambda^2 N_{n-1}^{1-2x} Var(\overline{Y}_n^{1-x}|\mathcal{F}_{n-1})$. Now according to item 2 of lemma 2.1.1,

$$\begin{aligned} N_{n-1}^{1-2x}\sigma_{n,1-x}^2 &= N_{n-1}^{-(1-\varepsilon)x} N_{n-1}^{1-(1+\varepsilon)x}\sigma_{n,1-x}^2 \\ &\leq O_\varepsilon(1) N_{n-1}^{-(1-\varepsilon)x} \end{aligned}$$

implying, since ε is arbitrary,

$$N_{n-1}^{1-2x}\sigma_{n,1-x}^2 = O(1) N_{n-1}^{-x}. \tag{17}$$

Using (16) and (17), we obtain $\sigma_{N_{n-1}}^{2obs} = O(1)N_{n-1}^{-x}$ and since $\sigma_{N_{n-1}}^{2obs} \leq m^2 + \sigma^2$ because $Y_{n,1}^{obs} \leq Y_{n,1}$, then there exists $0 < C < \infty$ such that $\sigma_{N_{n-1}}^{2obs} \leq C N_{n-1}^{-x}$.

□

Proposition 2 *Assume* $p_{N_n^{bef}} = \lambda(N_n^{bef})^{-x}$. *Then* $N_n m^{-n}$ *and* $N_n^{bef} m^{-n}$ *converge a.s. and in* L^2 *to a non degenerate and non negative random variable* W *such that* $0 \leq W < \infty$, $P(W > 0) > 0$ *and* $E(W|N_0) = [N_0(m^x - 1) - \delta\lambda E(W^{1-x}|N_0)](m^x - 1)^{-1}$.

Proof. Prove the result concerning N_n. The proof is similar concerning N_n^{bef}. The result is obtained by using Klebaner's theorem 1.7 ([8]) (according to lemma 2.1.2 and proposition 2.1.1 $|m_{*n} - m|$ and σ_{*n}^2 satisfy the assumptions of theorem 1.7). Calculate $E(W|N_0)$. Using (5) and $E(N_n|\mathcal{F}_{n-1}) = m_{*N_{n-1}} N_{n-1}$, we have $E(N_n|\mathcal{F}_{n-1}) = mN_{n-1} - \delta\lambda N_{n-1}^{1-x} m_{n,1-x}$ implying $E(N_n|N_0) = m^n N_0 - \delta\lambda \sum_0^{n-1} m^k a_{N_{n-1-k}} m^{(n-1-k)(1-x)}$, where $a_{N_n} = E((N_n m^{-n})^{1-x}\overline{Y}_{n+1}^{1-x}|N_0)$. Since $N_n m^{-n}$ and $\overline{Y}_n$ converge in L^2 to W and m

respectively, by Hölder inequality, $N_n m^{-n}\overline{Y}_n$ converges in L^1 to Wm and then by the standard result (R), $E(a_n|N_0)$ tends to $E((Wm)^{1-x}|N_0)$. Consequently

$$E(\frac{N_n}{m^n}|N_0) = N_0 - \delta\lambda m^{-(1-x)} \frac{\sum_0^{n-1} a_{N_{n-1-k}} m^{kx}}{\sum_0^{n-1} m^{kx}} \frac{\sum_0^{n-1} m^{kx}}{m^{nx}}$$

implying the result by Toeplitz's lemma.

We prove in the same way the convergence of $N_n^{bef} m^{-n}$ to W^{bef}. We show now that $W^{bef} \overset{a.s.}{=} W$. From (2)

$$\frac{N_n^{bef}}{m^n} = \frac{\sum_1^{N_{n-1}} Y_{n,i} m^{-1}}{N_{n-1}} \frac{N_{n-1}}{m^{n-1}}$$

which, using the strong law of large numbers and the a.s. convergence of $N_n m^{-n}$, converges a.s. to W on $\{W > 0\}$. Comparing this result with $\lim_n N_n^{bef} m^{-n} \overset{a.s.}{=} W^{bef}$ leads to $W^{bef} \overset{a.s.}{=} W$. □

Corollary 1 *Assume* $p_{N_n^{bef}} = \lambda(N_n^{bef})^{-x}$. *We have a.s. on* $\{W > 0\}$

$$0 < \Pi_1^\infty(1 - \delta p_{N_{k-1}}) < \infty \text{ and } 0 < \Pi_1^\infty(1 - \delta p_{N_{k-1}^{bef}}) < \infty.$$

Proof. First $\Pi_1^\infty(1 - \delta p_{N_{k-1}})$ exists because $\{\Pi_1^n(1 - \delta p_{N_{k-1}})\}_n$ is a monotonic sequence. Next $0 < \Pi_1^\infty(1 - \delta p_{N_{k-1}}) < \infty$ if $\sum |\ln(1 - \delta\lambda m^{-1} N_{k-1}^{-x} m_{k,1-x})| < \infty$, *i.e.* if $|\delta|\lambda m^{-1} \sum_k N_{k-1}^{-x} m_{k,1-x} < \infty$ which is satisfied a.s. on $\{W > 0\}$ since using lemma 2.1.1 and proposition 2.1.2,

$\sup_k (N_{k-1} N_k^{-1})^x m_{k+1,1-x} m_{k,1-x}^{-1} = m^{-x} < 1$, a.s. (D'Alembert's criterion). The proof is similar for the other relationship. □

Lemma 3 . *Assume* $p_n = \lambda(N_0 m \Pi_1^{n-1} m(1-\delta p_k))^{-x}$, $0 < x \leq 1$. *Then* $m(1-\delta p_n) > 1$, *for all* n, $0 < \Pi_1^\infty(1 - \delta p_n) < \infty$ *and* $\lim_n p_n = 0$.

Proof. First $m(1 - \delta p_1) > 1$ and $p_{n+1} p_n^{-1} = [m(1 - \delta p_n)]^{-x}$. Therefore assuming $m(1-\delta p_n) > 1$, then $p_{n+1} < p_n$ and $m(1-\delta p_{n+1}) > m(1-\delta p_n) > 1$, for all n, when $\delta = 1$. Consequently $\lim_n m(1-\delta p_n) \geq m(1-\delta p_1) > 1$ when $\delta = 1$, and $\lim_n m(1-\delta p_n) \geq m > 1$ when $\delta = -1$ or $\delta = 0$. $\{p_n\}_n$ being a bounded decreasing sequence in $[0, 1]$, $\lim_n p_n$ exists and is in $[0, 1]$. Next we show that $0 < \Pi_k(1 - \delta p_k) < \infty$. This is satisfied if $\sum_k |\ln(1 - \delta p_k)| < \infty$, that is if $|\delta| \sum_k p_k < \infty$. This last condition holds since $\overline{\lim}_n p_{n+1} p_n^{-1} = \lim_n [m(1-\delta p_n)]^{-x}$ is less than 1 (D'Alembert's criterion). Consequently $0 < \Pi_k(1 - \delta p_k) < \infty$ which implies $\lim_n p_n = 0$. □

Let $W_{N_0,n} = N_n(\Pi_1^n m_{*N_{k-1}})^{-1}$, $W_{N_0,n}^{bef} = N_n^{bef}(m\Pi_1^{n-1} m_{*N_{k-1}})^{-1}$, $W_{N_0,n}^{obs} = N_n^{obs}(m_{N_{n-1}}^{obs} \Pi_1^{n-1} m_{*N_{k-1}})^{-1}$.

Proposition 3 .

1. *Assume* $\{p_n\}_n$ *is deterministic. Then* $\{W_{N_0,n}\}_n$ *and* $\{W^{bef}_{N_0,n}\}_n$ *converge a.s. and in* L^2 *to a non degenerate random variable* W_{N_0}, $E(W_{N_0}|N_0) = N_0$. *Moreover* $N_n m_*^{-n}$ *and* $N_n^{bef}[mm_*^{n-1}]^{-1}$ *converge a.s. and in* L^2 *to* $W = \Pi_1^\infty[(1-\delta p_k)(1-\delta p)^{-1}]W_{N_0}$.

2. *Assume* $p_{N_n^{bef}} = \lambda(N_n^{bef})^{-x}$. *Then* $\{W_{N_0,n}\}_n$ *and* $\{W^{bef}_{N_0,n}\}_n$ *converge a.s. to a non degenerate random variable* $W_{N_0} = W[\Pi_1^\infty(1-\delta\lambda p_{N_{k-1}})]^{-1}$, $\{W_{N_0} > 0\} \overset{a.s.}{\supset} \{W > 0\}$ *with equality when* $\delta = 0$ *or* $\delta = -1$. *Moreover* $\{W_{N_0,n}\}_n$ *and* $\{W^{bef}_{N_0,n}\}_n$ *converge also in* L^2 *to* W_{N_0} *when* $x > \delta - \ln(\lambda^{-1}(m-1))(\ln m)^{-1}$. *In that case* $E(W_{N_0}|N_0) = N_0$.

Proof.

1. The case $p > 0$ is explained in Jacob and Peccoud ([6]). When $p_n = \lambda(N_0 m\Pi_1^{n-1}m(1-\delta p_k))^{-x}$ with $0 < x \leq 1$, using lemma 2.1.3, we show as for $p > 0$, that $W_{N_0,n}$ and $W^{bef}_{N_0,n}$ are non negative martingales with finite first two moments because $\lim_n \Pi_1^{n-1}m(1-\delta p_k) = \infty$, as $n \to \infty$. Finally, $0 < \Pi_1^\infty[(1-\delta p_k)(1-\delta p)^{-1}] < \infty$, implying $\lim_n N_n m_*^{-n} \overset{a.s.,L^2}{=} W$ and $\lim_n N_n^{bef}[mm_*^{n-1}]^{-1} \overset{a.s.,L^2}{=} W$.

2. When $\{p_{N_n^{bef}}\}_n$ is the random sequence $\{\lambda(N_n^{bef})^{-x}$, $\{W_{N_0,n}\}_n$ is still a non negative martingale (since $m_{*N_{k-1}} > 0$), with expectation N_0, and therefore converges a.s. to a non degenerate random variable W_{N_0}. Show now that $W_{N_0} \overset{a.s.}{=} \Pi_1^\infty[m_{*N_{k-1}}^{-1}m]W$ and that $\{W_{N_0} > 0\} \overset{a.s.}{\supset} \{W > 0\}$. By proposition 2.1.2, $W_{N_0,n} = N_n m^{-n}[\Pi_1^n(1-\delta p_{N_{k-1}})]^{-1}$ converges a.s. both to $W[\Pi_1^\infty(1-\delta p_{N_{k-1}})]^{-1}$ and to W_{N_0} implying $W_{N_0} = W[\Pi_1^\infty(1-\delta p_{N_{k-1}})]^{-1}$. Using corollary 2.1.1, $\{W > 0\} \subset \{W_{N_0} > 0\}$, with equality when $\delta = 0$ or $\delta = -1$, because $0 \leq W < \infty$ and $\Pi_1^\infty(1-\delta p_{N_{k-1}}) \geq 1$.

   Next using

$$W_{N_0,n} = \frac{1}{\Pi_1^n m_{*N_{k-1}}}\sum_1^{N_{n-1}}(Y_{*n,i} - m_{*N_{n-1}}) + W_{N_0,n-1}$$

   we obtain iteratively

$$E(W^2_{N_0,n}|N_0) = \sum_{k=1}^n E(\frac{\sigma^2_{*N_{k-1}}N_{k-1}}{[\Pi_1^k m_{*N_{l-1}}]^2}|N_0) + N_0^2.$$

   And by lemma 2.1.1, $m_{*N_{k-1}} \geq (m + \inf\{-\delta, 0\}\lambda m^{1-x}$ and by lemma 2.1.2, there exists a constant C such that $\sigma^2_{*N_{k-1}} \leq C$. Consequently

$$E(W^2_{N_0,n}|N_0) \leq C\sum_1^n E(\frac{N_{k-1}}{\Pi_1^{k-1}m_{*N_{l-1}}}|N_0)\frac{1}{(m+\inf\{-\delta,0\}\lambda m^{1-x}))^{k+1}} + N_0^2. \qquad (18)$$

Since $E(N_{k-1}(\Pi_1^{k-1}m_{*N_{l-1}})^{-1}|N_0) = N_0$ and assuming $m+\inf\{-\delta,0\}\lambda m^{1-x}) > 1$, then $\overline{\lim}_n E(W^2_{N_0,n}|N_0) < \infty$. Therefore, $W_{N_0,n}$ being a martingale with a finite second moment, it converges in L^2 to W_{N_0}.

Concerning $W^{bef}_{N_0,n}$, as previously since $W^{bef}_{N_0,n} = N^{bef}_n m^{-n}[\Pi_1^{n-1}(1-\delta p_{N_{k-1}})]^{-1}$, $W^{bef}_{N_0,n}$ converges a.s. to $W[\Pi_1^{\infty}(1-\delta p_{N_{k-1}})]^{-1} = W_{N_0}$. Next using

$$W^{bef}_{N_0,n} = \frac{1}{m\Pi_1^{n-1}(1-\delta p_{N_{k-1}})}\sum_{i=1}^{N_{n-1}}(Y_{n,i}-m) + W_{N_0,n-1},$$

yields, as for $W^2_{N_0,n}$,

$$\lim_n E((W^{bef}_{N_0,n} - W_{N_0,n-1})^2|N_0) = 0.$$

Therefore the convergence in L^2 of $W^{bef}_{N_0,n}$ follows from the convergence in L^2 of $W_{N_0,n-1}$.

□

2.2 Asymptotic behaviour of $\{N^{obs}_n\}_n$

Let $\tilde{p}_n = p_n$ when $p_{N^{bef}_n}$ is deterministic, and $\tilde{p}_n = \lambda m^{-nx}$, when $p_{N^{bef}_n} = \lambda(N^{bef}_n)^{-x}$. Then, when p_n is deterministic, $m^{obs}_{N_{n-1}}[m\tilde{p}_n]^{-1} = 1$ and when $p_{N^{bef}_n} = \lambda(N^{bef}_n)^{-x}$, $\lim_n m^{obs}_{N_{n-1}}[m\tilde{p}_n]^{-1} \overset{a.s.}{=} W^{-x}$.

Proposition 4 *Assume $x < 1$. Let $\tilde{x} = 0$ when p_n is deterministic and $\tilde{x} = x$ when $p_{N^{bef}_n} = \lambda(N^{bef}_n)^{-x}$.Then*

$$\lim_n \frac{N^{obs}_n}{m\tilde{p}_n m^{n-1}} \overset{a.s.,L^2}{=} W^{1-\tilde{x}}.$$

PROOF. Assume $p_{N^{bef}_n} = \lambda(N^{bef}_n)^{-x}$. The proof in the deterministic case is similar. According to (3)

$$\frac{N^{obs}_n}{m\tilde{p}_n m^{n-1}} = \frac{\sum_1^{N_{n-1}} Y^{obs}_{n,i}(m^{obs}_{N_{n-1}})^{-1}}{N_{n-1}}\frac{N_{n-1}}{m^{n-1}}\frac{m^{obs}_{N_{n-1}}}{m\tilde{p}_n}. \tag{19}$$

On the non extinction set, by the standard law of large numbers (the Kolmogorov condition is satisfied: $\sum_{N_{n-1}}\sigma^{2obs}_{N_{n-1}}(m^{obs}_{N_{n-1}})^{-2}N^{-2}_{n-1} \leq \sum_n O(1)n^{-(2-x)}$ converges for $x < 1$) and according to proposition 2.1.2, $N^{obs}_n[m\tilde{p}_n m^n]^{-1}$ converges a.s. to W^{1-x}. Next, we study the convergence in L^2. According to (19)

$$\begin{aligned} & E[(\frac{N^{obs}_n}{m\tilde{p}_n m^{n-1}} - (\frac{N_{n-1}}{m^{n-1}})^{1-x})^2|\mathcal{F}_{n-1}] \\ = \;& \frac{N_{n-1}\sigma^{2obs}_{N_{n-1}}}{[m\tilde{p}_n m^{n-1}]^2} + [\frac{m^{obs}_{N_{n-1}}N_{n-1}}{m\tilde{p}_n m^{n-1}} - (\frac{N_{n-1}}{m^{n-1}})^{1-x}]^2. \end{aligned} \tag{20}$$

Next

$$\frac{m^{obs}_{N_{n-1}}N_{n-1}}{m\tilde{p}_n m^{n-1}} = \frac{E([\sum_1^{N_{n-1}} Y_{n,i}m^{-1}]^{1-x}|\mathcal{F}_{n-1})}{m^{(n-1)(1-x)}}$$

which implies

$$E[(\frac{N^{obs}_n}{m\tilde{p}_n m^{n-1}} - (\frac{N_{n-1}}{m^{n-1}})^{1-x})^2|\mathcal{F}_{n-1}] = \frac{N_{n-1}\sigma^{2obs}_{N_{n-1}}}{[m\tilde{p}_n m^{n-1}]^2} + (\frac{N_{n-1}}{m^{n-1}})^{2(1-x)}(\frac{m_{n,1-x}}{m^{1-x}} - 1)^2.$$

By the same argument as item 2 of lemma 2.1.1, we have $N^{\frac{1-(1+\varepsilon)x}{2}}(m_{n,1-x}m^{-(1-x)} - 1)$ converges a.s. to 0 on the non extinction set, and since by item 1 of lemma 2.1.1, $(m_{n,1-x}m^{-(1-x)} - 1) < 2$, then $(m_{n,1-x}m^{-(1-x)} - 1) = O(1)N^{\frac{-(1-x)}{2}}$ with $O(1) < C'$, $0 < C' < \infty$ and therefore by lemma 2.1.1

$$E[(\frac{N^{obs}_n}{m\tilde{p}_n m^{n-1}_*} - (\frac{N_{n-1}}{m^{n-1}_*})^{1-x})^2|N_0] \leq C''E((\frac{N_{n-1}}{m^{n-1}})^{1-x}|N_0)\frac{1}{m^{(n-1)(1-x)}}$$

which tends to 0. □

Proposition 5 *1. Assume $p > 0$. Then $\{W^{obs}_{N_0,n}\}_n$ converges a.s. and in L^2 to W_{N_0}.*

2. *Assume $p_n = \lambda(N_0 m\Pi_1^{n-1}m(1-\delta p_k))^{-x}$, $0 < x < 1$. Then $\{W^{obs}_{N_0,n}\}_n$ converges a.s. and in L^2 to W_{N_0}.*
3. *Assume $p_{N^{bef}_n} = \lambda(N^{bef}_n)^{-x}$. Then $W^{obs}_{N_0,n}$ converges a.s. to W_{N_0}. Moreover if $\delta - \ln(\lambda^{-1}(m-1))(\ln m)^{-1} < x < 1$, $W^{obs}_{N_0,n}$ converges also in L^2.*
4. *Assume $p_n = \lambda(N_0 m\Pi_1^{n-1}m(1-\delta p_k))^{-1}$. On $\{W_{N_0} > 0\}$, N^{obs}_n converges in distribution to the Poisson distribution $\mathcal{P}(\lambda N_0^{-1}W_{N_0})$ with parameter $\lambda N_0^{-1}W_{N_0}$, and $W^{obs}_{N_0,n}$ converges in distribution to $\lambda^{-1}N_0 P(\lambda N_0^{-1}W_{N_0})$.*
5. *Assume $p_{N^{bef}_n} = \lambda(N^{bef}_n)^{-1}$. On $\{W_{N_0} > 0\}$, N^{obs}_n converges in distribution to the Poisson distribution $\mathcal{P}(\lambda)$ with parameter λ and $W^{obs}_{N_0,n}$ converges in distribution to $\lambda^{-1}W_{N_0}P(\lambda)$.*

Proof.

1. The proof is given in ([6]). See also proposition 2.2.1.
2. The proof is the same as for the case $p > 0$.
3. The proof is similar to proposition 2.2.1 proof.
4. The first result follows directly from $p_n N^{bef}_n = \lambda N_0^{-1} W^{bef}_{N_0,n}$ and from the convergence of $W^{bef}_{N_0,n}$. The second result follows directly from $W^{obs}_{N_0,n} = N^{obs}_n \lambda^{-1} N_0$.

5. The first result follows directly from $p_n N_n^{bef} = \lambda$ and from $\lim_n N_n^{bef} \overset{a.s.}{=} \infty$ on the non extinction set. Moreover $W_{N_0,n}^{obs} = N_n^{obs} W_{N_0,n}^{bef} \lambda^{-1}$ converges in distribution to $\mathcal{P}(\lambda) W_{N_0} \lambda^{-1}$.

□

Bibliography

[1] Y. S. Chow, H. Teicher. Probability Theory: Independence, Interchangeability, Martingales, Springer-Verlag, New-York, 1978.

[2] J. P. Dion, N. M. Yanev. Limit theorems and estimation theory for branching processes with an increasing random number of ancestors. *J. Appl. Prob.* **34**,2, (1997), 309-327.

[3] J. C. O'Souza, J. D. Biggins. The supercritical Galton-Watson process in varying environments. *Stoch. Proc. Appl.* **42** (1992), 39–47.

[4] T. Fujimagar. Controlled Galton-Watson process and its asymptotic behaviour. *Kodai Math. Sem. Rep.* **27** (1976), 11–18.

[5] S. C. Gupta, O. P. Srivastava, Singh Mahendra. Branching Process with Emigration – A Genetic Model. *Math. Biosci.* **111** (1992), 159–168.

[6] C. Jacob, J. Peccoud. Estimation of the parameters of a branching process from migrating binomial observations. *Adv. Appl. Prob.* (to appear in december 1998).

[7] S. V. Kaverin. A refinement of limit theorems for critical branching processes with emigration. *Theory Probab. Appl.* bf 35,3 (1990), 574-580.

[8] F. C. Klebaner. On population-size-dependent branching processes. *Adv. Appl. Prob.* **16** (1984), 30–55.

[9] D. Pierre Loti Viaud. A strong law and a central limit theorem for controlled Galton-Watson processes. *J. Appl. Prob.* **31** (1994), 22–37.

[10] V. A. Vatutin. A critical Galton-Watson branching process with emigration. *Theory Probab. Appl.* **22** (1977), 465-481.

[11] G. V. Vinokurov. On a critical Galton-Watson branching process with emigration. *Theory Probab. Appl.* **32**, 2 (1987), 350-353.

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