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THE MAXIMAL NUMBER OF PARTICLES IN A BRANCHING PROCESS WITH STATE-DEPENDENT IMMIGRATION

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The limiting behavior of the maximal number of particles in the first n generations of a Bienaymé-Galton-Watson branching process with immigration in the state zero is studied.

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1 Introduction

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. Assume that on this space are given the following random variables:

i) A set $X = \{X_i(n), i = 1, 2, ..., n = 1, 2, ...\}$ of independent, identically distributed (i.i.d.), non-negative, integer valued random variables (r.v.) with probability generating function (p.g.f.) $f(s) = \mathcal{E}s^{X_i(n)} = \sum_{k=0}^{\infty} p_k s^k, 0 \leq s \leq 1$.

ii) A set $Y = \{Y_n, n = 0, 1, 2, ...\}$ of i.i.d., positive , integer valued r.v. with p.g.f. $g(s) = \mathcal{E}s^{Y_n} = \sum_{k=1}^{\infty} q_k s^k, 0 \leq s \leq 1$, independent of X.

A Bienaymé-Galton-Watson (BGW) branching process with immigration only in the state zero, can be defined as follows:

$$(1) \qquad Z_0 = Y_0 > 0 \; a.s., \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_i(n+1) + I_{\{Z_n=0\}} Y_{n+1}, \quad n = 0, 1, 2, ...,$$

where, as usual, I_A is the indicator of A and $\sum_{n=1}^{0} \cdot = 0$.

The process Z_n starts with a positive random number of ancestors Y_0 at time $n = 0$ and evolves as a BGW process without any migration up to the moment when it visits the state zero. At the next moment a positive number Y_1 of particles immigrate, the process starts again, and so on.

We shall assume that $0 < p_0 < 1$. This condition together with $q_0 = 0$ ensure that $\{0\}$ is accessible from the positive part of the state space and that the minimal state space is aperiodic; see Foster (1971).

This model was first considered independently by Foster(1971) and Pakes(1971) and was studied in several papers later.

In the present note we shall be concerned with the behavior of the record value of the process Z_n in the first n generations

$$W_n = \max\{Z_0, Z_1, ..., Z_n\}, \quad n = 0, 1, 2, ... \quad . \tag{2}$$

The investigation of W_n might be motivated in different ways. There have been several recent works developing results for certain kinds of extremes in branching processes, and the investigation of the process W_n is perhaps plausible as a contribution to this program.(See e.g. Borovkov and Vatutin (1996), Rahimov and Yanev (1996) and references therein.)

Here we want to pointout that Borovkov and Vatutin (1996) have been investigated the analogical sequence for a critical BGW process without immigration. The main result obtained there is for the tail behavior of limiting distribution function (d.f.). On the other hand, the results obtained here (Theorems 1 and 2) give an explicit form of the limiting d.f.

The other motivation is the ability to use the results for randomly indexed random sequences to obtain the limiting behavior of some global characteristic of a regenerative process.

It is easily seen that the process Z_n is regenerative with regeneration epochs the moments when it visits the state zero, i.e. Z_n consists of a sequence of independent and identically distributed copies of a simple BGW processes with offspring and initial distributions $\{p_I\}$ and $\{q_i\}$ respectively.

Let us denote

$$S_0 = 0, \quad S_1 = \min\{n > 0 : Z_n = 0\}, \quad S_k = \min\{n > S_{k-1} : Z_n = 0\}, \quad k = 2, 3, ... ,$$

the embedded renewal process.

Obviously, the cycle lengths $T_k = S_k - S_{k-1}, k = 1, 2, ...$ have the same distribution as the life-period of a simple BGW process with offspring and initial distributions $\{p_i\}$ and $\{q_i\}$ respectively:

$$\mathcal{P}(T_k > n) = 1 - g(f_n(0)), \quad n = 0, 1, 2, ...$$

(See e.g. Sevastyanov (1971)).

2 Basic conditions and results

In what follows we will suppose the following conditions are fulfilled:

$$g'(1-) = \mu < \infty, \tag{3}$$

$$f(s) = s + (1-s)^{1+\alpha} L\left(\frac{1}{1-s}\right), \quad s \in [0,1], \tag{4}$$

where $\alpha \in (0,1]$ and $L(.)$ is a function slowly varying at infinity.

The second condition characterizes the process Z_n as a critical one with possibly infinite variance of the offspring of a particle.

Under conditions (3) and (4) (see Slack (1968))

$$\mathcal{P}(T_i > n) \sim \mu n^{-1/\alpha} L_1(n), \quad n \to \infty, \tag{5}$$

where $L_1(.)$ is a function slowly varying at infinity.

Let us denote

$$R(x) = \int_0^x \mathcal{P}(T_k > y)dy = \sum_{j \le x} (1 - g(f_j(0)).$$

It is clear that if $\alpha < 1$ then

$$R = \lim_{x \to \infty} R(x) = \mathcal{E}T_i < \infty,$$

but if $\alpha = 1$ then either $R < \infty$ or $R = \infty$ can occur.

So, we will consider, separately, the following two cases:

$$\alpha = 1, \quad \mathcal{E}T_i = \infty, \tag{6}$$

or

$$\alpha \le 1, \quad R = \mathcal{E}T_i < \infty. \tag{7}$$

The main results of the paper are the following theorems.

Theorem 1 *Under conditions (3), (4) and (6)*

$$\lim_{n \to \infty} \mathcal{P}\left(\frac{W_n}{\mu n / R(n)} \le x\right) = Q(x) = \begin{cases} \exp(-1/x) & x > 0, \\ 0 & x \le 0. \end{cases}$$

Theorem 2 *Under conditions (3), (4) and (7)*

$$\lim_{n \to \infty} \mathcal{P}\left(\frac{W_n}{\mu \alpha n / R} \le x\right) = Q(x) \ .$$

3 Proofs of the theorems

We need some preliminary results for the proofs of the theorems.

Let us denote $\nu(n) = \max\{k : S_k \leq n < S_{k+1}\}$ and define the sequence of i.i.d. r.v.

$$M_i = \max_{S_i < k < S_{i+1}} Z_k, \quad i = 0, 1, 2, \ldots \ .$$

So, the r.v. M_i is the global maxima of the simple BGW process which survives between $i-$th and $(i+1)-$st successive visits to the state $\{0\}$.

Denote

$$\overline{M}_n = \max_{1 \leq i \leq n} M_i.$$

the record value in the first n cycles.

Obviously, one has

$$\overline{M}_{\nu(n)} \leq W_n \leq \overline{M}_{\nu(n)+1} \ \ a.s. \ ,$$

and

$$\mathcal{P}(\overline{M}_{\nu(n)+1} \leq x) \leq \mathcal{P}(W_n \leq x) \leq \mathcal{P}(\overline{M}_{\nu(n)} \leq x). \tag{8}$$

Denote by $P(x) = \mathcal{P}(M_i \leq x), \ x \geq 0$. From Borovkov and Vatutin (1996) we get that if conditions (3), (4) hold then the d.f. $P(x)$ is a proper one and

$$1 - P(x) \sim \frac{\mu\alpha}{x}, \quad x \to \infty. \tag{9}$$

The inequalities (8) and combination of the following two lemmas allow us to prove the theorems.

Lemma 1 *If conditions (3), (4) hold, then*

$$\lim_{n\to\infty} \mathcal{P}\Big(\frac{\overline{M}_n}{\mu\alpha n} \leq x\Big) = Q(x) \ .$$

The proof follows from (9) (see Feller(1971), Sect VIII.8, Example b).

The next lemma is a version of a result of Dobrushin (1955), for the limit of randomly indexed, random sequences.

Lemma 2 *(Dobrushin(1955)) Assume for the sequence of r.v.'s ξ_n that*

$$\frac{\xi_n}{n^\gamma} \stackrel{d}{\to} \xi, \quad n \to \infty,$$

where $\gamma > 0$, and ξ is a r.v. with proper d.f..

Assume for the sequence of positive integer valued r.v.'s η_n, independent of ξ_n, that

$$\eta_n / a(n) \stackrel{d}{\to} \eta, \quad n \to \infty,$$

where $a(n) \uparrow \infty, \quad n \to \infty$ and η has proper d.f.

Then

$$\frac{\xi_{\eta_n}}{a(n)^\gamma} \xrightarrow{d} \overline{\xi}^\gamma \overline{\eta}, \quad n \to \infty \ ,$$

where $\overline{\xi}$ *and* $\overline{\eta}$ *are independent and have the same d.f.'s as* ξ *and* η *respectively.*

It is clear that Lemma 3 establishes the limit for the sequence

$$\mathcal{P}(\xi_{\eta_n} \le x) = \sum_{k=0}^{\infty} \mathcal{P}(\xi_k \le x)\mathcal{P}(\eta_n = k),$$

under suitable normalization, when n tends to infinity.

Proof of Theorem 1. Let (3), (4) and (6) holds. Denote by $N(y) = \int_0^y xd\mathcal{P}(T_i \le x)$. The relation

$$N(y) = R(y) - y\mathcal{P}(T_i > y), \quad y \ge 0,$$

(5),(6) and Seneta(1976), Problem 1.17, yield

$$N(y) \sim R(y), \quad y \to \infty, \tag{10}$$

and in this case $R(.)$ is a monotone increasing slowly varying function and $R(y) \uparrow \infty,\ y \to \infty$. (See also Feller (1971), Sect. VIII.9, Theorem 1).

Define the sequence b_n by the equation $n = b_n/N(b_n), \quad n \ge 1$. By the LLN (Feller (1971), Sect. VII.8, Theorem 2) we have

$$S_n/b_n \xrightarrow{p} 1, \quad n \to \infty,$$

which, together with the relation $\mathcal{P}(S_k \ge n) = \mathcal{P}(\nu(n) \le k)$ and (10), implies that

$$\frac{\nu(n)}{n/R(n)} \xrightarrow{p} 1, \quad n \to \infty. \tag{11}$$

(See also Kulkarni and Pakes (1986)).

For the d.f. of $\overline{M}_{\nu(n)}$ we have by the total probability formula

$$\mathcal{P}(\overline{M}_{\nu(n)} \le x) = \sum_{k=1}^{\infty} \mathcal{P}(\overline{M}_k \le x)\mathcal{P}(\nu(n) = k). \tag{12}$$

Applying Lemma 3 to (12) and taking into account (11) and Lemma 2 we get

$$\lim_{n\to\infty} \mathcal{P}\Big(\frac{\overline{M}_{\nu(n)}}{\mu n/R(n)} \le x\Big) = Q(x),$$

and

$$\lim_{n\to\infty} \mathcal{P}\Big(\frac{\overline{M}_{\nu(n)+1}}{\mu n/R(n)} \le x\Big) = Q(x) \ .$$

Finally, the last two relations and (8) complete the proof.

Proof of Theorem 2. Let (3), (4) and (7) hold. By the SLLN we have that

$$S_n/n \stackrel{a.s.}{\to} R^{-1}, \quad n \to \infty.$$

Hence

$$\frac{\nu(n)}{n/R} \stackrel{a.s.}{\to} 1, \quad n \to \infty. \tag{13}$$

Now applying Lemma 3 to (12) and taking into account (13) and Lemma 2 we get

$$\lim_{n\to\infty} \mathcal{P}\big(\frac{\overline{M}_{\nu(n)}}{\mu\alpha n/R} \le x\big) = Q(x),$$

which together with

$$\lim_{n\to\infty} \mathcal{P}\big(\frac{\overline{M}_{\nu(n)+1}}{\mu\alpha n/R} \le x\big) = Q(x),$$

and (8) complete the proof of the theorem.

An example. The critical BGW process with finite variance of the offspring of one particle satisfies the condition (6) with $\alpha = 1$ and $L(1/(1-s)) \to \sigma^2/2, s \uparrow 1$, where $\sigma^2 = f''(1-) < \infty$. In this case

$$\mathcal{P}(T_i > n) \sim \frac{2\mu}{\sigma^2 n}, \quad n \to \infty.$$

Hence

$$R(n) \sim \frac{2\mu}{\sigma^2} \log n, \quad n \to \infty.$$

Now Theorem 1 yields

$$\lim_{n\to\infty} \mathcal{P}\big(\frac{W_n}{\sigma^2 n/2 \log n} \le x\big) = Q(x).$$

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