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LARGE DISTINCT PART SIZES IN A RANDOM INTEGER PARTITION*

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A partition of a positive integer n is a way of writing it as the sum of positive integers without regard to order; the summands are called parts. The number of partitions of n, usually denoted by $p(n)$, is determined asymptotically by the famous partition formula of Hardy and Ramanujan [5]. We shall introduce the uniform probability measure P on the set of all partitions of n assumming that the probability $1/p(n)$ is assigned to each n-partition. The symbols E and Var will be further used to denote the expectation and variance with respect to the measure P. Thus, each conceivable numerical characteristic of the parts in a partition can be regarded as a random variable. Erdös and Lehner [2] were apparently the first who have studied random integer partitions by a probabilistic approach. Subsequent work by a number of authors provides considerable information about the structure of "typical" partition. (We refer the reader e.g. to [1], [9-13], [3], [4], [6] and [8]).

If κ is one of the $p(n)$ partitions of n and $s \geq 1$, let $Z_{s,n} = Z_{s,n}(\kappa)$ and $Y_{s,n} = Y_{s,n}(\kappa)$ denote the number of parts larger than $s-1$ that κ has, counted respectively with and without multiplicity. Wilf [14] observed that for most partitions of n, $Z_{0,n}$ exceeds $Y_{0,n}$. In particular, he showed that

$$E(Y_{0,n}) \sim (6n)^{1/2}/\pi \tag{1}$$

as $n \to \infty$, while Erdös and Lehner's result [2] obtained long ago, states that

$$E(Z_{0,n}) \sim \pi^{-1}(3/2)^{1/2} n^{1/2} \log n.$$

As a matter of fact, Erdös and Lehner did better by finding an appropriate normalization for $Z_{0,n}$ in order that there be a nontrivial limiting distribution; they showed that

$$\lim P[\pi Z_{0,n}/(6n)^{1/2} - \log \frac{(6n)^{1/2}}{\pi} < v] = e^{-e^{-v}}, -\infty < v < \infty. \tag{2}$$

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One year later this result was strengthened to a form of a local limit theorem by Auluck, Chowla and Gupta [1]. A recent paper of Hwang [6] supplied it by a better estimate on the rate of convergence. Fristedt [3] also obtained some further extensions and determined asymptotically the probability distribution of $Z_{s,n}$. The results summarized in the present review form a part of a project aimed at a closer investigation of differences that appear in the asymptotic behavior of $Y_{s,n}$ and $Z_{s,n}$. We first state below Fristedt's result for $Z_{s,n}$.

Theorem 1 [3]. *Suppose that $s = s_n$ is such that $s_n/n^{1/2} \to \infty$ and $\pi s_n/(6n)^{1/2} - \frac{1}{2}\log n \to -\infty$ as $n \to \infty$. Then, for any fixed v,*

$$\lim_{n\to\infty} P\{[Z_{s,n}+\pi^{-1}(6n)^{1/2}\log(1-\exp(-\pi s/(6n)^{1/2}))]/\pi^{-1/2}(6n)^{1/4}\exp(-\pi s/2(6n)^{1/2})<v\}$$
$$= (2\pi)^{-1/2}\int_{-\infty}^{v} e^{-w^2/2}dw.$$

Substantial extensions concerning the joint distribution of counts of parts with bounded sizes in a random partition were recently made by Pittel [8]. He also studied $Z_{s,n}$ and obtained results comparable with those of Szalay and Turan [9-11]. In particular, he proved that the distribution of $Z_{s,n}$ is asymptotically concentrated around a deterministic number as $n \to \infty$ in a range of s including the value $s = O(n^{1/2})$ (see his Thm. 2).

For different part sizes, it turns out that the random variable $Y_{0,n}$, appropriately normalized, converges weekly to a Gaussian random variable as well. Goh and Schmutz [4] proved this fact directly; it can be also deduced using a general method suggested in [7] (see Example 2), where an asymptotic expression for $Var(Y_{0,n})$ was also derived. We summarize all what is known for the asymptotic behavior of the total number $Y_{0,n}$ of the distinct part sizes in the next theorem.

Theorem 2 [14, 4, 7]. *As $n \to \infty$, (1) holds together with*

$$Var(Y_{0,n}) \sim (6n)^{1/2}(1/2\pi - 3/\pi^3).$$

Furthermore, for any fixed v,

$$\lim_{n\to\infty} P\{[Y_{0,n} - (6n)^{1/2}/\pi]/(6n)^{1/4}(1/2\pi - 3/\pi^3)^{1/2} < v\}$$
$$= (2\pi)^{-1/2}\int_{-\infty}^{v} e^{-w^2/2}dw.$$

The results of Theorem 2 are extended by the following limit theorem for $Y_{s,n}$.

Theorem 3 *If the integers $s = s_n$ are such that $s_n = \lambda(6n)^{1/2}/\pi + o(n^{1/4})$, where $0 \le \lambda < \infty$, then*

$$E(Y_{s,n}) \sim (6n)^{1/2}e^{-\lambda}/\pi = \mu_n(\lambda),$$
$$Var(Y_{s,n}) \sim (6n)^{1/2}[e^{-\lambda}(1 - e^{-\lambda}/2)/\pi - 3e^{-2\lambda}(\lambda+1)^2/\pi^3] = \sigma_n^2(\lambda),$$

and, for any fixed v,

$$\lim_{n\to\infty} P\{[Y_{s,n} - \mu_n(\lambda)]/\sigma_n(\lambda)\} = (2\pi)^{-1/2} \int_{-\infty}^{v} e^{-w^2/2} dw.$$

Obviously, the results of Theorem 2 follow immediately from those of Theorem 3. The main tool in our proof here is the saddle-point method. Furthermore, note that Theorem 1 establishes the convergence of $Z_{s,n}$ to a Gaussian distribution when s grows slightly faster than $n^{1/2}$, while in Theorem 3 we prove the same convergence for $Y_{s,n}$ assuming that s is exactly of order $n^{1/2}$. In a subsequent study we plan to describe the limiting distribution of $Z_{s,n}$ when $s = O(n^{1/2})$ and to make a closer examination of the change of the limiting distribution of $Z_{s,n}$ from Gaussian one (see Theorem 2) to an extreme-value (see (2)).

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