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FUNCTIONAL TRANSFER THEOREMS FOR MAXIMA OF STATIONARY PROCESSES*

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In this paper we discuss the problem of finding the limit process of sequences of continuous time random processes, which are constructed as properly affine transformed maxima of random number identically distributed random variables.

The max-increments of these processes are dependent.

First we work under the well known conditions $D(u_n)$ and $D'(u_n)$ of Leadbetter, Lindgren and Rootzen, (1983).

Further we investigate the case of moving average sequence. The distribution function of the noise components is assumed to have regularly varying tails or is subexponential and belongs to the max-domain of attraction of Gumbel distribution or belongs to the max-domain of attraction of Weibull distribution.

We work with random time-components which are a.s. strictly increasing to infinity. In particular their counting process is a mixed Poisson process or a renewal process with regularly varying tails with parameter $\beta \in (0, 1)$.

Here is proved that such sequences of random processes converges weakly to a compound extremal process.

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1. Introduction

In 1987 using point processes technics, Davis and Resnick (see e.g. S. Resnick [14]) showed that under certain conditions the study of the extremes of moving average sequence could be reduced to the study of the extremes of the sequence of independent identically distributed (iid) innovations. They obtained that Lamperty's Invariance principle for maxima remains valid for moving average sequence with innovations in the max-domain of attraction of Frechet distribution. In this case the right normalizations for the maxima of moving average sequence coincide with the appropriate normalizations for the maxima of sequence of iid innovations.

In 1991, in terms of minima, Davis and Resnick [4] investigated the extremes of moving averages of random variables (rv's) with finite endpoint. In their paper they mentioned that their results can be adapted to study of minima of moving average with noise components in the max-domain of attraction of Weibull distribution.

In 1997 P. Embrechts et al. [6] proved an analogous result for moving average sequence with subexponential noise in the max-domain of attraction of Gumbel distribution.

Using the results of Davis and Resnick [4], in 2008 P. Jordanova [8] obtained Invariance principle for maxima for finite or infinite moving average with innovations in the max-domain of attraction of Weibull distribution.

The proofs of all of above results pass through the following three steps:

- to obtain convergence of generating point processes;
- to prove continuity of the mapping between the generating point processes and corresponding extremal process;
- to use the Continuous mapping theorem.

It is well known that the following conditions $D_r(\mathbf{u}_n)$ and $D'(u_n)$ (See Leadbetter et al. (1983) [10]) guaranty that we do not go faraway from iid case.

We denote by \mathbf{u}_n the vector $(u_n(x_1), u_n(x_2), \dots, u_n(x_r))$.

Condition $D_r(\mathbf{u}_n)$: The sequence of random variables X_1, X_2, \dots is such that for any integers k, s and $1 \leq i_1 < \dots < i_k < j_1 < \dots < j_s \leq n$ such that $j_1 - i_k > l(n)$ we have

$$\begin{aligned} & |P(X_{i_1} \leq u_n(x_{i_1}), X_{i_2} \leq u_n(x_{i_2}), \dots, X_{i_k} \leq u_n(x_{i_k}), X_{j_1} \leq u_n(x_{j_1}), X_{j_2} \leq u_n(x_{j_2}), \\ & \dots, X_{j_s} \leq u_n(x_{j_s})) - P(X_{i_1} \leq u_n(x_{i_1}), X_{i_2} \leq u_n(x_{i_2}), \dots, X_{i_k} \leq u_n(x_{i_k})). \\ & \cdot P(X_{j_1} \leq u_n(x_{j_1}), X_{j_2} \leq u_n(x_{j_2}), \dots, X_{j_s} \leq u_n(x_{j_s}))| \leq \alpha_{n, l(n)}, \end{aligned}$$

where $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}, x_{j_1}, x_{j_2}, \dots, x_{j_s}, \}$ are in the set $\{x_1, x_2, \dots, x_r\}$ and $\alpha_{n,l(n)} \rightarrow 0$ as $n \rightarrow \infty$ for some sequence $l(n) = o(n)$.

Condition D'(u_n): The sequence of random variables X_1, X_2, \dots is such that the relation

$$\limsup_{n \rightarrow \infty} n \sum_{j=2}^{\lfloor \frac{n}{k} \rfloor} P(X_1 > u_n, X_j > u_n) \rightarrow 0$$

holds as $k \rightarrow \infty$.

They mean correspondingly that large value exceedances of the sequence $\{X_n\}_{n \in \mathbb{Z}}$ are asymptotically independent and separated in time. (See [6].) That is why we have no multiplicities of the limiting point process and it is Poisson and simple in time. These give us possibility to obtain the Invariance principle for maxima for strictly stationary sequences. See Lemma 3.1.

In this paper we consider conditions, providing the transfer of convergence property from the maxima of non-random number of dependent and identically distributed random variables to the maxima of random number. Having in mind Dobrushin's Theorem (Theorem 3.1.2 in [5]), we call these theorems Transfer theorems.

The case of iid rv's is investigated in series of papers of Pancheva and Jordanova (See e.g. [7] and [12]). Independently of them Satheesh et al. [14] investigate the properties of Λ -extremal processes. In [13], Pancheva et al. obtain necessary and sufficient condition for a compound extremal process to has independent max-increments and obtain some other properties of these processes. They also interpret their mathematical model as a particular insurance business and obtain upper and lower boundaries for probability of ruin.

The paper is organized as follows: in Section 2 we describe common conditions in our results, in Section 3 we discuss the case when the sequence of space points satisfy conditions $D(u_n)$ and $D'(u_n)$, in Section 4 we consider the case of moving average sequences.

Along the paper $\xrightarrow[n \rightarrow \infty]{fdd}$ stands for weak convergence of the finite dimensional distributions (fdd) of the random processes, and $\xrightarrow[n \rightarrow \infty]{d}$ for weak convergence of their one dimensional marginals. We denote by $\stackrel{fdd}{=}$ equality of all fdd's, by $\stackrel{d}{=}$ equality in distribution and by $x_0 = \inf\{x \in \mathbf{R} : H(x) > 0\}$, where H is a distribution function (df).

When the sequence of random processes $\{\eta_n\}_{n \in \mathbf{N}}$ in a metric space S converges weakly in the corresponding topology to a stochastic process η , we will write $\eta_n \Longrightarrow \eta$ in S .

We denote by $\mathcal{M}^*([0, \infty))$ the space of non-decreasing, right-continuous functions $y(t) : [0, \infty) \rightarrow [0, \infty)$, with finite left limits on $(0, \infty)$, endowed with the Skorokhod topology and by $\mathcal{M}([0, \infty)) \subset \mathcal{M}^*([0, \infty))$ containing only strictly increasing functions.

Because the extremal processes have non-decreasing sample paths the weak convergence of sequences of such random processes as random elements in $\mathcal{M}^*([0, \infty))$ coincides with convergence in J_1 -topology of Skorokhod. See e.g [18].

When we need to prove weak convergence of extremal processes the next two lemmas are very useful. The proof of the first of them could be found in [2]. The second one describes a particular case when composition is a continuous mapping. It is an immediate consequence of Theorem 13.2.4 of [18]. Another comprehensive treatment on this matter is [17].

Lemma 1.1. *Let $\{Z_n\}_{n \in \mathbf{N}}$ be a sequence of stochastic processes, whose path functions lie in $\mathcal{M}^*([0, \infty))$. If*

a) $Z_n \xrightarrow[n \rightarrow \infty]{fdd} Z$ and

b) Z is stochastically continuous,

then $Z_n \Longrightarrow Z$ in $\mathcal{M}^*([0, \infty))$, in the Skorokhod J_1 -topology.

Lemma 1.2. *For $n \in \mathbf{N}$ let $Z_n, Z, \theta_n,$ and θ be random processes with a.s. sample paths in $\mathcal{M}^*([0, \infty))$. Let $Z_n \Longrightarrow Z$ in $\mathcal{M}^*([0, \infty))$, where Z is stochastically continuous and $\theta_n \Longrightarrow \theta$ in $\mathcal{M}^*([0, \infty))$. Assume that for all $n \in \mathbf{N}$, Z_n and θ_n are independent, for all $t > 0$,*

$$P(\forall \epsilon > 0, \theta(t - \epsilon) < \theta(t + \epsilon) | Z(\theta(t) -) < Z(\theta(t))) = 1 \quad \text{and}$$

$$(1) \quad P(\theta(t -) = \theta(t) | Z(\theta(t) -) < Z(\theta(t))) = 1.$$

Then $Z_n \circ \theta_n \Longrightarrow Z \circ \theta,$ in $\mathcal{M}^*([0, \infty))$, $n \rightarrow \infty$.

If the process θ has almost surely (a.s.) continuous and strictly increasing sample paths, conditions (1) are automatically satisfied.

The statements are formulated in R^1 but it is not difficult to extend them in multidimensional case.

2. Description of the model

Let (Ω, \mathcal{A}, P) be a complete probability space with filtration $(\mathcal{A}_t)_{t \geq 0}$ and all P -null sets of \mathcal{A} are added to \mathcal{A}_0 .

We assume that all discussed random elements here are defined on (Ω, \mathcal{A}, P) .

Suppose $\{X_n\}_{n \in \mathbf{N}}$ is a strictly stationary sequence with common distribution function F .

We denote by $M_n = \max(X_1, \dots, X_n)$ the maxima of X_1, X_2, \dots, X_n and by Y_n the following continuous time process

$$(2) \quad Y_n(t) = \begin{cases} \frac{M_{[nt]} - a_n}{b_n} & t \geq n^{-1} \\ \frac{X_1 - a_n}{b_n} & 0 < t < n^{-1} \end{cases}, \quad t > 0,$$

where $a_n \in \mathbf{R}$, $b_n > 0$, $n \in \mathbf{N}$ and $[s]$ stands for the biggest integer, less than s .

Let

$$N_n = \left\{ \left(\frac{i}{n}, \frac{X_i - a_n}{b_n} \right), \quad i \in \mathbf{N} \right\},$$

$T_0 = 0$ and $0 < T_1 < T_2 < \dots$ be a.s. strictly increasing to infinity with counting process

$$N(t) = \max\{n \geq 0 : T_n \leq t\}.$$

We assume that the sequences $\{X_n\}_{n \in \mathbf{N}}$ and $\{T_n\}_{n \in \mathbf{N}}$ are independent, $\tilde{a}_n \in \mathbf{R}$, $\tilde{b}_n > 0$, $n \in \mathbf{N}$ and discuss the convergence of the following sequence of extremal processes with dependent max-increments

$$(3) \quad \tilde{Y}_n(t) = \begin{cases} \frac{M_{N(nt)} - \tilde{a}_n}{\tilde{b}_n} & t \geq \frac{T_1}{n} \\ \frac{X_1 - \tilde{a}_n}{\tilde{b}_n} & 0 < t < \frac{T_1}{n} \end{cases}, \quad t > 0,$$

associated with the point process

$$(4) \quad \tilde{N}_n = \left\{ \left(\frac{T_k}{n}, \frac{X_k - \tilde{a}_n}{\tilde{b}_n} \right) : k \in \mathbf{N} \right\}.$$

We denote by

$$E_\beta(t) = \inf\{x \geq 0 : S_\beta(x) > t\}$$

the hitting time process of the strictly stable Levy motion $\{S_\beta(t)\}_{t \geq 0}$, with

$$E e^{iu S_\beta(1)} = e^{-iu\beta\Gamma(-\beta)\cos\frac{\pi\beta}{2}}.$$

3. The case when conditions $D_r(u_n)$ and $D'(u_n)$ are satisfied

Additionally to the description of our model in Section 2 we assume that there exist constants $b_n > 0$ and $a_n \in \mathbf{R}$ such that

$$(5) \quad \lim_{n \rightarrow \infty} n\bar{F}(a_n + b_n x) = -\ln H(x), \quad x > x_0,$$

where H is an extreme value distribution.

In 1983 Leadbetter et al. [10] prove the following result about the weak convergence of point processes.

Leadbetter’s theorem 5.7.2 in [10]. *Suppose $\{X_n\}_{n \in \mathbf{N}}$ is a strictly stationary sequence with common distribution function F and there exist constants $b_n > 0$ and $a_n \in \mathbf{R}$ such that condition (5) holds.*

Let $u_n(x) = a_n + b_n x$. Assume that condition $D'(u_n(x))$ holds for all $x \in \mathbf{R}$ and for all $r \in \mathbf{N}$ and $(x_1, x_2, \dots, x_r) \in \mathbf{R}^r$ condition $D_r((u_n(x_1), u_n(x_2), \dots, u_n(x_r)))$ is satisfied.

Then $N_n \implies N_\infty$ in $M_p((0, \infty) \times (x_0, \infty])$, where $M_p((0, \infty) \times (x_0, \infty])$ is the space of all point measures on $(0, \infty) \times (x_0, \infty]$ equipped with an appropriate σ -algebra $\mathcal{M}_p((0, \infty) \times (x_0, \infty])$, $x_0 = \inf\{x : H(x) > 0\}$ and N_∞ is a homogeneous Poisson random measure on $(0, \infty) \times (x_0, \infty]$ with intensity μ_∞ , such that

$$\mu_\infty((0, t) \times (x, \infty]) = -t \ln H(x), \quad \text{for all } x > x_0.$$

In the following lemma we obtain that the Invariance principle for maxima [14] remains valid for such strictly stationary sequences.

Lemma 3.1. *Suppose $\{X_n\}_{n \in \mathbf{N}}$ is a strictly stationary sequence with common distribution function F and there exist constants $b_n > 0$ and $a_n \in \mathbf{R}$ such that condition (5) holds.*

Let $u_n(x) = a_n + b_n x$. Assume that condition $D'(u_n(x))$ holds for all $x \in \mathbf{R}$ and for all $r \in \mathbf{N}$ and $(x_1, x_2, \dots, x_r) \in \mathbf{R}^r$ condition $D_r((u_n(x_1), u_n(x_2), \dots, u_n(x_r)))$ is satisfied.

Then $Y_n \implies Y$ in $M^([0, \infty))$, where Y is an H -extremal process.*

Proof. The main tool used in this proof is the Continuous mapping theorem [1].

Resnick [14] proved that the mapping $T_1 : M_p([0, \infty) \times (x_0, \infty]) \rightarrow M^*([0, \infty))$ defined by

$$\left(T_1 \sum_k \varepsilon_{(t_k, y_k)} \right) (t) = \begin{cases} \vee_{k:t_k \leq t} y_k & t \geq t^* \\ y_1 & 0 < t < t^* \end{cases},$$

where $t^* = \sup \left\{ s > 0 : \sum_k \varepsilon_{(t_k, y_k)}((0, s] \times (x_0, \infty]) = 0 \right\}$, is a.s. continuous w.r.t. $\text{PRM}(dt \times d\nu)$, $\nu((x, \infty]) = -\ln H(x)$, $x > x_0$.

Now we use Leadbetter's theorem, Continuous mapping theorem and the right continuity of the processes at zero and obtain that for all $x > x_0$,

$$(T_1 N_n)(\cdot) \implies (T_1 N_\infty)(\cdot), \quad \text{in } M^*((0, \infty)),$$

where N_∞ is a homogeneous Poisson random measure on $(0, \infty) \times (x_0, \infty)$ with intensity μ_∞ , such that

$$\mu_\infty((0, t) \times (x, \infty]) = -t \ln H(x), \quad \text{for all } x > x_0.$$

The last means that

$$Y_n(\cdot) \implies Y(\cdot), \quad \text{in } M^*((0, \infty)),$$

where Y is an H -extremal process. □

Now we are ready to transfer the convergence property from the maxima of non-random number of affine transformed moving averages to the maxima of random number. We obtain the following theorem.

Theorem 3.1. *Suppose that the stationary sequence $\{X_n\}_{n \in \mathbf{N}}$, a_n and $b_n > 0$, $n \in \mathbf{N}$ are such that conditions $D(a_n + b_n x)$ and $D'(a_n + b_n x)$, $x \in \mathbf{R}$ hold.*

If there exists a random process θ with a.s. continuous and strictly increasing sample paths, such that

$$\frac{N(nt)}{n} \implies \theta(t), \quad \text{in } \mathcal{M}([0, \infty)), \quad n \rightarrow \infty,$$

then

$$\tilde{Y}_n(\cdot) \implies Y(\theta(\cdot)), \quad \text{in } \mathcal{M}^*([0, \infty)),$$

where Y is an extremal process, generated by H .

Proof. Conditions of Lemma 3.1 are satisfied, so $Y_n(\cdot) \implies Y(\cdot)$, in $M^*((0, \infty))$, where $Y(t)$ is an H -extremal process. Because of θ has a.s. continuous and strictly increasing sample paths, we can apply Lemma 1.2 and we complete the proof. □

Because of the limiting process is the same as in iid case its properties are investigated for example in [12], [13] and [15].

Corollary 3.1. *Let \tilde{N}_n be the point processes, defined in (4) and the counting process N , be a mixed Poisson process with random intensity θ and $E\theta < \infty$. Then*

$$\tilde{Y}_n(\cdot) \implies Y(\theta \cdot) \quad \text{in } \mathcal{M}^*([0, \infty)),$$

where Y is an H -extremal process.

Proof. Because of the counting process N is a mixed Poisson then

$$\frac{N(n \cdot)}{n} \implies \theta \cdot, \quad n \rightarrow \infty \quad \text{in } \mathcal{M}([0, \infty)).$$

The random process $\theta(t) = \theta \cdot t, t > 0$ has continuous and strictly increasing sample paths, so conditions of Theorem 3.1 are satisfied and we complete the proof. \square

If the time-points T_1, T_2, \dots constitute a renewal process, with time between renewals J_1, J_2, \dots , with finite mean $EJ_1 < \infty$, then the counting process N could be interpreted as mixed Poisson with constant intensity $\theta = \frac{P(J_1 > 0)}{EJ_1}$. An immediate consequence of Renewal theory and this result is that the limiting process in Corollary 3.1 is $Y\left(\frac{tP(J_1 > 0)}{EJ_1}\right)$. It is self-similar and max-stable.

When EJ_1 is not finite, we can not apply the above theorem. Analogously to Theorem 4.2 c) in [7], where the iid case is considered, we obtain the following theorem.

Theorem 3.2. *Let \tilde{N}_n be the point processes, defined in (4) with time-points T_1, T_2, \dots that constitute a renewal process with time between renewals J_1, J_2, \dots with df J and counting process N . Assume that $1 - J \in RV_{-\beta}, \beta \in (0, 1)$. Then*

a)

$$\tilde{Y}_n(\cdot) \implies \tilde{Y} = Y(E_\beta(\cdot)) \quad \text{in } \mathcal{M}([0, \infty)),$$

where Y is an H extremal process, $\tilde{a}_n \sim a_{(1-J)^{-1}(n)}, \tilde{b}_n \sim b_{(1-J)^{-1}(n)}$.

b) For all fixed $t > 0$

$$P(\tilde{Y}(t) < x) = \sum_{n=0}^{\infty} \frac{(\ln H(x) \cdot t^\beta)^n}{\Gamma(1 + n\beta)}, \quad x > 0.$$

Proof. a) By Lemma 3.1, $Y_n(\cdot) \implies Y(\cdot)$ with an H -extremal process Y .

This means that for $b_n^* \sim \frac{1}{1 - J(n)} \in RV_\beta$,

$$Y_{b_n^*}(\cdot) \implies Y(\cdot).$$

By Theorem 3.6 [11] of Meerschaert and Scheffler

$$\frac{N(n\cdot)}{b_n^*} \implies E_\beta(\cdot), \quad n \rightarrow \infty,$$

where $b_n^* \sim \frac{1}{1 - J(n)} \in RV_\beta$.

The sample paths of the processes $\frac{N(n\cdot)}{b_n^*}$ and $E_\beta(\cdot)$ are in $\mathcal{M}^*[0, \infty)$, but the sample paths of $E_\beta(t)$ are not a.s. strictly increasing. That is why we have to check condition (1). This means that E_β should be a.s. continuous and strictly increasing in every point $t_0 > 0$, such that $E_\beta(t_0)$ is a point of discontinuity of Y . When we interpret this for S_β and Y , they a.s. should not have simultaneous jumps. This is obviously true, because these processes are independent and stochastically continuous. So, condition (1) is satisfied. Now we apply Lemma 1.2. and complete the proof.

b) Let $t > 0$ and $x > 0$,

$$\begin{aligned} \mathbf{P}(Y \circ E_\beta(t) < x) &= \mathbf{P}(Y(t^\beta E_\beta(1)) < x) = \\ &= \int_0^\infty \mathbf{P}(Y(t^\beta z) < x) d\mathbf{P}(E_\beta(1) < z) = \\ &= \int_0^\infty H^{t^\beta z}(x) d\mathbf{P}(E_\beta(1) < z) = \\ &= \mathbf{E} \exp\{-(-\ln H(x))t^\beta E_\beta(1)\} \\ &= \mathbf{E} \exp\{-(-\ln H(x))t^\beta (S_\beta(1))^{-\beta}\}, \end{aligned}$$

where the last equality follows by Corrolary 3.2,(a) in [11].

In [3] is shown that $(S_\beta(1))^{-\beta}$ is Mittag-Leffler distributed.

So, we complete the proof. \square

Remark. In b) for $\beta = 1$ we get $Y(E_\beta(\cdot)) \stackrel{fdd}{=} Y(\cdot)$.

4. The moving average sequence case

Further we suppose that the random variables $\{X_n\}_{n \in \mathbf{N}}$ in the point process (4) have linear process representation.

Assume that there exists sequence $\{\xi_i\}_{i \in \mathbf{Z}}$ of iid rv's with df F such that $\{X_n\}_{n \in \mathbf{N}}$ have representations as a linear processes, i.e.

$$(6) \quad X_n = \sum_{j=-k}^k c_j \xi_{n-j}, \quad n \in \mathbf{N}, \quad k \leq \infty.$$

The sequence $\{c_i\}_{i \in \mathbf{Z}}$ of real numbers will be specified later.

If $k = \infty$ and this series is a.s. convergent, then the sequence $\{X_n\}_{n \in \mathbf{N}}$ is strictly stationary.

Case A) First we consider the case when $k = \infty$ and the noise components in (6) have df in the max-domain of attraction of the Frechet distribution.

Suppose that the tails of F are balanced in the sense that there exists $p \in (0, 1]$ and $\alpha > 0$, such that

$$(7) \quad \lim_{x \rightarrow \infty} \frac{P(\xi_1 > x)}{P(|\xi_1| > x)} = p, \quad \lim_{x \rightarrow \infty} \frac{P(\xi_1 \leq -x)}{P(|\xi_1| > x)} = 1 - p$$

and

$$(8) \quad P(|\xi_1| > x) \in RV_{-\alpha}.$$

We set $q = 1 - p$.

We assume that the sequence $\{c_i\}_{i \in \mathbf{Z}}$, of real numbers satisfies the following condition: There exists $\delta \in (0, \alpha \wedge 1)$ such that

$$(9) \quad \sum_{j=-\infty}^{\infty} |c_j|^\delta < \infty.$$

We will use the following notations

$$c^+ = \max\{c_j \vee 0, j \in \mathbf{Z}\} \quad \text{and} \quad c^- = \max\{-c_j \vee 0, j \in \mathbf{Z}\}.$$

The weak limit behaviour of various quantities related to the extremes of $\{X_n\}_{n \in \mathbf{N}}$ is discussed for example in [10] and [14].

By the Invariance principle for the maxima of a linear process with noise in the max-domain of attraction of the Frechet distribution (See e.g. Proposition 4.28 in [14]) we have that if either $c_+p > 0$ or $c_-q > 0$ then

$$(10) \quad Y_n \Longrightarrow Y_\alpha \quad \text{in } \mathcal{M}^*([0, \infty)),$$

where $\{Y_\alpha(t)\}_{t \geq 0}$ is an extremal process generated by the extreme value distribution

$$(11) \quad G_\alpha(x) = \exp\{-(c_+^\alpha p + c_-^\alpha q)x^{-\alpha}\}, \quad x > 0.$$

Moreover $a_n \sim 0$ and $b_n \sim F^{\leftarrow}\left(1 - \frac{1}{n}\right)$, $n \rightarrow \infty$.

Theorem 4.1. *Let T_1, T_2, \dots be time-points with counting process N . We suppose that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k = \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid and satisfy conditions (7) and (8) and the sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ satisfies condition (9) and either $c_+p > 0$ or $c_-q > 0$.*

Assume that there exists a random process θ , with a.s. continuous and strictly increasing sample paths, such that

$$\frac{N(n \cdot)}{n} \Longrightarrow \theta(\cdot), \quad n \rightarrow \infty \quad \text{in } \mathcal{M}([0, \infty)).$$

Then

$$\tilde{Y}_n(\cdot) \Longrightarrow Y_\alpha(\theta(\cdot)) \quad \text{in } \mathcal{M}^*([0, \infty)),$$

where Y is an extremal process, generated by the extreme value distribution G_α , defined in (11).

Moreover $\tilde{a}_n \sim 0$ and $\tilde{b}_n \sim F^{\leftarrow}\left(1 - \frac{1}{n}\right)$, $n \rightarrow \infty$.

Proof. By Proposition 4.28 in [14], under the conditions on $\{X_n\}_{n \in \mathbf{N}}$ we have

$$Y_n(\cdot) \Longrightarrow Y_\alpha(\cdot) \quad \text{in } \mathcal{M}^*([0, \infty)),$$

with Y_α , which is an extremal process, generated by the extreme value distribution (11), $a_n \sim 0$ and $b_n \sim F^{\leftarrow}\left(1 - \frac{1}{n}\right)$, $n \rightarrow \infty$.

Because of θ , with a.s. continuous and strictly increasing sample paths we can apply Lemma 1.2 and complete the proof. \square

Analogously to Corollary 3.1 we obtain the following result.

Corollary 4.1. *Let T_1, T_2, \dots be time-points with counting process N , which is a mixed Poisson process with random intensity θ and $E\theta < \infty$. The sequence $\{X_n\}_{n \in \mathbf{N}}$ has representation (6) where $k = \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid and satisfy conditions (8) and (7) and the sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ satisfies condition (9) and either $c_+p > 0$ or $c_-q > 0$. Then*

$$\tilde{Y}_n(\cdot) \Longrightarrow Y_\alpha(\theta \cdot) \quad \text{in } \mathcal{M}^*([0, \infty)),$$

where Y_α is an extremal process, generated by the extreme value distribution G_α , defined in (11), $\tilde{a}_n \sim 0$ and $\tilde{b}_n \sim F^{\leftarrow}\left(1 - \frac{1}{n}\right)$, $n \rightarrow \infty$.

Theorem 4.2. *Let T_1, T_2, \dots be a renewal process with time between renewals J_1, J_2, \dots with J and counting process N . Suppose that $1 - J \in RV_{-\beta}$, $\beta \in (0, 1)$. Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k = \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid and satisfy conditions (7) and (8) and the sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ satisfies condition (9) and either $c_+p > 0$ or $c_-q > 0$. Then*

$$\tilde{Y}_n(\cdot) \implies \tilde{Y}(\cdot) = Y_\alpha(E_\beta(\cdot)) \quad \text{in } \mathcal{M}([0, \infty)),$$

where Y_α is an extremal process, generated by the extreme value distribution G_α , defined in (11), $\tilde{a}_n \sim 0$ and $\tilde{b}_n \sim F^{\leftarrow}(J(n)) \in RV_{\beta\alpha-1}$.

Moreover, for all fixed $s > 0$, $\tilde{Y}(st) \stackrel{\text{fdd}}{=} s^{\beta/\alpha} \tilde{Y}(t)$ and

$$P(\tilde{Y}(t) < x) = \sum_{n=0}^{\infty} \frac{(-x^{-\alpha}t^\beta)^n}{\Gamma(1 + n\beta)}, \quad x > 0.$$

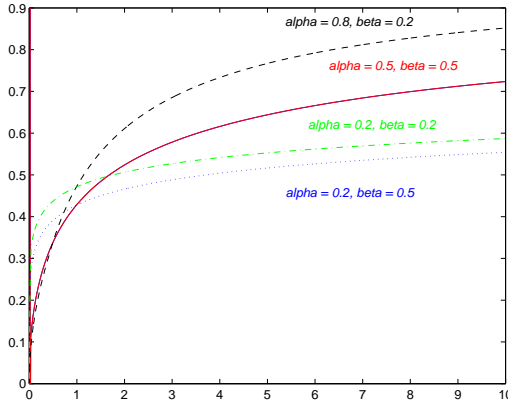


Figure 1: Distribution functions of $\tilde{Y}(1)$ for different parameters α and β

Proof. By Proposition 4.28 in [14],

$$Y_{\tilde{b}_n^*}(\cdot) \implies Y_\alpha(\cdot)$$

with Y_α , which is an extremal process, generated by the extreme value distribution (11) and $\tilde{b}_n^* \sim \frac{1}{1 - J(n)} \in RV_\beta$. As in the proof of Theorem 3.2 a), we apply

Theorem 3.6 [11] of Meerschaert and Scheffler and Lemma 1.2 and complete the proof. \square

The properties of the limiting process follow by Theorem 2.1 in [7].

Case B) Now we suppose that $k = \infty$ and $\{\xi_i\}_{i \in \mathbf{Z}}$ is subexponential noise with df F , which belongs to the max-domain of attraction of the Gumbel distribution

$$G(x) = \exp\{-e^{-x}\}, \quad x \in \mathbf{R}.$$

We will use the tail-balance condition: there exists $p \in (0, 1]$ such that

$$(12) \quad \lim_{x \rightarrow \infty} \frac{P(\xi_1 > x)}{P(|\xi_1| > x)} = p, \quad \lim_{x \rightarrow \infty} \frac{P(\xi_1 \leq -x)}{P(|\xi_1| > x)} = 1 - p.$$

But we will not use condition (8), in the case of regularly varying tails.

Condition (9) will be substituted by the following condition:

There exists $\delta \in (0, 1)$ such that

$$(13) \quad \sum_{j=-\infty}^{\infty} |c_j|^\delta < \infty.$$

Without lost of generality we assume that

$$(14) \quad \max_j |c_j| = 1$$

and we define the following quantities $k_+ = \text{card}\{j : c_j = 1\}$ and $k_- = \text{card}\{j : c_j = -1\}$.

We denote by Y^+ the extremal process associated with the Poisson point process

$$N^+ = \{(t_i^+, \eta_i^+) : i \in \{0, 1, \dots\}\}$$

with the mean measure $\mu^+((0, t] \times (x, \infty)) = te^{-x}$ and by Y^- the independent of it extremal process associated with the Poisson point process

$$N^- = \{(t_i^-, \eta_i^-) : i \in \{0, 1, \dots\}\}$$

with the mean measure

$$\mu^-((0, t] \times (x, \infty)) = t \frac{p}{q} e^{-x}.$$

Analogously to the regularly varying case when we use the Invariance principle for the maxima of a linear process with subexponential noise in the max-domain of attraction of Gumbel distribution (Theorem 5.5.11 [6]) we obtain the following results.

Theorem 4.3. *Let \tilde{N}_n be the point processes defined in (4). Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k = \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid, $\{\xi_i\}_{i \in \mathbf{Z}}$ have subexponential df's in the max-domain of attraction of Gumbel distribution and satisfy condition (12). The sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ is such that (13) and (14) hold.*

Assume that there exists a random process θ , with a.s. continuous and strictly increasing sample paths, such that

$$\frac{N(n \cdot)}{n} \implies \theta(\cdot) \quad n \rightarrow \infty \quad \text{in } \mathcal{M}([0, \infty)).$$

Then $\tilde{Y}_n(t) \implies Y(\theta(t))$, where

$$(15) \quad Y(t) = Y^+(t) \vee Y^-(t).$$

is a G-extremal process, generated by the extreme value distribution

$$G(x) = \exp\{-e^{-x} p^{-1}\}, \quad x \in \mathbf{R},$$

$$\tilde{a}_n \sim F^{\leftarrow} \left(1 - \frac{1}{n} \right) \text{ and}$$

$$\tilde{b}_n \sim n \int_{F^{\leftarrow}(1-\frac{1}{n})}^{\infty} \bar{F}(y) dy.$$

Corollary 4.2. *Let T_1, T_2, \dots be time-points with counting process N , which is a mixed Poisson process with random intensity θ and $E\theta < \infty$. Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k = \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid, $\{\xi_i\}_{i \in \mathbf{Z}}$ have subexponential df's in the max-domain of attraction of Gumbel distribution and satisfy condition (12). The sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ is such that (13) and (14) are satisfied.*

Then

$$\tilde{Y}_n(\cdot) \implies Y(\theta \cdot),$$

where Y is the G-extremal process, defined in (15), $\tilde{a}_n \sim F^{\leftarrow} \left(1 - \frac{1}{n} \right)$ and

$$\tilde{b}_n \sim n \int_{F^{\leftarrow}(1-\frac{1}{n})}^{\infty} \bar{F}(y) dy.$$

Theorem 4.4. Let \tilde{N}_n be the point process, defined in (4) with time-points T_1, T_2, \dots that constitute a renewal process with time between renewals J_1, J_2, \dots with df's J and counting process N . Suppose that $1 - J \in RV_{-\beta}$, $\beta \in (0, 1)$. Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k = \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid, $\{\xi_i\}_{i \in \mathbf{Z}}$ have subexponential df's in the max-domain of attraction of Gumbel distribution and satisfy condition (12). The sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ is such that (13) and (14) are satisfied.

Then

$$\tilde{Y}_n(\cdot) \implies \tilde{Y}(t) = Y(E_\beta(\cdot)) \quad \text{in } \mathcal{M}([0, \infty)),$$

where $\tilde{a}_n \sim F^\leftarrow(J(n))$,

$$\tilde{b}_n \sim \int_{F^\leftarrow(J(n))}^{\infty} \frac{\bar{F}(y)dy}{\bar{J}(n)}$$

and Y is the G -extremal process, defined in (15).

Moreover

$$P(\tilde{Y}(t) < x) = \sum_{n=0}^{\infty} \frac{(-e^{-x} p^{-1} t^\beta)^n}{\Gamma(1 + n\beta)}, \quad x \in \mathbf{R}.$$

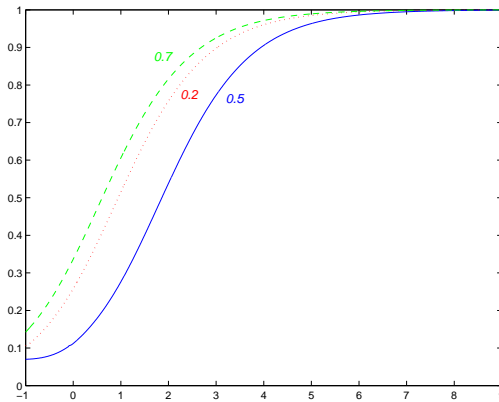


Figure 2: Distribution of $\tilde{Y}(1)$ for $\beta = 0.5$ and $p = 0.7$, $p = 0.2$ and $p = 0.5$ (correspondingly from up to down)

Case C) Now we suppose that $k < \infty$, $c_i > 0$, $i \in \mathbf{Z}$ and the df F of $\{\xi_i\}_{i \in \mathbf{Z}}$ belongs to the max-domain of attraction of the Weibul distribution

$$(16) \quad \Psi_\alpha(x) = \exp\{-(-x)^\alpha\}, \quad x \leq 0, \alpha > 0.$$

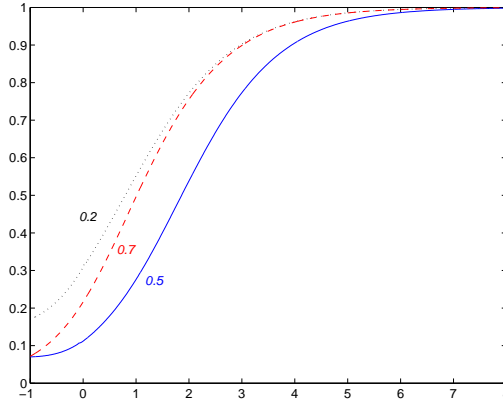


Figure 3: Distribution of $\tilde{Y}(1)$ for $p = 0.5$ and $\beta = 0.2, \beta = 0.7$ and $\beta = 0.5$ (correspondingly form up to down)

We denote $x_F^R = \sup\{x : F(x) < 1\}$ and

$$(17) \quad c(\alpha, k) = \frac{\Gamma^k(\alpha + 1)}{\Gamma(\alpha k + 1)} \prod_{i=1}^k c_i.$$

It is well known that if F belongs to the max-domain of attraction of the Weibul distribution, $x_F^R < \infty$.

Analogously to the previous cases when we use the Invariance principle for the maxima of such linear process (Theorem 2.1. [8]) we obtain the following results.

Theorem 4.5. *Let \tilde{N}_n be the point processes, defined in (4). Suppose $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k < \infty$. Let $\{\xi_i\}_{i \in \mathbf{Z}}$ be iid rv's with df F in the max-domain of attraction of $\Psi_\alpha, \alpha > 0$. $\{c_i\}_{i \in \mathbf{Z}}$ are positive real numbers.*

Assume that there exists a random process θ , with a.s. continuous and strictly increasing sample paths, such that

$$\frac{N(n \cdot)}{n} \Longrightarrow \theta(\cdot), \quad \text{in } \mathcal{M}([0, \infty)), \quad n \rightarrow \infty.$$

Then $\tilde{Y}_n(\cdot) \Longrightarrow Y(\theta(\cdot))$, where Y is a $\Psi_{k\alpha}$ -extremal process, generated by the extreme value distribution

$$(18) \quad \Psi_{k\alpha}(x) = \exp\{-c(\alpha, k)(-x)^{k\alpha}\}, \quad x \leq 0, \quad t > 0,$$

$\tilde{a}_n \sim x_F^R \sum_{j=1}^k c_j$, $\tilde{b}_n \sim x_F^R - F^{\leftarrow}(1 - n^{-1/k})$ and $c(\alpha, k)$ are defined in (17).

Corollary 4.3. *Let \tilde{N}_n be the point processes, defined in (4) with time-points T_1, T_2, \dots with counting process N , which is a mixed Poisson process with random intensity θ and $E\theta < \infty$. Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k < \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid with df's F in the max-domain of attraction of $\Psi_\alpha, \alpha > 0$ and positive real numbers $\{c_i\}_{i \in \mathbf{Z}}$.*

Then

$$\tilde{Y}_n(\cdot) \implies Y(\theta \cdot),$$

where Y is the $\Psi_{k\alpha}$ -extremal process, defined in (18), \tilde{a}_n and \tilde{b}_n could be chosen as in the previous theorem.

Theorem 4.6. *Let \tilde{N}_n be the point processes, defined in (4) with time-points T_1, T_2, \dots that constitute a renewal process with time between renewals J_1, J_2, \dots with df's J and counting process N . Suppose that $1 - J \in RV_{-\beta}, \beta \in (0, 1)$. Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k < \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid with df's F in the max-domain of attraction of $\Psi_\alpha, \alpha > 0$. $\{c_i\}_{i \in \mathbf{Z}}$ are positive real numbers.*

Then

$$\tilde{Y}_n(\cdot) \implies \tilde{Y}(t) = Y(E_\beta(\cdot)) \quad \text{in } \mathcal{M}([0, \infty)),$$

where $\tilde{a}_n \sim x_F^R \sum_{j=1}^k c_j$, $\tilde{b}_n \sim x_F^R - F^{\leftarrow}(1 - (1 - J(n))^{1/k})$ and Y is the $\Psi_{k\alpha}$ -extremal process, defined in (18).

Moreover

$$P(\tilde{Y}(t) < x) = \sum_{n=0}^{\infty} \frac{(-c(\alpha, k)(-x)^{k\alpha}t^\beta)^n}{\Gamma(1 + n\beta)}, \quad x \leq 0,$$

where $c(\alpha, k)$ are defined in (17).

Case D) Let $k = \infty$ and $\{\xi_i\}_{i \in \mathbf{Z}}$ be iid with df F in the max-domain of attraction of $\Psi_\alpha, \alpha > 0$. See (16).

Condition (9) will be substituted by the following conditions: For $i \in \mathbf{Z}$,

$$c_i > 0$$

and for some positive real number $q > 2$

$$c_i \sim O(j^{-q}), \quad j \rightarrow \infty.$$

If $c_{(1)} \geq c_{(2)} \geq \dots$ is the same sequence but ordered, we assume that for all $s \in (0, 1)$

$$\lim_{n \rightarrow \infty} s^n \sum_{j=s^{-n}}^{s^n} \frac{c_{(j)}^2}{c_{(n)}^2} = 0.$$

We call these conditions – “Conditions C”.

We denote

$$r(\lambda) = \sum_{j=1}^{\infty} \frac{c_j E(\xi_1 \exp\{\lambda c_j \xi_1\})}{E \exp\{\lambda c_j \xi_1\}}, \quad \lambda > 0$$

and $\nu = \sum_{j=1}^{\infty} c_j \xi_j$.

Analogously to the previous cases when we use the Invariance principle for the maxima of such linear process we obtain the following results.

Theorem 4.7. *Let \tilde{N}_n be the point processes, defined in (4). Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k = \infty$, $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid with df’s F , in the max-domain of attraction of Weibull df, Ψ_α , $\alpha > 0$. The sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ is such that Conditions C are satisfied.*

Assume that there exists a random process θ , with a.s. continuous and strictly increasing sample paths, such that

$$\frac{N(n \cdot)}{n} \implies \theta(\cdot) \quad n \rightarrow \infty \quad \text{in } \mathcal{M}([0, \infty)).$$

Then $\tilde{Y}_n(t) \implies Y(\theta(t))$, where Y is a extremal process, generated by the extreme value distribution

$$(19) \quad G_1(x) = \exp\{-e^{-x}\},$$

$$\tilde{a}_n \sim F_\nu^{\leftarrow} \left(1 - \frac{1}{n}\right) \quad \text{and} \quad \tilde{b}_n \sim (r^{\leftarrow}(F_\nu^{\leftarrow}(1 - n^{-1}))^{-1}.$$

Corollary 4.4. *Let T_1, T_2, \dots be time-points with counting process N , which is a mixed Poisson process with random intensity θ and $E\theta < \infty$. Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k = \infty$ and $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid with df’s F in the max-domain of attraction of Ψ_α , $\alpha > 0$. The sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ is such that Conditions C are satisfied.*

Then

$$\tilde{Y}_n(\cdot) \implies Y(\theta \cdot),$$

where Y , \tilde{a}_n and \tilde{b}_n are the same as in the previous Theorem.

Theorem 4.8. *Let T_1, T_2, \dots be a renewal process with time between renewals J_1, J_2, \dots with df's J and counting process N . Suppose that $1 - J \in RV_{-\beta}$, $\beta \in (0, 1)$. Assume that $\{X_n\}_{n \in \mathbf{N}}$ have representations (6) where $k = \infty$ and $\{\xi_i\}_{i \in \mathbf{Z}}$ are iid with df's F in the max-domain of attraction of Ψ_α , $\alpha > 0$. The sequence of real numbers $\{c_i\}_{i \in \mathbf{Z}}$ is such that Conditions C are satisfied.*

Then

$$\tilde{Y}_n(\cdot) \implies \tilde{Y}(t) = Y(E_\beta(\cdot)) \quad \text{in } \mathcal{M}([0, \infty)),$$

$\tilde{a}_n \sim F_\nu^{\leftarrow}(J(n))$, $\tilde{b}_n \sim (r^{\leftarrow}(F_\nu^{\leftarrow}(J(n))))^{-1}$ and Y is the G_1 -extremal process, defined in (19).

Moreover

$$P(\tilde{Y}(t) < x) = \sum_{n=0}^{\infty} \frac{(-e^{-xt^\beta})^n}{\Gamma(1 + n\beta)}, \quad x \in \mathbf{R}.$$

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