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THE LIMIT THEOREMS FOR TRANSPORTATION NETWORKS

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The questions of ergodicity and of existence of explicit formulas for the stationary distribution are examined for various types of transportation networks which can be viewed as polling models. Also several limit theorems are proved both for large symmetric and asymmetric networks.

1. Introduction

Among various queueing networks one can distinguish polling systems by the feature that servers, not clients, move around the network. The theory of polling systems develops actively which is demonstrated by the extensive literature [1, 2]. Transportation networks, where service consists in transporting clients from one of N nodes to another according to routing matrix $P = (P_{ij})$, can also be considered polling systems. There are cars for transporting clients, and their number is either constant and P is a stochastic matrix, or cars arrive at the network randomly and then P is a semistochastic matrix while $d_i = 1 - \sum_{j=1}^{N} P_{ij}$ is the probability of a car's going out right after its arrival to node i. In the first case such networks are called closed in respect of cars, in the second one — open. In

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respect of clients transportation networks are open i.e. clients arrive from outside randomly and after being served leave the network. Concerning cars' (clients') behaviour upon arrival to a node when there are no clients (cars) there, different assumptions are possible which leads to the great variety of models. Sometimes a control of one or another kind is introduced. For example, one can assume the presence of carriers that transport cars from one node to another even if there are no clients. In respect of carriers the network can be closed or open. In the latter case we arrive at models belonging to the class of G-networks [3].

2. Models description and notation

The flows of clients, cars, carriers (for open networks) are independent Poisson processes that do not depend on the travel times and have intensities, respectively, λ_i, a_i, b_i for node i $(i = \overline{1, N})$.

Clients choose a node of destination according to matrix $P = (P_{ij})$. In the case of a closed (with respect to cars) network the number of cars equals M and P is an ergodic matrix, $\pi = (\pi_1, \dots, \pi_N)$ being its stationary distribution.

If at the time of a client's arrival there are no cars in the node, the client joins the queue which has L_i $(0 \le L_i \le \infty)$ waiting spots. A car, that met clients in a node, takes one of them and goes to the node of destination, upon arrival where the client leaves the networks. If there are no clients in a node, a car queues; K_i is the number of parking lots in node i $(0 \le K_i \le \infty)$. The travel times between nodes are independent random variables with distribution function $B_{ij}(x)$ for path (i, j), $\beta_{ij} = \int\limits_0^\infty x\, dB_{ij}(x)$, $i, j = \overline{1, N}$. Let us consider the following models.

Model 1. N nodes, M cars, $L_i = 0$, $K_i = \infty$ $(i = \overline{1, N})$. P is a stochastic matrix. There exist explicit formulas for the stationary distribution.

Model 2. N nodes, random number of cars. $L_i = 0, K_i = \infty$ $(i = \overline{1, N})$, open for cars. P is a semistochastic matrix. Ergodicity condition:

$$\rho_i = c_i/\lambda_i < 1, \quad c_i = a_i + \sum_{k=1}^{N} c_k P_{ki}.$$

There exist explicit formulas for the stationary distribution.

Model 3. N nodes, M cars. $L_i = \infty$, $K_i = 0$ $(i = \overline{1, N})$, closed for cars. Ergodicity condition:

$$\rho_j = \frac{\lambda_j \beta}{M \pi_j} < 1.$$

If $B_{ij}(x) = 1 - e^{-\mu x}$, $P_{ij} = \pi_j$, then the stationary distribution is geometric one.

Model 4. $L_i = \infty$, $K_i = 0$, open for cars. Ergodicity condition:

$$\lambda_i < c_i, \quad c_i = a_i + \sum_{k=1}^{N} c_k P_{ki}.$$

There are no explicit formulas for the stationary distribution.

Model 5. L_i and K_i are voluntary, closed for cars. If $L_i = \infty$, $K_i = \infty$, there is no stationary distribution.

Model 6. L_i and K_i are voluntary, open for cars and carriers. If there are no cars in a node at the time a carrier arrives, then it is lost; otherwise, it takes one of the cars and transports it according to matrix P, leaving the network after that.

Let

$x_i(t)$ be the number of cars in node i;
$y_i(t)$ be the number of clients in node i;
$x_{ij}(t)$ be the number of cars on path (i, j)

and

$$q_i(t) = \begin{cases} -x_i(t), & \text{if } x_i(t) > 0,\ y_i(t) = 0; \\ 0, & \text{if } x_i(t) = 0,\ y_i(t) = 0; \\ y_i(t), & \text{if } x_i(t) = 0,\ y_i(t) > 0. \end{cases}$$

We shall consider process $X(t) = \{q_i(t), i = \overline{1,N},\ x_{ij}(t), i, j = \overline{1,N}\}$.

3. On explicit formulas for the stationary distribution

Theorem 1. *In model 1 (i.e. $L_i = 0$, $K_i = \infty$, M is fixed) there exists*

(1) $$\lim_{t\to\infty} \Pr\{x_i(t) = n_i, x_{ij}(t) = n_{ij}, i, j = \overline{1,N}\} = g(\vec{n}, \{n_{ij}\}) = C \prod_{k=1}^{N} \left(\frac{\pi_k}{\lambda_k}\right)^{n_k} \prod_{i=1}^{N}\prod_{j=1}^{N} \frac{(\pi_i P_{ij} \beta_{ij})^{n_{ij}}}{n_{ij}!},$$

if $\sum_{i=1}^{N} n_i + \sum_{i=1}^{N}\sum_{j=1}^{N} n_{ij} = M$, and 0 otherwise. C is a normalizing constant.

Proof. The model represents a closed Jackson network in respect of cars and consists of $N+N^2$ stations, N of them, which correspond to the nodes of a transportation network, being single-channel queues, and N^2 ones, which correspond to the paths, being infinite-channel queues. Formula (1) can be obtained in the way which is traditional for the queueing theory. By introducing auxiliary variables one should pass on to a Markov process and check that function

$$f\left(\vec{n},\{n_{ij}\},\{y_s^{(i,j)}, s=\overline{1,n_{ij}}\}\right) = g(\vec{n},\{n_{ij}\}) \prod_{i=1}^{N}\prod_{j=1}^{N}\prod_{s=1}^{n_{ij}} \frac{1-B_{ij}\big(y_s^{(i,j)}\big)}{\beta_{ij}}$$

satisfies the system of equations for the stationary distribution. □

Corollary 1. *For symmetric networks, i.e.*

$$\lambda_i=\lambda,\ P_{ij}=\frac{1}{N},\ \beta_{ij}=\beta \quad (i,j=\overline{1,N})$$

formula (1) implies

$$C=\lambda^M\left(\sum_{m=0}^{M} C_{N-1+m}^{m}\frac{\rho^{M-m}}{(M-m)!}\right)^{-1}, \text{ where } \rho=N\lambda\beta.$$

Theorem 2. *In model 2 ($L_i=0$, $K_i=\infty$, open for cars) process $X(t)$ has the proper limit distribution if and only if*

$$\rho_i=\frac{c_i}{\lambda_i}<1 \quad \forall i=\overline{1,N}, \tag{2}$$

where

$$c_i=a_i+\sum_{k=1}^{N} c_k P_{ki} \quad (i=\overline{1,N}). \tag{3}$$

If (2) holds,

$$\lim_{t\to\infty}\Pr\{x_i(t)=n_i, x_{ij}(t)=n_{ij}, i,j=\overline{1,N}\}=\prod_{k=1}^{N}\rho_k^{n_k}(1-\rho_k)\prod_{i,j=1}^{N}\frac{\tau_{ij}^{n_{ij}}}{n_{ij}!}e^{-\tau_{ij}}, \tag{4}$$

where $\tau_{ij}=c_iP_{ij}\beta_{ij}$.

P r o o f. Since $X(t)$ is a regenerative random process, and its regeneration points are the moments when there are no cars and no clients in the network, existence of a limit distribution follows from the Smith theorem. As the paths are infinite-channel queues and $\beta_{ij} < \infty$, processes $x_{ij}(t)$ are stochastically bounded. A node constitutes a single-channel queue with exponentially distributed service time and input rate c_i $(i = \overline{1, N})$, which is specified by traffic equations (3), so conditions (2) are necessary and sufficient for stochastic boundedness of $x_i(t)$ $(i = \overline{1, N})$. Stochastic boundedness of a regenerative process implies its ergodicity. Formulas (4) are proved in the same way as in Theorem 1 with the help of a Markov process obtained by introducing auxiliary variables. □

Theorem 3. *In model 3 ($L_i = \infty$, $K_i = 0$, M cars) the limit distribution of process $X(t)$ is proper if and only if*

(5) $$\rho_j = \frac{\lambda_j \beta}{M\pi_j} < 1,\ j = \overline{1, N},\ \beta = \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i P_{ij} \beta_{ij}.$$

If

(6) $$B_{ij}(x) = 1 - e^{-\mu x} \quad \text{and} \quad P_{ij} = \pi_j, \quad i, j = \overline{1, N},$$

then

(7) $$\lim_{t\to\infty} \Pr\{y_j(t) = k_j, j = \overline{1, N}\} = \prod_{j=1}^{N} \rho_j^{k_j}(1 - \rho_j).$$

P r o o f. Process $X(t)$, generally speaking, is not regenerative, provided functions $B_{ij}(x)$ are arbitrary, and more delicate methods are required to prove existence of a limit distribution. One of approaches, based on the Borovkov's theorem for multidimensional Markov processes [4] is proposed in [5]. Another approach bears on the fact, that sequence $\{\theta_{n+k}, r_{n+k}\}_{k=1}^{\infty}$, where θ_n is the interval between $(n-1)$-th and n-th trip completion for one of M cars, and r_n is the number of the node, where it arrives, converges to a stationary one as $n \to \infty$. Without loss of generality sequence $\{\theta_n, r_n\}$ itself might be considered stationary and metrically transitive. Let $t_n = \sum\limits_{k=1}^{n} \theta_k$, $t_0 = 0$, $\varkappa_i(n)$ be the number of clients, arriving at node i during time (t_{n-1}, t_n), and $X_n = \{y_j(t_n - 0), j = \overline{1, N}\}$. The following recurrent relation holds

$$X_n = [X_{n-1} + \vec{\varkappa}_n - e(r_n)]^+ = f(X_{n-1}, \vec{\varkappa}_n, r_n),$$

where $\vec{\varkappa}_n = (\varkappa_1(n), \dots, \varkappa_N(n))$ and $e(r)$ is the N-dimensional unit vector with one at r-th place. This relation means, that stationary ergodic sequence $\{\vec{\varkappa}_n, r_n\}$ is a control one with respect to X_n. Since $f(X, \varkappa, r)$ is monotone by X, by using the method, which was proposed by Loynes [6], one can establish existence of a limit distribution for $\{X_n\}$. If (and only if) (5) is true, then each of processes $y_j(t)$ is stochastically bounded, so the limit distribution is proper. As to its form, it does not break up into factors, unless (6) holds. The point is that under condition (6) the moments of trip completions of each car form a Poisson process and these processes are independent. Since $P_{ij} = \pi_j$ $(i = \overline{1,N})$, each node receives independent Poisson flows of cars with rate $M\frac{\pi_j}{\beta}$ for node j. Therefore, processes $\{y_j(t), j = \overline{1,N}\}$ are independent and represent the length of the queue in system $M|M|1|\infty$, whence (7) follows. If one of conditions (6) is broken, then, provided the travel time is exponentially distributed, the input flows of cars to the nodes are doubly stochastic Poisson processes, so the simple formula (7) does not hold. $\square$

Theorem 4. *In model 4 ($L_i = \infty$, $K_i = 0$, open for cars) the limit distribution of process $X(t)$ is proper if and only if*

$$\lambda_i < c_i \ (i = \overline{1,N}), \tag{8}$$

where $\{c_i,\ i = \overline{1,N}\}$ are given by (3). It does not break up into factors.

The proof of ergodicity is based on the fact that $X(t)$ is regenerative.

The flows of cars to the different nodes are dependent even under (6); they are doubly stochastic Poisson (if the travel times are distributed exponentially), the random intensity is

$$\lambda_j(t, \omega) = \sum_{i=1}^{N} \beta_{ij}^{-1} x_{ij}(t).$$

Model 4a. There exists the set of routes $J = \{\vec{i} = (i_1, \dots, i_k), k = 1, 2, \dots\}$, $\|\vec{i}\| = k$. $P(\vec{i})$ is the probability that a car, having arrived to the network, chooses route $\vec{i}$. $J^0 \subset J$ and $J^0 = \{\vec{i}\colon i_s \neq i_l \text{ for } s \neq l, s, l = \overline{1,k}\}$, $k = 1, 2, \dots$, i.e. J^0 is the set of routes without self-intersections. If the input flow to the network is Poisson with intensity a, then the flow of cars on route $\vec{i}$ is Poisson with parameter $aP(\vec{i})$, and the flow of cars to node j is Poisson with parameter

$$\widetilde{c}_j = a \sum_{\vec{i} \in J^0} P(\vec{i}) \sum_{s=1}^{\|\vec{i}\|} \delta_{i_s j}.$$

The traffic intensity of node j is

$$\widetilde{\rho}_j = \lambda_j / \widetilde{c}_j,$$

and if $\widetilde{\rho}_j < 1$, then $\Pr\{y_j = m\} = \widetilde{\rho}_j^m (1 - \widetilde{\rho}_j)$.

The models of this kind suit for describing separate districts of a city with bus systems.

In model 5, implying $L_i = \infty$, $K_i = \infty$, no stationary distribution exists [7]. The situation in networks, that are open for cars, is similar. For existence of a limit distribution it should be either $L_i < \infty$, or $K_i < \infty$, or the waiting time of clients (cars) should be limited, or a control should be introduced.

Theorem 5. *In model 6, while $L_i = \infty$, $K_i = \infty$ $(i = \overline{1,N})$, the limit distribution of process $X(t)$ is proper if and only if*

$$\lambda_j < c_j < \lambda_j + b_j \quad \forall j = \overline{1,N}, \tag{9}$$

where $c_j = a_j + \sum\limits_{k=1}^{N} c_k P_{kj}$.

P r o o f. Like in Theorem 2, process $X(t)$ is regenerative, and coordinates $x_{ij}(t)$ $(i, j = \overline{1,N})$ are stochastically bounded. This implies convergence $\frac{Y_i(t)}{t} \xrightarrow[t\to\infty]{\Pr} c_i$, where $Y_i(t)$ is the number of cars, arrived at node i during time t, intensities $\{c_i,\, i = \overline{1,N}\}$ being a solution of system (3). If there exists such $i = \overline{1,N}$ that $\lambda_i \geq c_i$, then the corresponding $q_i(t) \xrightarrow[t\to\infty]{\Pr} +\infty$. Let $\lambda_i < c_i$ $(i = \overline{1,N})$ but $c_j \geq \lambda_j + b_j$ for some $j = \overline{1,N}$. We denote $\widetilde{q}_j(t)$ the state of node j in the network without carriers and with input rate of clients $\lambda_j + b_j$. Then stochastic inequality $q_j(t) \leq \widetilde{q}_j(t)$ holds, and also $\widetilde{q}_j(t) \xrightarrow[t\to\infty]{\Pr} -\infty$. If $c_j < \lambda_j + b_j$ then the stochastic boundedness of $q_j(t)$ is established by contradiction, as it was done, for example, in [8]. $\square$

Example 1. *The symmetric network, open for cars:*

$$\lambda_j = \lambda,\ a_j = a,\ b_j = b,\ K_j = K,\ L_j = L,\ P_{ij} = \frac{\alpha}{N} \text{ for } i, j = \overline{1,N},\ \alpha \in (0,1), \tag{10}$$

α being the probability that a car, upon arrival to a node, stays in the network. Conditions (9) are equivalent to

$$\lambda < \frac{a}{1-\alpha} < \lambda + b. \tag{11}$$

One can find the limit distribution, if $L_j = 0$ ($j = \overline{1,N}$) and the travel times along the paths equal zero, by using the Gelenbe's results [3].

We see that evaluation of the stationary distribution for transportation networks is possible only in exceptional cases meanwhile these are the distributions that are necessary for the calculation of the operational characteristics which allow to find out how the parameters of a system affect the efficiency function. In this connection the task of obtaining asymptotic formulas for the stationary limit distribution arises in numerous limit cases: for heavy and light traffic, zero travel times, situation of a network of high dimensionality (many nodes, servers, long queues) and the like.

4. The limit theorems for large symmetric transportation networks

Here it is assumed that conditions (10) for open and $P_{ij} = \frac{1}{N}$ for closed networks (with respect to cars) are true. Random process

$$Q^N(t) = \left\{ q_j^N(t), j = \overline{1,N},\, n^N(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_{ij}^N(t) \right\}$$

is under question. Let $w_k^N(t) = N^{-1} \sum_{i=1}^{N} I\{q_i^N(t) = k\}$, $k \in [-K, L]$. The mean-value method prompts that in the limit as $N \to \infty$ the evolution becomes determinate, i.e. there exist functions $p_j(t)$ such that for any finite $t \geq 0$

$$\sup_{0 \leq s \leq t} \left| w_j^N(s) - p_j(s) \right| \xrightarrow[N \to \infty]{\text{Pr}} 0, \tag{12}$$

if

$$w_j^N(0) \xrightarrow[N \to \infty]{\text{Pr}} p_j(0). \tag{13}$$

4.1. The exponential distribution of travel time

If $B_{ij}(x) = 1 - e^{-\mu x}$, process $Q^N(t)$ constitutes a Markov chain with a countable set of states. The proof of convergence of $w_j^N(s)$ to the determinate functions $p_j(t)$ is based on the classical results on convergence of Markov processes in the event that their generators converge. The similar results for $K < \infty$, $L < \infty$ are obtained in [9]. Functions $p_j(t)$ satisfy a system of nonlinear equations, the invariant point of the system describing the stationary distribution of a single node, if the limit dynamic system is globally stable (on this topic see [10]). Here we shall give a result concerning the open for cars network with carriers and $K = \infty$, $L = \infty$.

Theorem 6. *In model 6 for any finite $t \geq 0$ convergence (12) takes place, and*

$$\sup_{0\leq s\leq t}\left|\frac{n^N(t)}{N} - M(t)\right| \xrightarrow[N\to\infty]{\Pr} 0,$$

if (13) holds and $\frac{n^N(0)}{N} \xrightarrow[N\to\infty]{\Pr} M(0)$. Functions $M(t)$ and $p_j(t)$ satisfy the system of equations

(14)

$$\begin{cases} p'_j(t) = -(a+\lambda+bI\{j<0\}+\alpha\mu M(t))p_j(t)+ \\ \qquad (\lambda+bI\{j\leq 0\})p_{j-1}(t)+(a+\alpha\mu M(t))p_{j+1}(t), \quad j=0,\pm1,\pm2,\dots \\ \\ M'(t) = (-\mu+\alpha\mu v(t))M(t)+(\lambda+b)u(t)+av(t), \end{cases}$$

where $u(t) = \sum_{j=1}^{\infty} p_{-j}(t)$, $v(t) = \sum_{j=1}^{\infty} p_j(t)$.

Proof. Since process $Y^N(t) = \{w_j^N(t), n^N(t), j = 0, \pm1, \dots\}$ is Markov, the convergence to dynamic system (14) is established, like in [11], from equivalence of convergence of translation semigroups and of their generators on a core of the domain. The proof is connected with certain difficulties, being, however, of a technical character. □

The stationary point of system (14) has the following shape

$$p_j = \rho_1^j C,\ p_{-j} = \rho_2^j C \text{ (for } j\geq 0);\ M = \frac{a}{\mu(1-\alpha)},$$

where $\rho_1 = \lambda(1-\alpha)a^{-1}$, $\rho_2 = a/(1-\alpha)(\lambda+b)$, $C = \frac{(1-\rho_1)(1-\rho_2)}{1-\rho_1\rho_2}$. This point specifies probability distribution p_j ($j = 0, \pm1, \pm2, \dots$) if and only if $\rho_1 < 1, \rho_2 < 1, C < \infty$, which is equivalent to (11), i.e.

$$\lambda < \frac{a}{1-\alpha} < \lambda + b.$$

Besides, this distribution does not depend on mean travel time β whereas for closed networks this dependence exists (see [12]).

4.2. General distribution of travel time

Process $Q^N(t)$ is not Markov, therefore proving results similar to Theorem 6 requires new approaches. One of them is proposed in [11]. First the asymptotic independence of the processes, describing states of the individual nodes, is established, and then on the basis of this independence the convergence of the flow of cars, arriving at a node, to Poisson one is proved. Then the limit evolution of each node can be specified by some modifications of system (14) depending on the shape of a model. One of the results herein concerns model 3. We shall give it in a simplified version. Let $V(t)$ be the renewal function of the renewal process connected with the arrival times of a car at any node and let us assume that $V'(t)$ exists.

Theorem 7. *In model 3 let*

$$p_j^N(t) = \Pr\{y_k^N(t) = j\},\ j = 0, 1, 2, \dots,\ k = \overline{1, N}.$$

If $N \to \infty$ *and* $\frac{M}{N} \to r$, *then*

$$p_j^N(t) \to p_j(t),$$

where $\{p_j(t), j \geq 0\}$ *is the solution of the system of equations*

$$p_j'(t) = -(\lambda + rV'(t)I\{j > 0\})p_j(t) + \lambda I\{j > 0\}p_{j-1}(t) + rV'(t)p_{j+1}(t). \tag{15}$$

P r o o f. One can easily make sure, that the input flow of cars $W_k^N(t)$ to any fixed node k, being a sum of independent sparse renewal processes, converges to the Poisson process with non-constant intensity $rV'(t)$. Then for all k process $y_k^N(t)$ weakly converges to process $y(t)$, which characterizes the number of clients in the single-node system receiving the same input flow of clients as any individual node of the network and the Poisson input flow of cars with intensity $rV'(t)$, because

$$y_k^N(t) = y_k^N(0) + Z_k(t) - W_k^N(t) - \min\Big(0, \inf_{0 \leq s \leq t}\big[y_k^N(0) + Z_k(s) - W_k^N(s)\big]\Big),$$

where $Z_k(t)$ is the number of clients that arrived at node k during time t. This proves the theorem. □

Since $\lim_{t\to\infty} V'(t) = \beta^{-1}$, the stationary point of system (15) has the same shape as in Theorem 3, so the stationary distribution does not depend on the form of $B(x)$ and is determined by mean value β, which ensures Dobrushin's Poisson hypothesis.

The similar results are also obtained for the more complicated model where the number of waiting spots for cars $K > 0$ (see [11]).

5. The limit theorems for asymmetric transportation networks

The generalization of the results of section 4. in the asymmetric situation may be done in different ways. One of them is used in [13]. The whole set of nodes is divided into finite number m of groups (districts) containing n_i nodes in each group and $\sum_{i=1}^{m} n_i = N$. The network is closed for cars and $\frac{M}{N} \to r$, $\frac{n_i}{N} \to d_i$ ($d_i > 0$) as $N \to \infty$. The routes are equiprobable in each group, namely, $\frac{\alpha_i}{n_i}$ is the probability that a car leaves the group, and $\frac{n_i - \alpha_i}{n_i^2}$ is the probability of transition from one node of the group to another. If a car leaves group i, then it goes with probability P_{ij} to group j, where routing is also equiprobable. All travel times have the exponential distribution. It turned out that in this case one can prove as well convergence to a dynamic system, find its invariant point and for some models establish convergence of the stationary measures of the systems with finite N to it. Another approach is based on the assumption that number of nodes N and routing matrix P are fixed while number of cars $M \to \infty$ and their travel times increase. This involves that input flows to the nodes become Poisson and independent, so, for example, in model 5 the convergence to N independent birth-and-death processes takes place. Thus, sequence S^M of networks with M cars and travel times having distribution function $B_{ij}\left(\frac{x}{M}\right)$ is under consideration. Here we shall give the results only for model 5 in case $K_j < \infty$, $L_j \le \infty$, $j = \overline{1,N}$. Let us assume that at the start time all nodes are empty, cars are in motion and $F_i\left(\frac{x}{M}\right)$ is the distribution function of the elapsed time till the first occurrence of a car in node i.

Theorem 8. *Let* $F_i'(0) = \gamma_i > 0$, $i = \overline{1,N}$, *exist. If* $\Pr\{q_j^M(0) = m\} \xrightarrow[M\to\infty]{} p_m^{(j)}(0)$, *then for any finite* $t \ge 0$

$$(16) \qquad \lim_{M\to\infty} \Pr\{q_j^M(t) = m_j - K_j\,, j = \overline{1,N}\} = \prod_{j=1}^{N} p_{m_j}^{(j)}(t),\ 0 \le m_j \le K_j + L_j,$$

where functions $p_m^{(j)}(t)$ *satisfy the system of equations*

$$\frac{dp_m^{(j)}}{dt} = -(\lambda_j I\{m < K_j + L_j\} + \gamma_j I\{m > 0\})p_m^{(j)} + \lambda_j I\{m > 0\}p_{m-1}^{(j)} + \gamma_j I\{m < K_j + L_j\}p_{m+1}^{(j)}, \quad 0 \le m \le K_j + L_j.$$

P r o o f. Let $y_{ij}^{(M)}(t)$ be the number of occurrences of car i in node j during time $(0,t)$ and $Y_j^{(M)}(t) = \sum_{i=1}^{M} y_{ij}^{(M)}(t)$. Since the number of nodes N is fixed, random processes $\{Y_j^{(M)}(t), j = \overline{1,N}\}$ are asymptotically independent as $M \to \infty$. With the help of the theorem on convergence of sums of indicators from [14] one establishes, that process $Y_j^{(M)}(t)$ weakly converges to the Poisson one with parameter γ_j as $M \to \infty$. Just as in Theorem 7, $q_j^{(M)}(t)$ is a continuous functional on the trajectories of processes $\big(Z_j(s), Y_j^{(M)}(s), s \in (0,t)\big)$, where $Z_j(s)$ is the number of clients that arrived at node j during time $(0,s)$, so (16) is true. □

If $\rho_j = \lambda_j/\gamma_j < 1 \;\forall j = \overline{1,N}$, then the convergence of the stationary distributions to function $\prod\limits_{j=1}^{N} g_j(m_j, \rho_j)$, where $g_j(m,\rho) = \dfrac{\rho^m(1-\rho)}{1-\rho^{K_j+L_j+1}}$ for $0 \le m \le K_j + L_j$, is valid.

In fact, these two approaches have the adjoining points. Each group of nodes with $n_j \to \infty$ can be interpreted as a generalized node, and then the system is composed of fixed number m of the generalized nodes. As the number of nodes in the group increases, the probability $\frac{\alpha_j}{n_j}$ of leaving a node vanishes, so the time, that a car spends in a generalized node, increases.

REFERENCES

[1] A. A. Borovkov and R. A. Schassberger. Ergodicity of a polling network *Stochastic Process. Appl.* (1994) **50**, 253–262.

[2] G. Fayolle and J. M. Lasgouttes. A state-dependent polling model with Markovian routing. In: *Stochastic Networks, IMA Volumes in Math. & its Applications.* Springer Verlag, (1995) **71**, 283–301.

[3] E. Gelenbe. Product-form queueing networks with negative and positive customers *J. Appl. Probab.* (1991) **28**, 656–663.

[4] A. A. Borovkov. Ergodicity and stability of stochastic processes. Moscow, Russia: *Editorial URSS, Institute of Mathematics Publisher* (1999), 440 p. (in Russian).

[5] A. A. Sergeev. Global stability of one polling model with travelling servers. In: *Voronezhskaya zimnyaya matematicheskaya shkola 2006. Transactions.* Voronezh, Russia, (2006), 92–95 (in Russian).

[6] R. M. Loynes. The stability of a queue with non-independent inter-arrival and service times *Proc. Camb. Phil. Soc.* (1962) **58(3)**, 497–528.

[7] L. G. Afanassieva, F. Delcoigne, and G. Fayolle. On polling systems where servers wait for customers *Markov Process. Related Fields.* (1997) **3(4)**, 527–545.

[8] J. Kiefer and J. Wolfowitz. On the theory of queues with many servers *Trans. Amer. Math. Soc.* (1955) **78**, 1–18.

[9] L. G. Afanassieva, G. Fayolle, and S. Yu. Popov. Models for transportation networks *J. Math. Sci.* (1997) **84(3)**, 1092–1103.

[10] V. I. Oseledets and D. V. Khmelev. Stochastic transportation networks and the stability of dynamical systems *Theory Probab. Appl.* (2002) **46(1)**, 154–161.

[11] A. A. Sergeev. Limit theorems for one class of polling models *Theory Probab. Appl.* (2006) **50(3)**, 1–10.

[12] L. G. Afanassieva, G. Fayolle, and L. V. Nazarov. Preliminary models for Moving Server Networks. *Rapport de recherche RR-2469*, INRIA, Le Chesnay, France, 1995.

[13] D. Khmelev. The limit theorems for non-symmetrical transportation networks *Fundam. Prikl. Mat.* (2001) **9(4)**, 1401–1407 (in Russian).

[14] B. A. Sevast'yanov. Poisson limit law for a scheme of sums of dependent random variables *Theory Probab. Appl.* (1973) **17(4)**, 695–699.

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