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A NEW CLASS OF PROCESSES FOR FORMALIZING AND GENERALIZING INDIVIDUAL-BASED MODELS: THE SEMI-SEMI-MARKOV PROCESSES

C. Jacob A.F. Viet [1]

Individual-based models are a "bottom-up" approach for calculating empirical distributions at the level of the population from simulated individual trajectories. We build a new class of stochastic processes for mathematically formalizing and generalizing these simulation models according to a "top-down" approach, when the individual state changes occur at countable random times. We allow individual population-dependent semi-Markovian transitions in a non closed population such as a branching population. These new processes are called Semi-Semi-Markov Processes (SSMP) and are generalizations of Semi-Markov processes. We calculate their kernel and their probability law, and we build a simulation algorithm from the kernel.

1. Introduction

The number of individual-based models used in population dynamics with a complex structure have considerably increased thanks to the increase of the computers capacity and to the popularization of informatics ([9]) concomitant to the will of development of complex systems analysis. These models, based on a "bottom-up

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approach" ([9]) allow the calculus of empirical distributions at the level of the population from simulated individual trajectories. The advantage of this approach is that it generally seems *a priori* much easier to build an individual-based model than a population-state-variable model when the individuals are marked by personal characteristics and/or history or when the next state-change is driven by complex rule decisions depending on the current state of the population. Nevertheless, the drawback of such models is the lack of mathematical formalism ([7]). Such attempts are rare while a general rigorous mathematical formalism of these models at the population level is necessary at least for having a good readability of the different model components independently of the programming language in which it is written ([7]) and for validating the simulation approach. The notion of Generalized Semi-Markov Process (GSMP) was proposed for formalizing the event-driven simulation approach in the area of operational research ([8]). But in this framework, individuals were assumed to be independent and the formalism remained closed to the simulation approach. In addition, a general algorithmic strategy using multi-agents architecture under a discrete-time simulation mode is described in [7]. But this approach remains a bottom-up one.

The simplest individual-based models with countable random jump times, consist of N i.i.d. (identically and independently distributed) individual renewal processes describing the times of arrival of an event. In this setting the behavior of the total sum of arrivals until some time t when each process is a Poisson one with rate λ (the inter-arrival time is exponential with parameter λ), is a Poisson variable with parameter $tN\lambda$. In the nontrivial case of more heavy tailed distributions of the inter-arrival times, the asymptotic behavior of this resulting process and of some related quantities was studied for infinite life-duration of each event ([18]) and for heavy-tailed life duration of the event ([14]). The case of branching processes when the arrival times are procreation times, is also well-known especially when the individual transitions (offspring or death) are Markovian (exponential distribution). Less studied is the case of age-dependent (non exponential) transitions. The most famous and studied age-dependent models are the Sevastyanov's process, the Bellman-Harris process (particular case of the previous one) and the Crump-Mode-Jagers process ([11]).

The objective of this paper is to built a new class of stochastic processes for rigorously formalizing at the level of the population (top-down approach) individual-based models. We allow the evolution of the population according to births and deaths (branching population). The process is defined by a set of individual random characteristics undergoing semi-Markovian transitions that may be population-dependent. We call these new processes semi-semi-Markov processes (SSMP) since they are generalizations of semi-Markov processes and have in the general case a longer memory than these processes.

In section 2, we remind the main definitions and properties of the classical semi-Markov processes. In section 3, we define the semi-semi-Markov processes (SSMP). These processes are first defined for a closed population of random individual characteristics evoluating according to semi-Markovian and population-dependent transitions. Then they are generalized to any branching population. The proofs follow the same general way as in the semi-Markov setting. We show that the complexity increases when passing from the individual level to the population level. In the general case of semi-Markovian individual transitions, the process at the population level is a semi-semi-Markov process (SSMP). But in the particular case of Markovian individual transitions, the SSMP is reduced to a classical Markov process with rate equal to the sum of the individual rates. We give the simulation algorithm associated with the kernel and calculate the probability law of the process.

We assume in this paper that the individual transition laws are time homogeneous, semi-Markovian and may be population-dependent. But the strategy of elaboration of the kernel (and therefore of the simulation algorithm), of the transition rates and the direct calculus of the probability law of the process, remain valid under more complex individual histories.

2. Classical semi-Markov Process (SMP) for one individual

We recall here the main definitions and properties of this class of process (see for example [4], [3], [6], [10], or [12]). Let $\{X_n(\omega)\}$ be a random process taking values in a countable space $\mathcal{X}$ and $\{T_n(\omega)\}$ be a random process taking values in $\mathbb{R}^+$. Assume that the process $\{X_n(\omega), T_n(\omega)\}$ is an homogeneous Markov Renewal Process (MRP), that is, denoting (X_n, T_n) for $(X_n(\omega), T_n(\omega))$, it satisfies, for all $n \geq 0$,

$$P(X_{n+1} = j, T_{n+1} - T_n \leq \tau | X_n, ..., X_0, T_n, ..., T_0) = \\ P(X_1 = j, T_1 - T_0 \leq \tau | X_0). \tag{1}$$

Let $\{X_t(\omega)\}$ be a random process taking values in $\mathcal{X}$, defined from the MRP by

$$X_t(\omega) \stackrel{def.}{=} X_{n_t}(\omega) 1_{\{n_t(\omega) = \sup\{n : T_n(\omega) \leq t\}\}}, \tag{2}$$

where "$def.$" means "by definition". Process $\{X_t(\omega)\}$ (denoted $\{X_t\}$) is constant on each interval $[T_n, T_{n+1}[$ and undergoes state transitions at times $\{T_n\}$ in accordance with the Markov chain $\{X_n\}$. Therefore it spends a random sojourn time $\Delta T_{n+1} \stackrel{def.}{=} T_{n+1} - T_n$ in the state defined by X_n until the time T_{n+1} of the $(n+1)$th transition. This process is an homogeneous process called semi-Markov process because it is Markovian at the transition times (and not necessarily at all times).

Let $Q_{i,j}(\tau) = P(X_1 = j, T_1 - T_0 \leq \tau | X_0 = i)$. According to (1) and (2), $\{Q_{i,j}(\tau)\}_{i,j,\tau}$, called the semi-Markov kernel of the process, defines the law of $\{X_t\}$. Writing ΔT_1 for $T_1 - T_0$, $Q_{i,j}(.)$ can also be written when $\lim_{\tau \to \infty} Q_{i,j}(\tau) \neq 0$,

$$(3) \qquad Q_{i,j}(\tau) = P(\Delta T_1 \leq \tau | X_1 = j, X_0 = i) P(X_1 = j | X_0 = i)$$

$$(4) \qquad \stackrel{not.}{=} F_{i,j}(\tau) P(i,j),$$

where "not." means "denoted", $F_{i,j}(\tau)$ is the cdf of the sojourn time in i before jumping in the state j, and $P(i,j) = \lim_{\tau \to \infty} Q_{i,j}(\tau)$ is the probability of transition of the embedded Markov chain $\{X_n\}$. Relationship (4) allows simulations of trajectories of the process according to an event-driven simulation algorithm. Let i be the current state of $\{X_n\}$, then choose the next state j according to the law $\{P(i,j)\}_j$, and then simulate the sojourn time τ in i before jumping in j according to the law $F_{i,j}(.)$. In some applications, it may be more natural to deal with the transition rates that are defined given the only past of the process, contrary to $F_{i,j}(.)$ defined given the past and the next jump state.

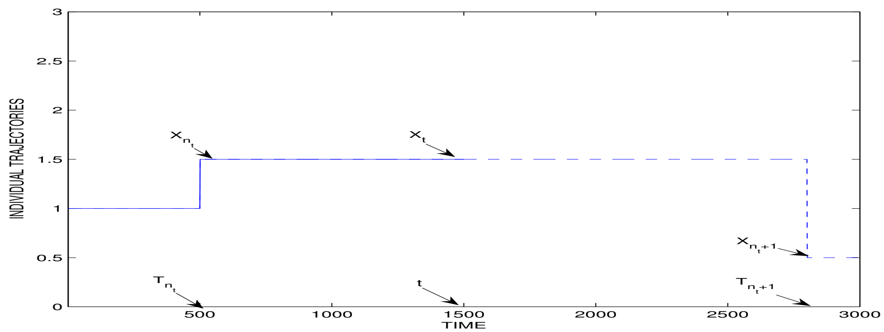

Figure 1: Trajectory of a semi-Marvov process; the past until the present time $T_n = t_n$ is in a continuous lines while the future is represented by a dashed line.

2.1. Transition rates

Define, in the same way as in [3], the transition rates $\lambda_{i,j}(\tau)$ as the probability per time unit that the process enters in the state j just after a time lag τ in the state i, given that there was no transition from i during this lag:

$$\lambda_{i,j}(\tau) = \lim_{\Delta\tau\to 0} \frac{P(\Delta T_1 \in (\tau, \tau+\Delta\tau), X_1 = j | X_0 = i, \Delta T_1 > \tau)}{\Delta\tau}$$
$$\overset{homog.}{=} \lim_{\Delta\tau\to 0} \frac{P(X_{\tau+\Delta\tau} = j | X_s = i, \forall s \leq \tau)}{\Delta\tau}.$$

From the definition of $\lambda_{i,j}(\tau)$ and $Q_{i,j}(\tau)$, one get

(5) $$\lambda_{i,j}(\tau) = \frac{\dot{Q}_{i,j}(\tau)}{1 - Q_i(\tau)}$$

where $\dot{Q}_{i,j}(\tau) = dQ_{i,j}(\tau)/d\tau$ and $Q_i(\tau) = \sum_j Q_{i,j}(\tau) = P(\Delta T_1 \leq \tau | X_0 = i) \overset{not.}{=} F_i(\tau)$, is the cdf of the time spent in state i. One also have, when $F_{i,j}(.)$ is absolutely continuous with respect to the Lebesgue's measure (see section 3.3.):

(6) $$Q_{i,j}(\tau) = \int_0^\tau \lambda_{i,j}(u) exp(-\int_0^u \lambda_i(s) ds) du$$

where $\lambda_i(s) = \sum_k \lambda_{i,k}(s)$ (see section 3.3.). According to (5) and (6), the knowledge of $\{Q_{i,j}(\tau)\}_{i,j,\tau}$ is equivalent to that of $\{\lambda_{i,j}(\tau)\}_{i,j,\tau}$.

The Markov process is a particular case of semi-Markovian processes. It is an homogeneous process without memory implying that the transition rates are independent of the transition time, *i.e.* $\lambda_{i,j}(\tau) = \lambda_{i,j}$, for any τ. Therefore (6) is reduced to $Q_{i,j}(\tau) = \lambda_{i,j}\lambda_i^{-1}(1 - \exp(-\lambda_i\tau)) \overset{def.}{=} F_{i,j}(\tau)P(i,j)$ leading to $F_i(\tau) \overset{def.}{=} Q_i(\tau) = 1 - \exp(-\lambda_i\tau) = F_{i,j}(\tau)$, $P(i,j) = \lambda_{i,j}\lambda_i^{-1}$.

2.2. Renewal equations

The marginal law of the process may be calculated by using the renewal equations also called backward equations. Let $P_{i,j}(t) = P(X_t = j | X_0 = i)$. Then, for any

$t, i, j,$

$$\begin{aligned} P_{i,j}(t) &= P(\Delta T_1 > t | X_0 = i)\delta_{i,j} + \\ & \int_0^t \sum_{k\in\mathcal{X}} dP(X_s = k, \Delta T_1 = s | X_0 = i) P(X_t = j | X_s = k) \\ &= (1 - Q_i(t))\delta_{i,j} + \int_0^t \sum_{k\in\mathcal{X}} dQ_{i,k}(s) P_{k,j}(t-s), \end{aligned} \tag{7}$$

where $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ otherwise. The first term of (7) describes the event that state i has not been left until t and contributes only to $P_{i,i}(t)$. The second term describes the event, that state i is not left until a time s. At this time a transition to some state k occurs, and then the process moves from k to j in $t-s$ time units. This system of equations may also be written using the convolution operator: $\mathbf{P} = (\mathbf{I} - \mathbf{Q}^{\Sigma}) + \mathbf{Q} * \mathbf{P}$, where $\mathbf{I}$ is the identity matrix, $\mathbf{P}[i,j] = P_{i,j}(.)$, $\mathbf{Q}[i,j] = Q_{i,j}(.)$, $\mathbf{Q}^{\Sigma}$ is the diagonal matrix with $\{Q_i(.)\}_i$ on the diagonal. By induction, defining $\mathbf{Q}^{*(n+1)} = \mathbf{Q}^{*n} * \mathbf{Q}$ with $\mathbf{Q}^{*0} = \mathbf{I}$, one solution of this system is

$$\mathbf{P} = \sum_{n=0}^{\infty} \mathbf{Q}^{*n} * (\mathbf{I} - \mathbf{Q}^{\Sigma}), \tag{8}$$

and when $\mathcal{X}$ is finite, this solution is the unique one.

When $\mathcal{X}$ is finite, system (7) leads to a recursive calculus algorithm of $\{P_{i,j}(.)\}_{i,j}$ using a discretization of the time. Moreover upper and lower bounds of the renewal equations may also lead to analytical approximations of $\{P_{i,j}(.)\}_{i,j}$ ([13]). In the particular Markovian setting, the solution $\mathbf{P}(t)$ of the renewal equation (7), defined by $\mathbf{P}(t)[i,j] = P_{i,j}(t)$, for all i, j, is exponential:

$$\mathbf{P}(t) = \exp(\mathbf{\Lambda} t), \tag{9}$$

where $\exp(\mathbf{\Lambda} t) = \sum_{k\geq 0} \mathbf{\Lambda}^k t^k / k!$ can be calculated using different methods ([15]), and $\mathbf{\Lambda}[i,j] = \lambda_{i,j}$, $j \neq i$, $\mathbf{\Lambda}[i,i] = -\lambda_i = -\sum_{j\neq i} \lambda_{i,j}$.

2.3. Asymptotic behavior

The asymptotic behavior of $P_{i,j}(t)$, solution of (7) as $t \to \infty$, exists under some additionnal assumptions such as the transition matrix P of $\{X_n\}$ is irreducible, aperiodic and recurrent, and $0 < m_i < \infty$, where $m_i = \int_0^\infty (1 -$

$Q_i(t))dt$, $i \in \mathcal{X}$, which implies that m_i is the mean sojourn time in i: $m_i = \int t dQ_i(t) \stackrel{def.}{=} E(\Delta T_1 | X_0 = i)$. Then there exists a unique invariant probability $\nu = (\nu_1, \nu_2, \nu_3, \ldots)$ for P, that is $\nu P = \nu$, and, for any i, j, the asymptotic behavior of $\{P_{i,j}(t)\}_t$ and that of $\{X_n\}$ are given by

$$\lim_{t \to \infty} P_{i,j}(t) = \frac{\nu_j m_j}{\sum_k \nu_k m_k}; \lim_{n \to \infty} P(X_n = j | X_0 = i) = \nu_j.$$

Therefore $\lim_{n \to \infty} P(X_n = j | X_0 = i)$ represents the mean sojourn time in j for the embedded chain, and $\lim_{t \to \infty} P_{i,j}(t)$ represents the mean sojourn time in the state j of the semi-Markovian process.

3. Semi-semi-Markov Processes for closed populations

3.1. Elaboration of the kernel of the process

Let Ω be a finite population of individuals described by a finite set of stochastic characteristics which are determined by environment-dependent MRPs $\{\{(X_m^{(c,l)}(\omega_l), T_m^{(c,l)}(\omega_l))\}_m\}_{(c,l)}$ (assumptions $(A1)$ to $(A3)$). The environment is here the population and $X_m^{(c,l)}(\omega_l)$ (which represents a random characteristic of individual ω_l) takes values in $\mathcal{X}_l^c$ and $T_m^{(c,l)}(\omega_l)$ (which represents a random transition time) takes values in $\mathbb{R}^+$. An individual characteristic may be for example the individual health state, the group or family to which the individual belongs, or the individual spatial localization. But it may also represent a set of random linked characteristics of an individual or a random characteristic that concerns several linked individuals such as the existence of the couple l relative to two given individuals. In this case $\mathcal{X}_l^c = \{existence\, of\, the\, couple\, l, non\, existence\}$. Other examples may concern the potential vertices of a random graph defined by their connected edges, or the potential double helices of an RNA. Consequently the notion of individual characteristic must be taken here in a broad sense.

Let $\{(\mathcal{X}_n(\Omega), \mathcal{T}_n(\Omega))\}$ be the "renewal" process of the population and $\{\mathcal{X}_t(\Omega)\}$ be the semi-semi-Markov process both defined from $\{\{(X_m^{(c,l)}(\omega_l), T_m^{(c,l)}(\omega_l))\}_m\}_{(c,l)}$

by:

$$\mathcal{X}_t(\Omega) \stackrel{def.}{=} \mathcal{X}_{n_t}(\Omega) \tag{10}$$

$$n_t(\Omega) \stackrel{def.}{=} \sum_{(c,l)} m_{c,l,t}(\omega_l) \tag{11}$$

$$m_{c,l,t}(\omega_l) \stackrel{def.}{=} \sup\{m : T_m^{(c,l)}(\omega_l) \leq t\} \tag{12}$$

$$\mathcal{X}_{n_t}(\Omega) \stackrel{def.}{=} \{X_{m_{c,l,t}}^{(c,l)}(\omega_l)\}_{(c,l)} \tag{13}$$

$$\mathcal{T}_{n_t}(\Omega) \stackrel{def.}{=} \sup_{(c,l)}\{T_{m_{c,l,t}}^{(c,l)}(\omega_l)\} = \sup_{(c,l)} \sup_m \{T_m^{(c,l)}(\omega_l) \leq t\}. \tag{14}$$

The quantity n_t is therefore the number of jumps of the population process until t, $\mathcal{T}_{n_t}(\Omega)$ is the last jump time of the population before t defined by the last individual jump time before t, and $\{\mathcal{T}_n(\Omega)\} = \{T_m^{(c,l)}(\omega_l)\}_{m,(c,l)}$ is the set of jump times of $\{\mathcal{X}_t(\Omega)\}$ defined by the set of individual jump times. Denote $\mathcal{X}$ the state space of the process $\{\mathcal{X}_t(\Omega)\}$ which is also the state space of $\{\mathcal{X}_n(\Omega)\}$. Denote $T_m^{(c,l)}$, $X_m^{(c,l)}$, $\mathcal{T}_n$, $\mathcal{X}_n$ instead of $T_m^{(c,l)}(\omega_l)$, $X_{T_m^{(c,l)}(\omega_l)}^{(c,l)}(\omega_l)$, $\mathcal{T}_n(\Omega)$, $\mathcal{X}_{\mathcal{T}_n(\Omega)}(\Omega)$. Denote also $\mathcal{F}_n(I)$ for the past knowledge $\{\mathcal{X}_n = I, \mathcal{X}_{n-1} = I_{n-1}, ..., \mathcal{X}_0 = I_0, \mathcal{T}_n = t_n, \mathcal{T}_{n-1} = t_{n-1}, ..., \mathcal{T}_0 = t_0\}$ until time $\mathcal{T}_n$. According to (10) to (14), the law of the process $\{\mathcal{X}_t\}$ is defined by the law of $\{(\mathcal{X}_n, \mathcal{T}_n)\}$, itself defined by the set of transition probabilities (kernels): for all I, J, n,

$$P(\mathcal{X}_{n+1} = J, \Delta\mathcal{T}_{n+1} \leq \tau | \mathcal{F}_n(I)) =$$

$$P(\Delta\mathcal{T}_{n+1} \leq \tau | \mathcal{X}_{n+1} = J, \mathcal{F}_n(I)) P(\mathcal{X}_{n+1} = J | \mathcal{F}_n(I)) \stackrel{not.}{=}$$

$$F_{\mathcal{F}_n(I),J}(\tau) P(\mathcal{F}_n(I), J) \stackrel{not.}{=} Q_{\mathcal{F}_n(I),J}(\tau) \tag{15}$$

which have to be determined from the law of the set $\{\{(X_m^{(c,l)}, T_m^{(c,l)})\}_m\}_{(c,l)}$. We assume two kinds of individual transitions: those which may happen in an independent parallel way given the past, implying that the most rapid will determine the next jump of the population, the laws of the following transitions being defined again when this new state occurs, and those which are exclusive, implying that the jump state must be chosen *a priori* according to some probability law.

Let i_l^c, j_l^c be any couple of potential states of the random characteristic $X_{\cdot}^{(c,l)}$. We assume here that the individual characteristics are defined in such a way that the next transitions $\{i_l^c \to j_l^c\}_{(c,l)}$ given $\mathcal{F}_n(I)$ may occur independently, while the set of transitions $\{i_l^c \to j_l^c\}_{j_l^c}$ for a given individual characteristic (c,l) are exclusive, that is, incompatible. For simplifying the notations we use in this section 3. the notation (l) or l instead of (c,l). We assume that all transitions may be population-dependent. Define similarly as in (12) and for all l,

$$T_{m_{l,n}}^{(l)} = \sup_m \{T_m^{(l)} \leq \mathcal{T}_n\}$$

which is the last transition time before $\mathcal{T}_n$ concerning the individual characteristic l. For simplifying the notations we denote $T_{m_n}^{(l)}$ instead of $T_{m_{l,n}}^{(l)}$. Define also

$$R_n^{(l)} = T_{m_n+1}^{(l)} - \mathcal{T}_n,\ S_n^{(l)} = \mathcal{T}_n - T_{m_n}^{(l)},\ \Delta T_{m_n+1}^{(l)} = T_{m_n+1}^{(l)} - T_{m_n}^{(l)}.$$

Then $R_n^{(l)}$, $S_n^{(l)}$, $\Delta T_{m_n+1}^{(l)}$ are respectively the residual waiting time of l in its current state from $\mathcal{T}_n$ until its potential next jump at time $T_{m_n+1}^{(l)}$, the time spent by l in its current state until $\mathcal{T}_n$, and the time between the two consecutive jump times $T_{m_n}^{(l)}$, $T_{m_n+1}^{(l)}$. By definition $\Delta T_{m_n+1}^{(l)} = S_n^{(l)} + R_n^{(l)}$, and by definition of $\mathcal{T}_n$, there exists l such that $\mathcal{T}_n = T_{m_n}^{(l)}$, implying $R_n^{(l)} = \Delta T_{m_n+1}^{(l)}$, and $S_n^{(l)} = 0$ for this l (cf fig. 2).

Assume N individual characteristics. Let $(I,J) \in \mathcal{X}^2 = [\Pi_l \mathcal{X}_l]^2$, where $I = \{i_1, ..., i_l, ..., i_N\}$ and $J = \{i_1, ..., j_l, ..., i_N\}$, that is the transition $I \to J$ concerns the individual characteristic l. We also denote (I, J_l) for (I,J) and, for any l', $s_n^{(l')} = t_n - t_{m_n}^{(l')}$ for the value taken by $S_n^{(l')}$, where t_n, $t_{m_n}^{(l')}$ are the respective values taken by $\mathcal{T}_n$, $T_{m_n}^{(l')}$. Assume the following hypotheses, for any $\{i_l, j_l, \tau_l\}_l$, I, and given $\mathcal{F}_n(I)$,

1. $(A1)$ (independency of the next individual jumps):

$$dP(\{X_{m_n+1}^{(l)} = j_l, R_n^{(l)} = \tau_l\}_l | \mathcal{F}_n(I)) = \Pi_l dP(X_{m_n+1}^{(l)} = j_l, R_n^{(l)} = \tau_l | \mathcal{F}_n(I)).$$

2. $(A2)$ (memory needed for the individual residual waiting times $\{R_n^{(l)}\}_l$): for each l, $R_n^{(l)}$ depends at most on the time $s_n^{(l)}$ already spent in the individual current state i_l, on the corresponding current state I of the population at time t_n and on the next individual jump state j_l. In particular, $R_n^{(l)}$ is independent of $\{s_n^{(l')}\}_{l' \neq l}$. Since by definition $\mathcal{T}_n = T_{m_n}^{(l)} + S_n^{(l)}$ and $S_n^{(l)}$ has the value $s_n^{(l)}$, then

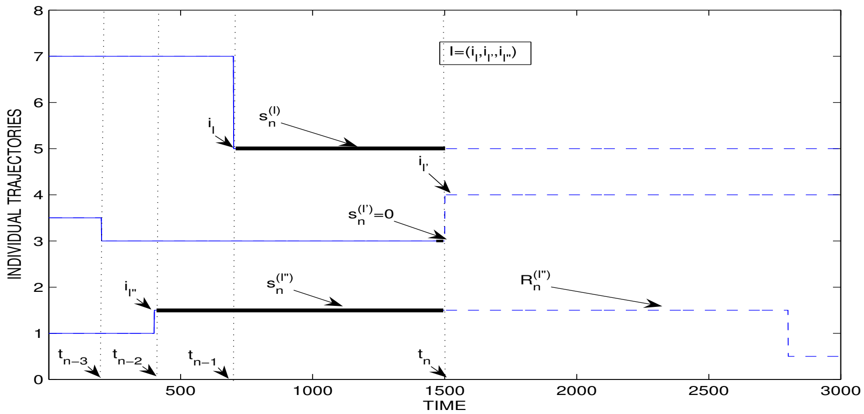

Figure 2: Trajectories of 3 individual characteristics l, l', l''. Until the current time $t_n = 1500$ characterized by the current state $I = \{i_l, i_{l'}, i_{l''}\}$, the trajectories are represented by continuous lines while the future from t_n is represented by dashed lines. The vertical lines show the jump times of the population. The individual trajectories are not synchronized.

$(A2)$ combined with $(A1)$ implies

$$dP(R_n^{(l)} = s | X_{m_n+1}^{(l)} = j_l, \{R_n^{(l')}\}_{l' \neq l}, \mathcal{F}_n(I)) =$$

$$dP(\Delta T_{m_n+1}^{(l)} = s + s_n^{(l)} | X_{m_n+1}^{(l)} = j_l, \mathcal{X}_{T_{m_n}^{(l)} + s_n^{(l)}} = I, \Delta T_{m_n+1}^{(l)} > s_n^{(l)}) =$$

$$dP(\Delta T_{m_n+1}^{(l)} = s + s_n^{(l)} | X_{m_n+1}^{(l)} = j_l, \mathcal{X}_{T_{m_n}^{(l)} + s_n^{(l)}} = I,$$

$$\mathcal{X}_r = I, \forall r \in [T_{m_n}^{(l)}, T_{m_n}^{(l)} + s_n^{(l)}], \Delta T_{m_n+1}^{(l)} > s_n^{(l)}) =$$

$$\frac{dP(\Delta T_{m_n+1}^{(l)} = s + s_n^{(l)} | \mathcal{X}_{T_{m_n}^{(l)}} = I, X_{m_n+1}^{(l)} = j_l)}{dP(\Delta T_{m_n+1}^{(l)} > s_n^{(l)} | \mathcal{X}_{T_{m_n}^{(l)}} = I, X_{m_n+1}^{(l)} = j_l)} = \frac{dF_{i_l|I,j_l}^{(l)}(s + s_n^{(l)})}{1 - F_{i_l|I,j_l}^{(l)}(s_n^{(l)})}, \tag{16}$$

where $F_{i_l|I,j_l}^{(l)}(.)$ is the cdf of the individual sojourn time in i_l before jumping in j_l, when the state of the whole population is I and remains unchanged during this sojourn time (constant environment), this cdf being assumed independent of n.

Notice that if $(A2)$ is not checked, then we replace $(A2)$ by $dP(R_n^{(l)} =$

$s|X^{(l)}_{m_n+1} = j_l, \mathcal{F}_n(I)) = dG^{(l)}_{i_l|\mathcal{F}_n(I),j_l}(s)$ which is no more time homogeneous.

3. $(A3)$ (memory needed for the individual jump probabilities): for each l,

$$(17)\quad P(X^{(l)}_{m_n+1} = j_l|\mathcal{F}_n(I)) = P(X^{(l)}_{m_n+1} = j_l|\mathcal{X}_{\mathcal{T}_n} = I) \overset{not.}{=} P^{(l)}(i_l|I, j_l),$$

where $\sum_{j_l \in \mathcal{X}_l} P^{(l)}(i_l|I, j_l) = 1$ if $\mathcal{X}_l(I) \neq \emptyset$, or 0 if $\mathcal{X}_l(I) = \emptyset$, where $\mathcal{X}_l(I) = \{j_l \in \mathcal{X}_l : P^{(l)}(i_l|I, j_l) \neq 0\}$. As in (16), $P^{(l)}(i_l|I, j_l)$ is a probability of an individual Markov chain with constant environment I.

4. $(A4)$ (definition of the next population jump as corresponding to the smallest individual residual time):

$$(18)\quad Q_{\mathcal{F}_n(I),J_l}(\tau) \;=\; P(\min_{l'}\{R^{(l')}_n\} = R^{(l)}_n, R^{(l)}_n \leq \tau, X^{(l)}_{m_n+1} = j_l|\mathcal{F}_n(I)).$$

We moreover assume for simplifying the presentation that the cdf of the individual sojourn times are absolutely continuous with respect to the Lebesgue's measure.

In the next proposition, we calculate the population kernel from the residual individual kernels defined as:

$$(19)\quad Q^{(l)|s^{(l)}_n}_{i_l|I,j_l}(\tau) \overset{def.}{=} P(R^{(l)}_n \leq \tau, X^{(l)}_{m_n+1} = j_l|\mathcal{F}_n(I))$$

$$(20)\quad = F^{(l)|s^{(l)}_n}_{i_l|I,j_l}(\tau) P^{(l)}(i_l|I, j_l),$$

where, for $P^{(l)}(i_l|I, j_l) \neq 0$,

$$(21)\quad F^{(l)|s^{(l)}_n}_{i_l|I,j_l}(\tau) = \frac{F^{(l)}_{i_l|I,j_l}(s^{(l)}_n + \tau) - F^{(l)}_{i_l|I,j_l}(s^{(l)}_n)}{1 - F^{(l)}_{i_l|I,j_l}(s^{(l)}_n)}$$

is the cdf of the residual individual transition time $i_l|I \to j_l$ from t_n given the time $s^{(l)}_n$ already spent in i_l by l until t_n, and we set $F^{(l)|s^{(l)}_n}_{i_l|I,j_l}(\tau) = 0$, for all $\tau < \infty$ when $P^{(l)}(i_l|I, j_l) = 0$.

Notice that when $s^{(l)}_n = 0$, then the kernel is reduced to the classical semi-Markovian kernel, except that it is population dependent.

Proposition 1. *Assume that the individual transition laws satisfy* $(A1)$ *to* $(A3)$ *and that the population kernel is defined by* $(A4)$. *Then*

$$(22)\quad dQ_{\mathcal{F}_n(I),J_l}(\tau) = \Pi_{l' \neq l}(1 - \sum_{j_{l'} \in \mathcal{X}_{l'}(I)} Q^{(l')|s^{(l')}_n}_{i_{l'}|I,j_{l'}}(\tau)) dQ^{(l)|s^{(l)}_n}_{i_l|I,j_l}(\tau).$$

Proof. The result is directly deduced from (18), (19) and $(A1)$. □

Consequences. We have

$$P(\mathcal{F}_n(I),J_l) = \int_0^\infty dQ_{\mathcal{F}_n(I),J_l}(\tau)$$

$$(23)\qquad = \int_0^\infty \Pi_{l'\neq l}\Big(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau)\Big)dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau),$$

$$dF_{\mathcal{F}_n(I),J_l}(\tau) = \frac{dQ_{\mathcal{F}_n(I),J_l}(\tau)}{P(\mathcal{F}_n(I),J_l)}$$

$$(24)\qquad = \frac{\Pi_{l'\neq l}\Big(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau)\Big)dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau)}{\int_0^\infty \Pi_{l'\neq l}\Big(1-\sum_{j_{l'}\in\mathcal{X}_{l'}(I)} Q^{(l')|s_n^{(l')}}_{i_{l'}|I,j_{l'}}(\tau)\Big)dQ^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau)}.$$

Remarks.

1. If instead of absolutely continuous distributions with respect to Lebesgue measure, we consider distributions with the possibility of ties, then we must generalize (22) by considering $R_n^{(l')} \geq R_n(l)$ instead of $R_n^{(l')} > R_n(l)$, $l' \neq l$.

2. According to the proposition, the kernel is defined by the time-homogeneous residual individual kernels $\{Q^{(l)|s_n^{(l)}}_{i_l|I,j_l}(.)\}$ and therefore is homogeneous and may be denoted

$$(25)\qquad Q_{\mathcal{F}_n(I),J_l}(.) = Q_{s_n;I,J_l}(.), \text{ where } s_n = \{s_n^{(l)}\}_l.$$

3. Define $\mathcal{Y}_n = (\mathcal{X}_n,\mathcal{S}_n)$, where $\mathcal{S}_n = \{S_n^{(l)}\}_l$. Define $\Delta I_{h+1} = \{\Delta I^{(l)}_{h+1}\}_l$, where $\Delta I^{(l)}_{h+1} = 0$ if and only if the state of l at time $\mathcal{T}_{h+1}$ is the same as its state at $\mathcal{T}_h$, denote $s(s_h,\Delta t_{h+1},\Delta I_{h+1}) = \{[s_h^{(l)}+\Delta t_{h+1}]1_{\{\Delta I^{(l)}_{h+1}=0\}}\}_l$, the set of spent times at time t_{h+1} defined from the set at time t_h. Then, for $s_{h+1} = s(s_h,\Delta t_{h+1},\Delta I_{h+1})$, for all $h \leq n$, $\{(\mathcal{Y}_n,\mathcal{T}_n)\}$ satisfies

$$dP(\mathcal{Y}_{n+1} = (I_{n+1},s_{n+1}),\Delta\mathcal{T}_{n+1} = \Delta t_{n+1}|\{\mathcal{Y}_h = (I_h,s_h),\mathcal{T}_h = t_h\}_{h\leq n}) =$$
$$dP(\mathcal{X}_{n+1} = I_{n+1},\Delta\mathcal{T}_{n+1} = \Delta t_{n+1}|\mathcal{X}_n = I_n,\mathcal{S}_n = s_n) =$$
$$dP(\mathcal{Y}_{n+1} = (I_{n+1},s_{n+1}),\Delta\mathcal{T}_{n+1} = \Delta t_{n+1}|\mathcal{Y}_n = (I_n,s_n)).$$

Consequently $\{(\mathcal{Y}_n, \mathcal{T}_n)\}$ is a MRP. Assuming absolutely continuous kernels with respect to the Lebesgue's measure, then for any $(s_n, s(s_n, \Delta t_{n+1}, \Delta I_{n+1}))$, we have either $\|s_{n+1} - s_n\| > 0$ or $s_{n+1} = s_n = 0$, where $\|s_{n+1} - s_n\| = \sup_l [s_{n+1}^{(l)} - s_n^{(l)}] 1_{s_{n+1}^{(l)} - s_n^{(l)} \geq 0}$. Therefore, for any $((s_n, I_n), (I_{n+1}, s_{n+1}))$, the kernel of $\{(\mathcal{Y}_n, \mathcal{T}_n)\}$ is, when $s_{n+1} = s_n = 0$, equal to $Q_{I_n, I_{n+1}}(.)$ (semi-Markov kernel), and when $\|s_{n+1} - s_n\| > 0$,

$$\begin{aligned}
&dP(\mathcal{Y}_{n+1} = (I_{n+1}, s_{n+1}), \Delta\mathcal{T}_{n+1} = \Delta t_{n+1} | \mathcal{Y}_n = (I_n, s_n)) \overset{not.}{=} \\
&dQ^{\mathcal{Y}}_{(I_n, s_n), (I_{n+1}, s_{n+1})}(\Delta t_{n+1}) = \\
&1_{\|s_{n+1} - s_n\|}(\Delta t_{n+1}) [dQ_{s_n; I_n, I_{n+1}}(\|s_{n+1} - s_n\|) 1_{\{(s_n, s_{n+1}) \in \Delta\mathcal{S}\}}] \overset{not.}{=} \\
&dF^{\mathcal{Y}}_{(I_n, s_n), (I_{n+1}, s_{n+1})}(\Delta t_{n+1}) P^{\mathcal{Y}}((I_n, s_n), (I_{n+1}, s_{n+1})),
\end{aligned}$$

where $1_{\|s_{n+1} - s_n\|}(\Delta t_{n+1}) = 1$, if $\Delta t_{n+1} = \|s_{n+1} - s_n\|$, and is null otherwise (dirac distribution), and $\Delta\mathcal{S}$ is the set of (s_n, s_{n+1}) such that $s_{n+1}^{(l)} - s_n^{(l)}$ is a positive constant for any l except one for which this difference is negative or null; $F^{\mathcal{Y}}_{(I_n, s_n), (I_{n+1}, s_{n+1})}(.) = 1_{\|s_{n+1} - s_n\|}(.)$ is the cdf of the sojourn time in (I_n, s_n) until the next jump into (I_{n+1}, s_{n+1}), and $P^{\mathcal{Y}}(., .)$ is the transition probability of $\{\mathcal{Y}_n\}$.

4. If $(A2)$ and $(A3)$ are replaced by a long memory assumption such that "$|I$" and "$|s_n^{(l)}$" must be replaced by "$|\mathcal{F}_n(I)$", then the kernel of the process, defined by (22) is no more time homogeneous and (25) is no more valid.

Corollary 1. *Assume the particular case* (Exp) *defined by individual Markovian transitions, that is, for all* τ, i_l, j_l, c, l, I, $F^{(l)}_{i_l | I, j_l}(\tau) = 1 - \exp(-\lambda_{i_l|I}\, \tau)$, *where* $\lambda_{i_l|I} = 0$ *when* $\mathcal{X}_l(I) = \emptyset$. *Then the SSMP is a MP (Markov process) defined, for all* I *such that* $\sum_{l'} \lambda_{i_{l'}|I} \neq 0$, *by*

$$dQ_{I, J_l}(\tau) \quad = \quad \lambda_{i_l|I} P^{(l)}(i_l | I, j_l) \exp(-\sum_{l'} \lambda_{i_{l'}|I}\, \tau) d\tau$$

Consequences. When $\sum_{l'} \lambda_{i_{l'}|I} \neq 0$ (otherwise, set $F_{I, J_l}(\tau) = 0$, $P(I, J_l) = 0$),

$$dF_{I, J_l}(\tau) \quad = \quad dF_I(\tau) = (\sum_{l'} \lambda_{i_{l'}|I}) \exp(-\sum_{l'} \lambda_{i_{l'}|I}\, \tau) d\tau \tag{26}$$

$$P(I, J_l) \quad = \quad P^{(l)}(i_l | I, j_l) \frac{\lambda_{i_l|I}}{\sum_{l'} \lambda_{i_{l'}|I}}. \tag{27}$$

Proof. Using (22), we get $dQ_{I,J_l}(\tau) = \lambda_{i_l|I} P^{(l)}(i_l|I, j_l) \exp(-\sum_{l'} \lambda_{i_{l'}|I}\, \tau) d\tau$ from which (26) and (27) are deduced. □

This corollary generalizes to population-dependent processes the well-known case of superposition of individual Poisson processes: take N i.i.d. Poisson processes with inter-arrival times exponentially distributed with rate λ. For each individual, the only possible transition at time $T_k^{(l)}$ is $k-1 \to k$ (arrival of the kth event). Then at time t, if $\mathcal{N}_t$ is the sum of the individual counting processes,

$$P(\mathcal{N}_t = k) = P(\mathcal{T}_k \leq t, \mathcal{T}_{k+1} > t) = F^{*k}(t) - F^{*(k+1)}(t),$$

where, according to (26), $F(t) = \exp(-N\lambda t)$ (inter-arrival population time distribution). Therefore $F^{*n}(.)$ is $Gamma(n, N\lambda)$ and $F^{*k}(t) - F^{*(k+1)}(t)$ is the probability for a $Poisson(N\lambda t)$ variable to be equal to k.

Consequence. Under (Exp), if $\sum_J P(\mathcal{F}_n(I), J) \neq 0$ (I is not an absorbing state), then $m_I \overset{(Exp)}{=} E(\Delta\mathcal{T}_1|\mathcal{X}_0 = I) = [\sum_{l'} \lambda_{i_{l'}|I}]^{-1}$.

3.2. Event-driven simulation algorithm

In a classical semi-Markov jump process the simulation is done in the following way: we first determine the next jump state j according to the probability $P(i,j)$, and then we determine the time of the transition $i \to j$ according to the law $F_{i,j}(.)$.

Consider now a semi-semi-Markov process. We propose an event-driven recursive simulation algorithm generated by the kernel of the process. Assume that at the nth jump, the process is in the state $I = (i_1, ..., i_l, ..., i_N)$ and the time at this jump is t_n. Then the next jump state with the associated transition time are determined by the kernel $Q_{\mathcal{F}_n(I),J}(.) = F_{\mathcal{F}_n(I),J}(.) P(\mathcal{F}_n(I), J)$, where $F_{\mathcal{F}_n(I),J}(.)$ and $P(\mathcal{F}_n(I), J)$ satisfy (24) and (23). We define the state J at the next jump and the corresponding jump time t_{n+1} in the following way. For each i_l in I satisfying $\mathcal{X}_l(I) \neq \emptyset$, and such that the next state j_l and the associated jump time $t_{m_n+1}^{(l)}$ satisfying $t_{m_n}^{(l)} \leq t_n < t_{m_n+1}^{(l)}$, are either not yet simulated or such that their transition laws depend not only on i_l but also on the current state I of the population,

1. First choose $j_l \in \mathcal{X}_l(I)$ according to the probability law $\{P^{(l)}(i_l|I, j_l)\}_{j_l}$;

2. Then simulate according to the law of $R_n^{(l)}$ defined by (16), and independently from the other waiting times, a residual waiting time $r_n^{(l)}$ in i_l from t_n until the jump into j_l; deduce $t_{m_n+1}^{(l)} = r_n^{(l)} + t_n$.

3. Keep the simulations $\{j_l, t^{(l)}_{m_n+1}\}_l$ in memory.

Then the minimum time $t^{(l)}_{m_n+1}$ among the set of all simulated jump times $\{t^{(l')}_{m_n+1}\}_{l'}$ defines the next jump time and the next population state J_l.

3.3. Transition rates

Define the individual transition rate from $i_l|I$ to j_l:

$$\lambda^{(l)}_{i_l|\mathcal{F}_n(I),j_l}(\tau) \stackrel{def.}{=} \lim_{\Delta\tau\to 0} \frac{P(R^{(l)}_n \in (\tau, \tau+\Delta\tau), X^{(l)}_{m_n+1} = j_l | X^{(l)}_{m_n} = i_l, R^{(l)}_n > \tau, \mathcal{F}_n(I))}{\Delta\tau}.$$

Then we directly get

$$\lambda^{(l)}_{i_l|\mathcal{F}_n(I),j_l}(\tau) = \frac{\dot{Q}^{(l)|s^{(l)}_n}_{i_l|I,j_l}(\tau)}{1 - \sum_{j\in\mathcal{X}_l} Q^{(l)|s^{(l)}_n}_{i_l|I,j}(\tau)} \stackrel{not.}{=} \lambda^{(l)|s^{(l)}_n}_{i_l|I,j_l}(\tau). \tag{28}$$

Define now the transition rate from I to J_l relative to the population:

$$\lambda_{\mathcal{F}_n(I),J}(\tau) \stackrel{def.}{=} \lim_{\Delta\tau\to 0} \frac{P(\Delta\mathcal{T}_{n+1} \in (\tau, \tau+\Delta\tau), \mathcal{X}_{n+1} = J | \mathcal{F}_n(I), \Delta\mathcal{T}_{n+1} > \tau)}{\Delta\tau}.$$

Then

$$\begin{aligned} \lambda_{\mathcal{F}_n(I),J}(\tau) &= \lim_{\Delta\tau\to 0} \frac{P(\mathcal{X}_{t_n+\tau+\Delta\tau} = J | \mathcal{X}_{t_n+s} = I, \forall s \le \tau, \mathcal{F}_n(I))}{\Delta\tau} \\ &= \frac{\dot{Q}_{s_n;I,J}(\tau)}{1 - \sum_{J'} Q_{s_n;I,J'}(\tau)} \stackrel{not.}{=} \lambda_{s_n;I,J}(\tau). \end{aligned} \tag{29}$$

Proposition 2. *For all i_l, j_l, I, l such that $P^{(l)}(i_l|I, j_l) \neq 0$, assume that $F^{(l)}_{i_l|I,j_l}(.)$ is absolutely continuous with respect to the Lebesgue's measure. Then, for all $\mathcal{F}_n(I), J, \tau$,*

$$Q_{\mathcal{F}_n(I),J}(\tau) = \int_0^\tau \lambda_{\mathcal{F}_n(I),J}(u) \exp(-\int_0^u \sum_J \lambda_{\mathcal{F}_n(I),J}(s)ds)du \tag{30}$$

$$Q^{(l)|s^{(l)}_n}_{i_l|I,j_l}(\tau) = \int_0^\tau \lambda^{(l)|s^{(l)}_n}_{i_l|I,j_l}(u) \exp(-\int_0^u \sum_j \lambda^{(l)|s^{(l)}_n}_{i_l|I,j}(s)ds)du \tag{31}$$

Proof. The proofs are the same for $Q^{(l)|s_n^{(l)}}_{i_l|I,j_l}(.)$ and $Q_{\mathcal{F}_n(I),J}(.)$. So we only prove (30).

Summing (29) on J, we get

$$\sum_J \lambda_{\mathcal{F}_n(I),J}(\tau) = \frac{\sum\limits_J \dot{Q}_{\mathcal{F}_n(I),J}(\tau)}{1-\sum\limits_J Q_{\mathcal{F}_n(I),J}(\tau)}. \tag{32}$$

The solution of (32) is, as in the semi-Markov frame,

$$1-\sum_J Q_{\mathcal{F}_n(I),J}(\tau) = \exp(-\int_0^\tau \sum_J \lambda_{\mathcal{F}_n(I),J}(s)ds). \tag{33}$$

But $Q_{\mathcal{F}_n(I),J}(\tau)$ may be written

$$\begin{aligned} Q_{\mathcal{F}_n(I),J}(\tau) &= \int_0^\tau P(\mathcal{X}_{n+1}=J,\Delta\mathcal{T}_{n+1}\in(u,u+du)|\mathcal{F}_n(I),\Delta\mathcal{T}_{n+1}>u). \quad (34)\\ P(\Delta\mathcal{T}_{n+1}>u|\mathcal{F}_n(I)) &= \int_0^\tau \lambda_{\mathcal{F}_n(I),J}(u).(1-\sum_J Q_{\mathcal{F}_n(I),J}(u))du. \end{aligned}$$

Using (33) and (35), we get (30). □ The proposition shows that the knowledge of $\{Q_{\mathcal{F}_n(I),J}(.)\}_{\mathcal{F}_n(I),J}$ is equivalent to that of $\{\lambda_{\mathcal{F}_n(I),J}(.)\}_{\mathcal{F}_n(I),J}$.

Corollary 2. *Assume the particular case (Exp) defined in corollary 1. Then*

1. $\lambda^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau)$ *is independent of* τ *and* $s_n^{(l)}$. *We denote* $\lambda^{(l)|s_n^{(l)}}_{i_l|I,j_l}(\tau) \overset{not.}{=} \lambda_{i_l|I,j_l}$ *and we have* $\lambda_{i_l|I,j_l} = \lambda_{i_l|I}P^{(l)}(i_l|I,j_l)$.
2. *The population transition rates are independent of the time and*

$$\begin{aligned} \lambda_{I,J_l} &= \lambda_{i_l|I}P^{(l)}(i_l|I,j_l) = \lambda_{i_l|I,j_l} \\ \lambda_I &= \sum_J \lambda_{I,J} = \sum_l \lambda_{i_l|I} \end{aligned}$$

The proof is directly deduced from (29) and corollary 1.

Consequence. In the Markovian setting, we directly get $\mathbf{P}(t) = \exp(\mathbf{\Lambda}t)$ from (9), where $\mathbf{P}(t)[I,J] = P_{I,J}(t)$ and $\mathbf{\Lambda}[I,J] = \lambda_{I,J}$, for all I, J.

3.4. Probability law of $\{\mathcal{X}_s\}_{s\in]0,t]}$

The probability law of the process in a given time interval $]0,t]$ is equivalent to the law of $(\{(\mathcal{X}_{\mathcal{T}_l},\mathcal{T}_l)\}_{l=1,n_t},n_t)$. Let n, $0<t_1<t_2<\ldots<t_n\leq t$ and $\{I_l\}_{0\leq l\leq n}$. Then

$$
\begin{aligned}
&dP(\{(\mathcal{X}_{\mathcal{T}_h}=I_h,\mathcal{T}_h=t_h)\}_{h=1,n_t},n_t=n|\mathcal{F}_0(I_0))=\\
&dP(\{(\mathcal{X}_{\mathcal{T}_h}=I_h,\mathcal{T}_h=t_h)\}_{h=1,n},\mathcal{T}_{n+1}>t|\mathcal{F}_0(I_0))=\\
&P(\mathcal{T}_{n+1}>t|\mathcal{F}_n(I_n))\Pi_{l=1}^n dP(\mathcal{X}_h=I_h,\mathcal{T}_h=t_h|\mathcal{F}_{h-1}(I_{h-1}))=\\
&[1-\sum_J Q_{s_n;I_n,J}(t-t_n)]\Pi_{h=1}^n dQ_{s_{h-1};I_{h-1}),I_h}(t_h-t_{h-1}) \qquad (35)
\end{aligned}
$$

3.5. Marginal probability law of $\{\mathcal{X}_t\}$

We can directly calculate the marginal probability law of the process using (35) and

$$
P(\mathcal{X}_t=J|\mathcal{F}_0(I_0))=P(\sqcup_n\{\mathcal{X}_{\mathcal{T}_n}=J,\mathcal{T}_n\leq t,\mathcal{T}_{n+1}>t\}|\mathcal{F}_0(I_0)) \qquad (36)
$$

or we can iteratively calculate it using the renewal theory.

3.5.1. Renewal equations.

Define an "initial" time t_0 and initial conditions

$$
\mathcal{F}_0(I_0)=\{\mathcal{X}_{\mathcal{T}_0}=I_0,\mathcal{T}_0=t_0,\mathcal{S}_0=s_0\}\stackrel{not.}{=}\mathcal{F}_{t_0,\Delta_0}(I_0);\ s_0\stackrel{not.}{=}\Delta_0.
$$

We denote the set of individual spent times $s(s_h,\Delta t_{h+1},\Delta I_{h+1})$ by $\Delta_{t_{h+1}-t_h}$, $h\geq 0$. We also denote $P(\mathcal{X}_t=J|\mathcal{F}_{t_0,\Delta_0}(I_0))=\mathbf{P}_{\Delta_0}[I_0,J](t-t_0)=\mathbf{P}[(I_0,\Delta_0),J](t-t_0)$, since due to the homogeneity of the process and its memory,
$P(\mathcal{X}_t=J|\mathcal{F}_{t_0,\Delta_0}(I_0))=P(\mathcal{X}_{t-t_0}=J|\mathcal{X}_0=I_0,\mathcal{S}=s_0)$. In the same way $P(\mathcal{X}_t=J|\mathcal{F}_{t_0,\Delta_0}(I_0),\mathcal{X}_1=I_1,\mathcal{T}_1=t_1)=P(\mathcal{X}_{t-t_1}=J|\mathcal{X}_0=I_1,\mathcal{S}=s(s_0,\Delta t_1,\Delta I_1))$. Therefore we also denote $P(\mathcal{X}_t=J|\mathcal{F}_{t_0,\Delta_0}(I_0),\mathcal{X}_1=I_1,\mathcal{T}_1=t_1)=$
$\mathbf{P}_{\Delta_{t_1-t_0}}[I_1,J](t-t_1)=\mathbf{P}[(I_1,\Delta_{t_1-t_0}),J](t)$,
$Q_{s_0;I_0,J}(t)=\mathbf{Q}_{\Delta_0}[I_0,J](t)=\mathbf{Q}^{\mathcal{Y}}[(I_0,s_0),(I_1,s(s_0,t,J-I_0))](t)$,
$[\sum_{J'}Q_{s_0;I_0,J'}(t)]\delta_{I_0,J}=\mathbf{Q}^{\Sigma}_{\Delta_0}[I_0,J](t)=\mathbf{Q}^{\Sigma}[(I_0,s_0),J](t)$, where these quantities are null, for $J\neq I_0$.

Proposition 3. *The marginal probability law of the process is given by the* *following backward equations in the matricial form:*

$$\mathbf{P}_{\Delta_0}(t-t_0) = \mathbf{I}-\mathbf{Q}^{\Sigma}_{\Delta_0}(t-t_0)+\int\limits_{t_1\in(t_0,t)} d\mathbf{Q}_{\Delta_0}(t_1-t_0)\mathbf{P}_{\Delta_{t_1-t_0}}(t-t_1), t\geq t_0 \tag{37}$$

equivalent to

$$\mathbf{P}=(\mathbf{I}-\mathbf{Q}^{\Sigma})+\mathbf{Q}^{\mathcal{Y}}*\mathbf{P}. \tag{38}$$

Therefore (38) leads by induction to (39), unique solution when $\mathcal{X}$ is finite:

$$\mathbf{P}=\sum_{n\geq 0}\mathbf{Q}^{\mathcal{Y}*n}*(\mathbf{I}-\mathbf{Q}^{\Sigma}) \tag{39}$$

Then $\mathbf{P}(t)$ may be approximated by $[\sum\limits_{n\geq 0}^{N_t}\mathbf{Q}^{\mathcal{Y}*n}*(\mathbf{I}-\mathbf{Q}^{\Sigma})](t)$, where $N_t<\infty$ increases with t, and (39) is reduced to (8) when $Q_{s_n;I,J}(.)$ is independent of s_n (semi-Markov kernel). Notice that (39) may also be obtained using directly (36) since $\mathcal{T}_{n+1}=\sum\limits_{k=1}^{n+1}\Delta\mathcal{T}_k$ and $dP(\sum\limits_{k=1}^{n+1}\Delta\mathcal{T}_k=u|I_0,s_0)=(\mathbf{Q}^{\mathcal{Y}*n}*d\mathbf{Q}^{\Sigma})[(I_0,s_0)](u)$.

Proof. As in the semi-Markov frame, we may write

$$P(\mathcal{X}_t=J|\mathcal{F}_{t_0,\Delta_0}(I_0))=P(\Delta\mathcal{T}_1>t-t_0|\mathcal{F}_{t_0,\Delta_0}(I_0))\delta_{I_0,J}+$$
$$\sum_{I_1\neq I_0}\int\limits_{t_1\in(t_0,t)}P(\mathcal{X}_t=J|\mathcal{F}_{t_0,\Delta_0}(I_0),\mathcal{X}_{\mathcal{T}_1}=I_1,\mathcal{T}_1=t_1).$$

$$dP(\mathcal{X}_{\mathcal{T}_1}=I_1,\Delta\mathcal{T}_1=t_1-t_0|\mathcal{F}_{t_0,\Delta_0}(I_0)). \tag{40}$$

Then the result follows from (40) using the time-homogeneity of the process, its memory and the notations defined at the beginning of the paragraph. □

Corollary 3. *Let* $\dot{\mathbf{Q}}_{\Delta_0}$ *be the derivative of* $\mathbf{Q}_{\Delta_0}$ *when this derivative exists, that is when all the individual sojourn time laws are absolutely continuous with respect to the Lebesgue's measure. Let* $\{a_i\}_i$ *be a deterministic sequence function of* h *depending on the chosen numerical integration scheme. Then the discretization of (37) using* $t-t_0=nh$, $t_1-t_0\in\{ih\}_{i\leq n}$, *leads to the solution* $\mathbf{P}_n=\mathbf{R}_n^{-1}\mathbf{B}_n$, *where* $\mathbf{P}_n=(\mathbf{P}^t_{\Delta_0}(nh),...,\mathbf{P}^t_{\Delta_{(n-1)h}}(h))^t$, $\mathbf{B}_n=(\mathbf{B}^t_{\Delta_0,n},...,\mathbf{B}^t_{\Delta_{(n-1)h},n})^t$,

$$\mathbf{R}_n = \begin{pmatrix} \mathbf{R}_{\Delta_0}(0) & \mathbf{R}_{\Delta_0}(h)\ldots\mathbf{R}_{\Delta_0}((n-1)h) \\ 0 & \mathbf{R}_{\Delta_h}(0)\ldots\mathbf{R}_{\Delta_h}((n-2)h) \\ . & .\ \ \ldots\ldots\ldots \\ 0 & 0\ldots\ldots\ldots\mathbf{R}_{\Delta_{(n-1)h}}(0) \end{pmatrix}$$

$$\mathbf{R}_{\Delta}(ih) = \mathbf{I}\delta_{0,i} - a_i\dot{\mathbf{Q}}_{\Delta}(ih)(1-\delta_{0,i}), i = 0, ..., n-1$$

$$\mathbf{B}_{\Delta_{jh},n} = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_{jh}}((n-j)h), j = 0, ..., n-1,$$

where $\delta_{0,i} = 1$ *if* $i = 0$ *and is 0 otherwise.*

Proof. The discretization of (37) using $t - t_0 = nh$, $t_1 - t_0 \in \{ih\}_{i\leq n}$, leads to

$$\mathbf{P}_{\Delta_0}(nh) = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_0}(nh) + \sum_{i=1}^{n-1} a_i\dot{\mathbf{Q}}_{\Delta_0}(ih)\mathbf{P}_{\Delta_{ih}}((n-i)h), \tag{41}$$

equivalent to

$$\sum_{i=0}^{n-1}\mathbf{R}_{\Delta_0}(ih)\mathbf{P}_{\Delta_{ih}}((n-i)h) = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_0}(nh). \tag{42}$$

Then using (42) with $(n-j, \Delta_{jh})$ instead of (n, Δ_0), $j = 0, \ldots, n-1$, we get

$$\sum_{i=0}^{n-j-1}\mathbf{R}_{\Delta_{jh}}(ih)\mathbf{P}_{\Delta_{(i+j)h}}((n-j-i)h) = \mathbf{I} - \mathbf{Q}^{\Sigma}_{\Delta_{jh}}((n-j)h)$$

which leads to the result. □

3.6. Asymptotic behavior

Recall that $\{(\mathcal{Y}_n, \mathcal{T}_n)\}$, where $\mathcal{Y}_n = (\mathcal{X}_n, \mathcal{S}_n)$, $\mathcal{T}_n = \sum_{h\leq n}\Delta\mathcal{T}_h$ is a MRP. According to the Markov renewal theory which deals with the asymptotic behavior of functionals of MRP (or MRW) (see [1]), if $\{\mathcal{Y}_n\}$ has a unique stationary probability measure ν, and if $0 < \mu < \infty$, where $\mu = \int_{(I,s)} m_{(I,s)}d\nu(I,s)$, $m_{(I,s)} = E(\Delta\mathcal{T}_1|\mathcal{Y}_0 = (I,s))$ (mean time spent in I given the set s of individual spent times), then, defining $A_t = t - \mathcal{T}_{n_t}$ the associated age process, for some appropriate function $g(.,.)$,

$$\lim_t E[g(\mathcal{Y}_t, A(t))|(I_0, s_0)] =$$

$$\mu^{-1} \int_{(I,s)\in\mathcal{Y}} \int_{\tau\in(0,\infty)} g((I,s),\tau) P_{(I,s)}(\Delta\mathcal{T}_1 > \tau) d\tau d\nu(I,s). \tag{43}$$

Then, for $g((\mathcal{X}_{n_t}, \mathcal{S}_{n_t}), u) = 1_J(\mathcal{X}_{n_t})$, where $J \in \mathcal{X}$, and since we assumed $m_{I,s} < \infty$ which implies $\int_{\tau\in(0,\infty)} P_{(I,s)}(\Delta\mathcal{T}_1 > \tau) d\tau = m_{(I,s)}$, (43) becomes

$$\lim_t P(\mathcal{X}_t = J|I_0, s_0) = \frac{\int_s m_{(J,s)} d\nu(J,s)}{\sum_I \int_s m_{(I,s)} d\nu(I,s)}.$$

It remains to show the existence of $\nu(.)$ (the problem has been moved from the asymptotic behavior of $\{\mathcal{X}_t\}$ to that of $\{(\mathcal{X}_n, \mathcal{S}_n)\}$).

4. Semi-semi-Markov processes for branching populations.

We generalize here the SSMP previously defined for a closed population to a branching population. In this case, a SSMP becomes a SSMBP (Semi-Semi-Markovian Branching Process).

4.1. Semi-semi-Markovian Branching Process for individuals with a pregnancy period

Assume for each individual a single random characteristic B (Branching) which is his physiological status together with the physiological status of his newborns, each physiological status taking values in $\mathcal{P} = \{pregnant, not\,pregnant, R\}$, where R means removed from the population by death or emigration. Each individual is labelled by his time $\mathcal{T}_j$ of birth and the number u of the individual among the set of individuals born at this time. Then $(\mathcal{T}_j(\omega), label(\omega))$ is equivalently represented by $(\mathcal{T}_j(\omega), l)$, where $l = (j, u)$, and we denote ω_l such a labelled individual. Then when ω_l gives birth at time $\mathcal{T}_n$ to $\widetilde{Y}_{n,l}$ newborns, ω_l undergoes the transition denoted $i_l^p \to j_l^B$, where $i_l^p = pregnant$ and $j_l^B = \{j_l^p, \{j_{l'}^p\}_{l' \in \mathcal{L}(\widetilde{Y}_{n,l})}\}$, where $j_l^p = (not\,pregnant)$, $j_{l'}^p = (not\,pregnant)$, and $\mathcal{L}(\widetilde{Y}_{n,l})$ is the set of labels of the newborns of ω_l ($\widetilde{Y}_{n,l}$ represents both the set of newborns of ω_l at time $\mathcal{T}_n$ and the size of this set).

Denote also Ω the population understood as an abstract entity, and $\Omega_n(\Omega)$, the composition of the population at time $\mathcal{T}_n$. Contrary to a closed population, this composition changes with time. At each time $\mathcal{T}_n$, each variable $Z_{\mathcal{T}_n}(\Omega)$ that we will define is in fact function of $\Omega_n(\Omega)$, that is there exists some variable $\widetilde{Z}$ such that $Z_{\mathcal{T}_n}(\Omega) = \widetilde{Z}_{\mathcal{T}_n}(\Omega_{\mathcal{T}_n}(\Omega))$, but for simplifying the notations we only write $Z_{\mathcal{T}_n}(\Omega)$. Denote also $\{\mathcal{X}_t(\Omega)\}$ the individual-based branching process; $\mathcal{X}_t(\Omega)$ takes values in $\mathcal{X} = \{\{(P_l)\}_{l\in\mathcal{S}(\mathcal{L})}\}_{\mathcal{S}(\mathcal{L})}$, where $P_l \in \mathcal{P}$, for any l, $\mathcal{S}(\mathcal{L})$ is any finite set of labels, and $\mathcal{X}_t(\Omega)$ is defined by

$$\mathcal{X}_t(\Omega) \stackrel{def.}{=} \mathcal{X}_{n_t}(\Omega) \tag{44}$$

$$n_t(\Omega) \stackrel{def.}{=} \sum_{\omega_l\in\Omega_{n_t-1}(\Omega)} m_{p,l,t}(\omega_l) \tag{45}$$

$$m_{p,l,t}(\omega_l) \stackrel{def.}{=} \sup\{m : T_m^{(p,l)}(\omega_l) \le t\},\ \omega_l \in \Omega_{n_t-1}(\Omega) \tag{46}$$

$$\mathcal{X}_{n_t}(\Omega) \stackrel{def.}{=} \{X_{m_{p,l,t}}^{(B,l)}(\omega_l)\}_{\omega_l\in\Omega_{n_t-1}(\Omega)} \tag{47}$$

$$X_{m_{p,l,t}}^{(B,l)}(\omega_l) \stackrel{def.}{=} \{X_{m_{p,l,t}}^{(p,l)}(\omega_l) \ne R, \{X_0^{(p,l')}(\omega_{l'}) \ne R\}_{\omega_{l'}\in\widetilde{Y}_{n_t,l}}\} \tag{48}$$

$$T_0^{(p,l')}(\omega_{l'}) \stackrel{def.}{=} T_{m_{p,l,t}}^{(p,l)}(\omega_l),\ \omega_{l'} \in \widetilde{Y}_{n_t,l},\ \omega_l \in \Omega_{n_t-1}(\Omega) \tag{49}$$

$$\mathcal{T}_{n_t}(\Omega) \stackrel{def.}{=} \sup_{\omega_l\in\Omega_{n_t-1}(\Omega)} \{T_{m_{p,l,t}}^{(p,l)}(\omega_l)\}. \tag{50}$$

Of course $\Omega_{n_t}(\Omega)$ which is the set of individuals alive at $\mathcal{T}_{n_t}(\Omega)$ is not constant since an individual born at $T_0^{(p,l)}(\omega_l)$ exists only from his birth date $T_0^{(p,l)}(\omega_l)$ until his removing time. Assuming that the set of individual transitions defined by $\{X_m^{(p,l)}, T_m^{(p,l)}\}_{m,l}$ satisfies $(A1)$ to $(A3)$ and that $(A4)$ is checked, then the kernel of the process is given by (22), where $F_{i_l|I,j_l}^{(l)}(.)$ is replaced by $F_{i_l^p|I,j_l^B}^{(B,l)}(.)$ and $P^{(l)}(i_l|I,j_l)$ is replaced by $P^{(B,l)}(i_l^p|I,j_l^B)$. When $i_l^p \to j_l^B$ concerns the transition from *pregnant* to *not pregnant*, then $F_{i_l^p|I,j_l^B}^{(B,l)}(.) = F_{pregnant,not\,pregnant}^{(B,l)}(.)$ (cdf of the pregnancy period), and $P^{(B,l)}(i_l^p|I,j_l^B)$ is the probability for ω_l to give birth to $\widetilde{Y}_{n,l}$ newborns at his next "jump" among the states $\{alive\, with\, \widetilde{Y}\, newborns\}_{\widetilde{Y}}, R\}$. When the transition is *pregnant* $\to R$, then $F_{i_l^p|I,j_l^B}^{(B,l)}(.) = F_{pregnant,R}^{(B,l)}(.)$ (cdf of the time in the pregnancy state before dying or emigrating), and $P^{(B,l)}(i_l^p|I,j_l^B)$ is the probability for ω_l in pregnancy to be removed at his next "jump" among the possibilities $\{alive\,with\,\widetilde{Y}\,newborns\}_{\widetilde{Y}}, R\}$, and in a similar way for the other possible transitions, when $i_l^p \neq pregnant$. Then $\{\mathcal{X}_t(\Omega)\}$ is a semi-semi-Markovian branching process.

4.2. Spread of a disease in a branching population structured in groups

We consider the case of a population of marked individuals undergoing three random characteristics, the previous branching one B, the health state H, and the membership group G. An example of this process is given in ([17]) where the authors study the spread of the bovine viral diarrhoea within a dairy herd. The R state is considered here as an exit group called R meaning that the animal is removed from its current group. The process is defined in the same way as the previous simple branching process with (47), (48), (49) replaced by (when forgetting the notations (ω_l) and $(\omega_{l'})$):

$$\mathcal{X}_{n_t}(\Omega) \overset{def.}{=} \{(X^{(B,l)}_{m_{p,l,t}}, X^{(h,l)}_{m_{h,l,t}}, X^{(g,l)}_{m_{g,l,t}})1_{\{X^{(g,l)}_{m_{g,l,t}} \neq R\}}\}_{\omega_l \in \Omega_{n_t-1}(\Omega)}$$

$$X^{(B,l)}_{m_{p,l,t}} \overset{def.}{=} \{X^{(p,l)}_{m_{p,l,t}}, \{(X_0^{(p,l')}, X_0^{(h,l')}, X_0^{(g,l')})1_{\{X_0^{(g,l')} \neq R\}}\}_{\omega_{l'} \in \widetilde{Y}_{n_t,l}}\},$$

$$\omega_l \in \Omega_{n_t-1}(\Omega)$$

$$T_0^{(c,l')} \overset{def.}{=} T^{(p,l)}_{m_{p,l,t}},\ \omega_{l'} \in \widetilde{Y}_{nt,l},\ \omega_l \in \Omega_{n_t-1}(\Omega),\ c \in \{p,h,g\}$$

where, when $\widetilde{Y}_{n_t,l} = 0$, the transition is defined in the same way as in a closed population, and when there exists $\omega_l \in \Omega_{n_t-1}(\Omega)$ such that $\widetilde{Y}_{n_t,l} \neq 0$, generalizing the definition of the previous simple branching SSMP, we must define the state of the newborns for each of the three characteristics and not only for the physiological state.

Assuming that $(A1)$ to $(A4)$ are checked, then the kernel of the process is given by (22), where as previously, when $c = B$, $F^{(c,l)}_{i_l|I,j_l}(.)$ and $P^{(c,l)}(i_l|I,j_l)$ are replaced by $F^{(B,l)}_{i_l^p|I,j_l^B}(.)$ and $P^{(B,l)}(i_l^p|I,j_l^B)$ in which $j_l^B = \{j_l^p, \{j_{l'}^p, j_{l'}^h, j_{l'}^g\}_{l' \in \widetilde{Y}_{n,l}}\}$. When $c = g$, $F^{(c,l)}_{i_l|I,j_l}(.)$ and $P^{(c,l)}(i_l|I,j_l)$ concern the transitions between groups that may depend on the initial physiological status of the individual l, and when $c = h$, they concern the health state changes that may depend on the number of infectives in each group and on the possibility of transmission of these infectives to ω_l.

The memory $s_n^{(c,l)}$ of the sojourn time of the characteristic (c, l) in the current state i_l until t_n is the sojourn time of the individual, already spent in his current physiological status (for example *pregnant*) until t_n, when $c = p$. It is the sojourn time already spent by l in his current group c until t_n, when $c = g$, and it is the sojourn time already spent by l in his current health state c, when $c = h$.

4.3. Branching bisexual (two-sex) population

Bisexual processes which concern sexual reproduction take into account the mating process in the evolution of the population. They have been introduced in discrete time by [5] and in continuous time by [2] and [16]. In these models, the pregnancy period is not taken into account. We may build as previously an individual-based generalization of these processes, allowing to take into account waiting times such as the waiting time before forming a couple and the pregnancy period. A state of the process may be described by the set of characteristics of each individual and the set of characteristics of each potential couple. A transition may concern an individual (mortality) or a couple of individuals (formation of a couple, mortality of a couple, pregnancy of a couple).

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