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(G,λ)-EXTREMAL PROCESSES AND THEIR RELATIONSHIP WITH MAX-STABLE PROCESSES

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The study of G-extremal processes was initiated by S. Resnick and M. Rubinovich (1973). Here we transform these processes by a non-decreasing and right-continuous function $\lambda : [0,\infty) \to [0,\infty)$ and investigate relationship between (G,λ)-extremal processes and max-stable processes. We prove that for the processes with independent max-increments if one of the following three statements is given, the other two are equivalent:

a) **Y** is a max-stable process;

b) **Y** is a (G,λ)-extremal process;

c) **Y** is a self-similar extremal process.

1. Introduction

In 1973 S. Resnick and M. Rubinovich [9] gave the following definition.

Definition 1.1. *Let G be a non-degenerate distribution function (d.f.) on $[0,\infty)$. A random process $Y : [0,\infty) \to [0,\infty)$ with almost sure (a.s.) non-decreasing and right-continuous sample paths, whose finite dimensional distributions (f.d.d's) satisfy equality*

(1) $\mathbb{P}(Y(t_1)<x_1,\ldots,Y(t_k)<x_k) G^{t_1}(\min\{x_1,\ldots,x_k\}) G^{t_2-t_1}(\min\{x_2,\ldots,x_k\})\ldots$

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$$\ldots G^{t_k - t_{k-1}}(x_k), \quad k \in N, \quad 0 \le t_1 < \cdots < t_k, \quad x_1 \in \mathbb{R}, \ldots, x_k \in \mathbb{R}$$

is called a G-**extremal process.**

As is known every distribution function (d.f.) on $\mathbb{R}$ is max-infinitely divisible and this means that for every d.f. G on $\mathbb{R}$ and for all $t > 0$, $G^*(x) = G^t(x)$ is again a d.f. On $\mathbb{R}^d$ this is not longer the case. So, if G is a d.f. on $\mathbb{R}^d$ this does not imply that for all $t > 0$, $G^*(\mathbf{x}) = G^t(\mathbf{x})$ is again a d.f.

In her paper [6], Section 3 E. Pancheva discuses G-extremal processes

$$\mathbf{Y} : [0, \infty) \to [0, \infty)^d.$$

She supposes that G is a max-infinitely divisible d.f. on $[0, \infty)^d$. Here we will follow her definition. We transform these processes by a non-decreasing and right-continuous function $\lambda : [0, \infty) \to [0, \infty)$ and investigate relationship between (G, λ)-extremal processes and max-stable processes. It turns out that provided the processes are self-similar these two concepts are equivalent.

The paper is organized as follows: in §2 we remind the concept of max-stable processes, in §3 we define (G, λ)-extremal processes and give some of their properties. Main results are proved in §4.

Let $D_{[0,\infty)}$ be a set of non-decreasing, right-continuous, piecewise constant functions from $[0, \infty)$ to $[0, \infty)^d$. If we endow $D_{[0,\infty)}$ with Skorohod's J_1-topology and consider the weak convergence of extremal processes of this function space we will denote this convergence by $\stackrel{J_1}{\Longrightarrow}$, elsewhere if we consider the weak convergence of sequences of extremal processes as random elements in the functional space $D_{[0,\infty)}$ endowed with weak topology we will denote this convergence by $\Longrightarrow$ (see e.g. Billingsley [2]).

The notations $\stackrel{\text{f.d.d.}}{=}$ and $\stackrel{\text{d.}}{=}$ are set correspondingly for equality of all f.d.d's of the processes and equality in distribution. We use boldface notation for vectors

$$\mathbf{x} = (x^{(1)}, \ldots, x^{(d)})$$

and all equalities, inequalities and maxima between vectors we understand componentwise.

Suppose H is a non-decreasing function. Along the paper

$$H^{\leftarrow}(y) = \inf\{s : H(s) > y\},$$

means the right-continuous inverse of H and $G(\mathbf{x}) = \mathbb{P}(\mathbf{Y}(1) < \mathbf{x})$.

2. Max-stable extremal processes

In 1984 L. de Haan [4] obtains a spectral representation for max-stable processes which are continuous in probability. In 1990 E. Gine, M.G.Hahn and P. Vatan [5] treat almost surely continuous max-stable and max-infinitely divisible processes with respect to (w.r.t.) affine normalizations. The max-infinitely divisible extremal processes with time intersections on $\mathbb{R}^d$ were introduced in 1996 by A.Balkema and E. Pancheva [1]. In 2000 E.Pancheva [6] defines max-stable processes w.r.t. arbitrary continuous one-parameter group (c.o.g.).

Let $\mathbf{Y} : [0, \infty) \to [0, \infty)^d$ be a non-degenerate extremal process and let

$$\mathcal{L} = \{\mathbf{L}_s : s > 0\}$$

be a c.o.g. with respect to the composition such that for all $s > 0$ $\mathbf{L}_s \in \mathcal{L}$ and

$$L_s^{(i)}(x^{(i)}) : \text{Supp } Y^{(i)}(1) \to \text{Supp } Y^{(i)}(1)$$

are strictly increasing and continuous in s and $x^{(i)}, i = 1, \dots, d$ functions.

Definition 2.1. *An extremal process* $\mathbf{Y} : [0, \infty) \to [0, \infty)^d$ *is called* **a max-stable w.r.t.** $\mathcal{L}$ *if for all* $n \in \mathbf{N}$

$$\mathbf{L}_n(\mathbf{Y}) \overset{\text{f.d.d.}}{=} \mathbf{Y}_1 \vee \dots \vee \mathbf{Y}_n,$$

where $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ *are independent copies of* $\mathbf{Y}$*, Briefly* $\mathbf{Y} \in MS(\mathcal{L})$*.*

As is known, (see [1]) the f.d.d's of an extremal process (with independent max-increments) are completely determined by its df. So, if $\mathbf{Y}$ is an extremal process a necessary and sufficient condition for $\mathbf{Y}$ to be max-stable is all of its moments intersections $\mathbf{Y}(t)$ to be max-stable random vectors, i.e. for all $t > 0$, $\mathbf{x} \in \mathbf{R}^d$

$$\mathbb{P}(\mathbf{L}_s(\mathbf{Y}(t)) < \mathbf{x}) = \mathbb{P}^s(\mathbf{Y}(t) < \mathbf{x}), \forall s > 0.$$

In this sense the next two properties follow immediately by the properties of the max-stable random vectors (see e.g. S.Resnick [8]).

Property 2.1. *If* $\mathbf{X} : [0, \infty) \to [0, \infty)^d$ *and* $\mathbf{Y} : [0, \infty) \to [0, \infty)^d$ *are independent max-stable extremal processes with respect to* $\mathcal{L}$*, then* $\mathbf{X} \vee \mathbf{Y} \in MS(\mathcal{L})$*.*

The class of max-stable random processes is invariant with respect to continuous and strictly monotone space-changes.

Property 2.2. *If* $\mathbf{Y} : [0, \infty) \to [0, \infty)^d$ *is a max-stable extremal processes with respect to* $\mathcal{L}$ *and for all* $i = 1, \dots, d$*,* $U^{(i)}(x^{(i)}) : \text{Supp } Y^{(i)}(1) \to \text{Supp } Y^{(i)}(1)$ *are continuous and strictly increasing functions, then* $\mathbf{U} \circ \mathbf{Y} \in MS(\widetilde{\mathcal{L}})$*, where* $\widetilde{\mathcal{L}} = \{\mathbf{U} \circ \mathbf{L}_s \circ \mathbf{U}^{\leftarrow} : s > 0\}$*.*

The notion lower curve of an extremal process was introduced in 1996 by A.Balkema and E. Pancheva [1].

Definition 2.2. *Let* $\mathbf{Y} : [0,\infty) \to [0,\infty)^d$ *be an extremal process. The right-continuous and non-decreasing function* $\mathbf{C} : [0,\infty) \to [0,\infty)^d$ *is called* **a lower curve of Y** *if for all* $t > 0$ $\mathbf{C}(t)$ *is the maximal vector such that*

$$\mathbb{P}(Y^{(i)}(t) \geq C^{(i)}(t)) = 1, \quad \forall i = 1,\ldots,d.$$

It is uniquely determined by the process.

Property 2.3. *If* $\mathbf{Y} : [0,\infty) \to [0,\infty)^d$ *is a max-stable extremal process then its lower curve* $\mathbf{C}$ *is a constant.*

Proof. Assume that there exist $0 < t < t_1$ such that $\mathbf{C}(t) < \mathbf{C}(t_1)$.

By the properties of max-stable random vectors is known that if $\mathbf{Y}(t) \in MS(\mathcal{L})$, then for all $i = 1,\ldots,d$

$$L_n^{\leftarrow(i)}(x^{(i)}) = F^{\leftarrow}_{Y^{(i)}(t_1)}(F^n_{Y^{(i)}(t_1)}(x^{(i)})) = F^{\leftarrow}_{Y^{(i)}(t)}(F^n_{Y^{(i)}(t)}(x^{(i)})), \tag{2}$$

where $F_{Y^{(i)}(t_1)}(x^{(i)}) = \mathbb{P}(Y^{(i)}(t_1) < x^{(i)})$ is a continuous function of $x^{(i)}$.

By the definition of lower curve, max-stability of $\mathbf{Y}$ and (2)

$$0 = F^n_{Y^{(i)}(t)}(C^{(i)}(t))) = \mathbb{P}(L_n^{(i)}(Y^{(i)}(t)) < C^{(i)}(t)) =$$

$$= F_{Y^{(i)}(t)}(F^{\leftarrow}_{Y^{(i)}(t_1)}(F^n_{Y^{(i)}(t_1)}(C^{(i)}(t)))) = F_{Y^{(i)}(t)}(C^{(i)}(t_1)) > 0.$$

So, we come to contradiction. □

Proposition 2.4. *The max-stability of the extremal process* $\mathbf{Y} : [0,\infty) \to [0,\infty)^d$ *does not imply neither its stochastic continuity, nor homogeneity of its max-increments.*

To prove this property we consider the following counterexample.

Counterexample: Let $\alpha > 0$. We define an extremal process Y by its d.f.

$$\mathbb{P}(Y(t) < x) = \begin{cases} 0 & x \leq 0 \\ e^{-[t]x^{-\alpha}} & x > 0 \end{cases} \qquad t \geq 0,$$

where $[t]$ means the integer part of t.

This process obviously has no homogeneous max-increments.

It is max-stable because all its time-intersections are max-stable with respect to the c.o.g. $\mathcal{L} = \{L_s : s > 0\}$, where $L_s(x) = s^{1/\alpha}x$.

Y is not stochastically continuous because its d.f. has discontinuities in every $t \in \{1, 2, \ldots\}$. For example in $t2$, $\varepsilon > 0$ and $t_n \nearrow 2, n \to \infty$.

$$\lim_{n\to\infty} \mathbb{P}(\mathbf{Y}(2) - \mathbf{Y}(t_n) > \varepsilon)\mathbb{P}(\mathbf{Y}(2) - \mathbf{Y}(1) > \varepsilon) =$$

$$= 1 - \int_0^\infty \mathbb{P}(U_Y(1,2] < \varepsilon + y) d\mathbb{P}(Y(1) < y) = 1 - \int_0^\infty e^{-(\varepsilon+y)^{-\alpha}} de^{-y^{-\alpha}} > 0.$$

Here we denote by $U_Y(1,2]$ the max-inctrements of the process over the time interval $(1,2]$.

The last statement is very natural because the max-stability is a property of the state components, whereas stochastic continuity is a property of the time-component.

Property 2.5. *If* $\mathbf{Y} : [0,\infty) \to [0,\infty)^d$ *is a max-stable extremal processes with respect to* $\mathcal{L}$ *and* $\lambda : [0,\infty) \to [0,\infty)$ *is a non-decreasing and right-continuous function, then* $\mathbf{Y} \circ \lambda \in MS(\mathcal{L})$.

Proof. Let $\mathbf{Y}_1(t), \ldots, \mathbf{Y}_n(t)$ be independent copies of $\mathbf{Y}(t)$, $t > 0$ and $\mathbf{x} > \mathbf{0}$. By the properties of maxima and by max-stability of $\mathbf{Y}$

$$\mathbb{P}(\mathbf{L}_n^{\leftarrow}(\mathbf{Y}_1(\lambda(t)) \vee \cdots \vee \mathbf{Y}_n(\lambda(t))) < \mathbf{x}) =$$

$$\mathbb{P}(\mathbf{L}_n^{\leftarrow}(\mathbf{Y}_1(\lambda(t))) < \mathbf{x}, \ldots, \mathbf{L}_n^{\leftarrow}(\mathbf{Y}_n(\lambda(t))) < \mathbf{x}) =$$

$$= \mathbb{P}^n(\mathbf{L}_n^{\leftarrow}(\mathbf{Y}(\lambda(t))) < \mathbf{x})\mathbb{P}(\mathbf{Y}(\lambda(t)) < \mathbf{x}).$$

□

Property 2.6. *Let* $\mathbf{Y} : [0,\infty) \to [0,\infty)^d$ *be an extremal process with homogeneous max-increments and* $\mathbf{Y}(1) \in MS(\mathcal{L})$*, then the process* $\mathbf{Y} \in MS(\mathcal{L})$.

Proof. Because the max-increments of $\mathbf{Y}$ are independent and homogeneous it is sufficiently to prove max-stability only for its distribution functions.

Let $n \in N$, $t > 0$, $\mathbf{x} > \mathbf{0}$ and $\mathbf{Y}_1(t), \ldots, \mathbf{Y}_n(t)$ be independent copies of $\mathbf{Y}(t)$. By the properties of maxima, homogeneity of max-increments and max-stability of $\mathbf{Y}(1)$ we obtain

$$\mathbb{P}(\mathbf{L}_n(\mathbf{Y}_1(t) \vee \cdots \vee \mathbf{Y}_n(t)) < \mathbf{x}) = \mathbb{P}^n(\mathbf{L}_n(\mathbf{Y}(t)) < \mathbf{x}) =$$

$$= \mathbb{P}^{nt}(\mathbf{L}_n(\mathbf{Y}(1)) < \mathbf{x}) = \mathbb{P}^t(\mathbf{Y}(1) < \mathbf{x}) = \mathbb{P}(\mathbf{Y}(t) < \mathbf{x}).$$

□

3. (G, λ)-extremal processes

In this section we will generalize the G-extremal processes.

Let G be a non-degenerate, max-infinitely divisible d.f. We suppose that $\lambda : [0, \infty) \to [0, \infty)$ is a non-decreasing and right-continuous, nonconstant function that is defined and finite for all finite $t \geq 0$ and $\lambda(1) = 1$. Without lost of generality we suppose that $\operatorname{Supp} G \subseteq [0, \infty)^d$, $\inf\{s : G^{(i)}(s) > 0\} = 0$ for all $i = 1, \ldots, d$ and $\lambda(0) = 0$.

Definition 3.1. *A random process* $\mathbf{Z} : [0, \infty) \to [0, \infty)^d$ *with a.s. non-decreasing and right-continuous sample paths, which satisfies condition*

$$\mathbf{Z} \overset{f.d.d.}{=} \mathbf{Y} \circ \lambda, \tag{3}$$

where $\mathbf{Y}$ *is an* G*-extremal process is called* **a** (G, λ)**-extremal process.**

Having in mind (1) and comments after it we can see that condition (3) is equivalent to the condition

$$\mathbb{P}(\mathbf{Z}(t_1){<}\mathbf{x}_1, \ldots, \mathbf{Z}(t_k){<}\mathbf{x}_k) G^{\lambda(t_1)}(\min\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}) G^{\lambda(t_2)-\lambda(t_1)}(\min\{\mathbf{x}_2, \ldots, \mathbf{x}_k\}) \ldots$$

$$\ldots G^{\lambda(t_k)-\lambda(t_{k-1})}(\mathbf{x}_k), \quad k \in N, \quad 0 \leq t_1 < \cdots < t_k, \quad \mathbf{x}_1 \in \mathbb{R}^d, \ldots, \mathbf{x}_k \in \mathbb{R}^d.$$

Restriction $\lambda(1) = 1$ is imposed in order to achieve uniqueness in the relationship between the couple $(\mathbf{Y}, \lambda)$ and the process $\mathbf{Z}$. Otherwise we would have that for all constants $c > 0$ $\mathbf{Z}$ is (G, λ)-extremal process if and only if it is $(G^{1/c}, c.\lambda)$-extremal process.

In contrast to max-stable extremal processes (G, λ)-extremal processes have not necessary max-stable time-intersections.

Further in this section we suppose that $\mathbf{Z} : [0, \infty) \to [0, \infty)^d$ is a (G, λ)-extremal process.

Property 3.1. *If* $G \in MS(\mathcal{L})$ *then* $\mathbf{Z} \in MS(\mathcal{L})$.

Proof. Let $t > 0$ and $\mathbf{x} > \mathbf{0}$.

$$\mathbb{P}(\mathbf{Z}(t) < \mathbf{L}_s^{\leftarrow}(\mathbf{x})) = G^{\lambda(t)}(\mathbf{L}_s^{\leftarrow}(\mathbf{x})) G^{s\lambda(t)}(\mathbf{x}) = \mathbb{P}^s(\mathbf{Z}(t) < \mathbf{x}).$$

□

It if not difficult to prove the following properties.

Property 3.2. *If* $\widetilde{\lambda} : [0, \infty) \to [0, \infty)$ *is a non-decreasing and right-continuous function, that is defined and finite for all finite* $t > 0$, $\widetilde{\lambda}(1) = 1$ *and* $\widetilde{\lambda}(0) = 0$, *then* $\mathbf{Z} \circ \widetilde{\lambda}$ *is a* $(G, \lambda \circ \widetilde{\lambda})$*-extremal process.*

Property 3.3. $\operatorname{Supp} \mathbf{Z}(1) \equiv \operatorname{Supp} \mathbf{Z}(t)$ *and the lower curve of* $\mathbf{Z}$ *is a constant.*

Property 3.4. *If* $\mathbf{U}(\mathbf{x}) : \operatorname{Supp} \mathbf{Z}(1) \to \operatorname{Supp} \mathbf{Z}(1)$ *is a strictly increasing and continuous in each coordinate componentwise function, then* $\mathbf{U} \circ \mathbf{Z}$ *is a* $(G \circ \mathbf{U}^{\leftarrow}, \lambda)$*-extremal process. Particularly:*

1.) If the d.f. G is continuous and strictly increasing in each coordinate and $\mathbf{G}(\mathbf{x}) = (G^{(1)}(\mathbf{x}^{(1)}), \ldots, G^{(1)}(\mathbf{x}^{(1)}))$*, then* $\mathbf{G} \circ \mathbf{Z} \in (U_0, \lambda)$*-extremal process, where* U_0 *is a d.f. of a vector with uniformly distributed coordinates and the same dependence function as the dependance function of* G*.*

2.) If $G \equiv U_0$*, where* U_0 *is a d.f. of a vector with uniformly distributed coordinates and if* $F^{(i)}, i = 1, \ldots, d$ *are d.fs., concentrated on* $[0, \infty)$*, then* $\mathbf{F}^{\leftarrow} \circ \mathbf{Z} \in (F, \lambda)$*-extremal process, where* F *is a d.f. with the same dependence function as the dependence function of* U_0 *and with marginals* $F^{(1)}, \ldots, F^{(d)}$*.*

Property 3.5. *Let* $G_1, G_2, \ldots, G_n, \ldots$ *be max-infinitely divisible distribution functions,* $\lambda_n : [0, \infty) \to [0, \infty)$ *be non-decreasing and right-continuous functions that are finite for all finite* t*,* $\lambda_n(0) = 0$ *and* $\lambda_n(1) = 1$*. We suppose also that*

1.) $\mathbf{Y}_n : [0, \infty) \to [0, \infty)^d$ *are* (G_n, λ_n)*-extremal processes,*

2.) $G_n \stackrel{w}{\to} G, n \to \infty$*, where* G *is a non-degenerate d.f. and*

3.) $\lambda_n \stackrel{w}{\to} \lambda, n \to \infty$*, where* λ *is finite for all finite* t*.*

Then $\mathbf{Y}_n \Longrightarrow \mathbf{Y}$*, where* $\mathbf{Y}$ *is a* (G, λ)*-extremal process.*

If we additionally suppose that λ *is a continuous function then* $\mathbf{Y}_n \stackrel{J_1}{\Longrightarrow} \mathbf{Y}$*.*

Let us remind the fact that the class of max-infinitely divisible random vectors is closed w.r.t. weak limit. To prove the first part of the last property it is sufficient to observe that the convergence of sequences of d.f's of extremal processes on dense set implies the weak convergence of the sequences of processes (see A. Balkema and E. Pancheva [1]). The second part of this property follows by convergence of f.d.d's, stochastically continuity of the limiting process and monotonicity of its sample paths, see Theorem 3 in [3].

4. Relationship between (G, λ) and max-stable extremal processes

In order to give relationship between (G, λ) and max-stable extremal processes we will involve self-similar processes.

Let $\sigma_s(t)$ and $L_s^{(i)}, i = 1, \ldots, d$ be continuous and strictly increasing functions, $\sigma_s(t) \to 0, t \to 0$, $L_s^{(i)}(t) \to 0, t \to 0$, $\sigma_s(t) \to \infty, t \to \infty$ and $L_s^{(i)}(t) \to \infty, t \to \infty$.

Definition 4.1. *We call the extremal process* $\mathbf{Y} : [0, \infty) \to [0, \infty)^d$ *a* **self-similar** *w.r.t. continuous one-parameter group* $\Gamma = \{(\sigma_s, \mathbf{L}_s) : s > 0\}$ *(briefly* $\mathbf{Y} \in SS(\Gamma)$*) if for all* $t > 0$ *its d.f.* $g(t, \mathbf{x})$ *satisfies the equality*

$$g(t, \mathbf{x}) = g(\sigma_s(t), \mathbf{L}_s^{\leftarrow}(\mathbf{x})) \quad \forall s > 0, \quad \mathbf{x} > \mathbf{0}.$$

This is equivalent to $\mathbf{L}_s(\mathbf{Y}(t)) \stackrel{d.}{=} \mathbf{Y}(\sigma_s(t))$, for all $s > 0$ and $t > 0$.

Particularly, if $\Gamma = \{(st, s^{\alpha}\mathbf{x}) : s > 0\}$, then the extremal process is called **self-similar with exponent** α.

The class of self-similar processes is very extensive. It includes Brownian motion (with $\alpha = 1/2$), α-stable Levy motion (with exponent $1/\alpha$), max-stable extremal process with homogeneous max-increments and Frechet d.f. (with exponent $-1/\alpha$) and so on. Many properties of such processes are investigated in [6] and [10]. In her paper [6] E.Pancheva shows that

1.) the class of self-similar extremal processes with homogeneous max-increments coincides with the intersection of the class of G-extremal processes and the class of MS-processes and

2.) an extremal process $\mathbf{Y}$ with homogeneous max-increments is self-similar if and only if the random vector $\mathbf{Y}(1)$ is max-stable.

The next theorem is a slight generalizations of her results. It clears question about relationship between max-stable and (G, λ)-extremal processes. To prove it we obtain the following lemma.

Lemma 4.1. *Let $\mathbf{Z}$ be an extremal process and $\lambda : [0, \infty) \to [0, \infty)$ be a continuous and strictly increasing function, such that $\lambda(1) = 1$ and $\lambda(0) = 0$. If one of the following three statements is given, the other two are equivalent.*

a) $\mathbf{Z}(1)$ *is MS($\mathcal{L}$), where $\mathcal{L}\{\mathbf{L}_s(\mathbf{x}) : s > 0\}$ is a c.o.g. with respect to the composition;*

b) $\mathbb{P}(\mathbf{Z}(t) < \mathbf{x}) = \mathbb{P}^{\lambda(t)}(\mathbf{Z}(1) < \mathbf{x})$, $t > 0$, $\mathbf{x} \in R^d$;

c) $\mathbf{Z} \in SS(\mathcal{H})$, *where $\mathcal{H}\{(\lambda^{\leftarrow}(\lambda(t)s), \mathbf{L}_s(\mathbf{x})) : s > 0\}$, i.e. for all $s > 0$*

$$\mathbb{P}(\mathbf{Z}(\lambda^{\leftarrow}(\lambda(t)s)) < \mathbf{x})\mathbb{P}(\mathbf{Z}(t) < \mathbf{L}_s^{\leftarrow}(\mathbf{x})).$$

P r o o f. First we prove that a) and b) imply c). We need to show that $\mathcal{H}$ is a c.o. group. For $s > 0, t > 0$ and $y > 0$

$$\sigma_s \circ \sigma_y(t) = \lambda^{\leftarrow}(\lambda(\lambda^{\leftarrow}(\lambda(t)y))s) = \lambda^{\leftarrow}(\lambda(t)ys)\sigma_{sy}(t)$$

$$\sigma_s^{\leftarrow}(\cdot) = (\lambda^{\leftarrow}(\lambda(\cdot)s))^{\leftarrow}\sigma_{s^{-1}}(\cdot).$$

We apply consecutively b), a), b) and obtain

$$\mathbb{P}(\mathbf{Z}(\lambda^{\leftarrow}(\lambda(t)s)) < \mathbf{x})\mathbb{P}^{\lambda(t)s}(\mathbf{Z}(1) < \mathbf{x})\mathbb{P}^{\lambda(t)}(\mathbf{Z}(1) < \mathbf{L}_s^{\leftarrow}(\mathbf{x})) = \mathbb{P}(\mathbf{Z}(t) < \mathbf{L}_s^{\leftarrow}(\mathbf{x})).$$

Now let us go on to the proof of the implication "a) and c) $\Longrightarrow$ b)". By c) for all $s > 0$

$$\mathbb{P}(\mathbf{Z}(\lambda^{\leftarrow}(\lambda(t)s)) < \mathbf{x})\mathbb{P}(\mathbf{Z}(t) < \mathbf{L}_s^{\leftarrow}(\mathbf{x})).$$

For $t = 1$

$$\mathbb{P}(\mathbf{Z}(\lambda^{\leftarrow}(s)) < \mathbf{x})\mathbb{P}(\mathbf{Z}(1) < \mathbf{L}_s^{\leftarrow}(\mathbf{x})).$$

By a) we have that $\mathbb{P}(\mathbf{Z}(1) < \mathbf{L}_s^{\leftarrow}(\mathbf{x}))\mathbb{P}^s(\mathbf{Z}(1) < \mathbf{x})$. So, because of λ is a strictly increasing function we obtain

$$\mathbb{P}(\mathbf{Z}(t) < \mathbf{x})) = \mathbb{P}^{\lambda(t)}(\mathbf{Z}(1) < \mathbf{x}).$$

In the end we have to prove "b) and c) $\Longrightarrow$ a)". Again by c) applied for $t = 1$

$$\mathbb{P}(\mathbf{Z}(\lambda^{\leftarrow}(s)) < \mathbf{x})\mathbb{P}(\mathbf{Z}(1) < \mathbf{L}_s^{\leftarrow}(\mathbf{x})).$$

From the other side by b)

$$\mathbb{P}(\mathbf{Z}(\lambda^{\leftarrow}(s)) < \mathbf{x}) = \mathbb{P}^s(\mathbf{Z}(1) < \mathbf{x}).$$

So $\mathbb{P}(\mathbf{Z}(1) < \mathbf{L}_s^{\leftarrow}(\mathbf{x}))\mathbb{P}^s(\mathbf{Z}(1) < \mathbf{x})$. □

*Notes:*1.) b) means that $\mathbf{Z}$ is a (G,λ)-extremal process, where G is the d.f. of $\mathbf{Z}(1)$.

2.) If $\mathbf{L}_s(\mathbf{x}) = s^{\alpha}.\mathbf{x}$, then all coordinates of all time-intersections of $\mathbf{Z}$ have distribution of Frechet.

3.)$\mathbf{Z} \circ \lambda^{\leftarrow}$ is an extremal process with homogeneous max-increments and is SS($\mathcal{H}^*$), where $\mathcal{H}^* = \{(ts, \mathbf{L}_s(\mathbf{x})) : s > 0\}$.

We use this lemma and Property 3.1 and obtain our next result.

Theorem 4.1. *Let* $\mathbf{Z}$ *be an extremal process, let* G *be a max-infinitely divisible d.f., let* $\lambda : [0,\infty) \to [0,\infty)$ *be a strictly increasing and continuous function, such that* $\lambda(0) = 0$ *and* $\lambda(1) = 1$. *Let* $\mathcal{L}\{L_s(\mathbf{x})) : s > 0\}$ *be a c.o.g. with respect to the composition.*

If one of the following three statements is given, the other two are equivalent:

a) $\mathbf{Z} \in MS(\mathcal{L})$ *with d.f.* G *of* $\mathbf{Z}(1)$*;*

b) $\mathbf{Z}$ *is a* (G,λ)*-extremal process;*

c) $\mathbf{Z} \in SS(\mathcal{H})$, *where* $\mathcal{H}\{(\lambda^{\leftarrow}(\lambda(t)s), \mathbf{L}_s(\mathbf{x})) : s > 0\}$.

We summarize our results on the Chart 4.1. The parallelogram CDEF presents the set of extremal processes that satisfy condition $\mathbf{Z}(1) \in MS(\mathcal{L})$. AEFD is for the set of (G,λ) - extremal processes. The set BFDE presents $SS(\mathcal{H})$ extremal processes and the pentagon GHDEF presents $MS(\mathcal{L})$ extremal processes.

It is interesting to know how the intersection of the discussed sets of processes appears in a weak limit.

In the next theorem we discuss a sequence of continuous and strictly increasing componentwise functions $\{\sigma_n(t,\mathbf{x})\} = \{(\tau_n(t), \mathbf{u}_n(\mathbf{x}))\}$ that satisfy conditions

$$\begin{aligned} \mathbf{u}_n^{\leftarrow} \circ \mathbf{u}_{[ns]}(\mathbf{x}) &\to \mathbf{U}_s(\mathbf{x}), \\ \tau_n^{\leftarrow} \circ \tau_{[ns]}(t) &\to \sigma_s(t), \quad s > 0. \end{aligned}$$

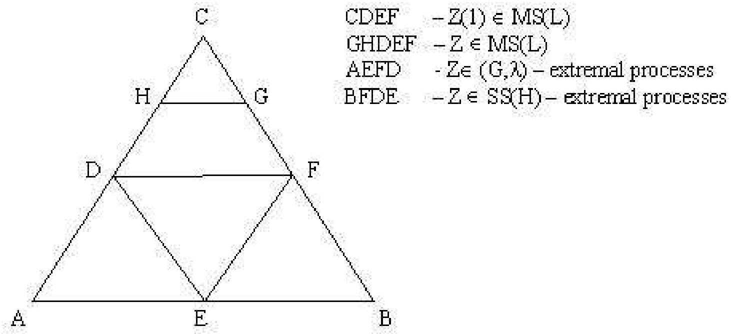

Figure 1: Chart 4.1.

The limiting functions $\mathbf{U}_s(\mathbf{x})$ and $\sigma_s(t)$ are also continuous and strictly increasing in each component. Such sequences are called regular.

Theorem 4.2 *The class of the limiting processes for regularly normalized sequences of maxima*

$$\bigvee_{i=1}^{k(\tau_n(t))} \mathbf{u}_n^{\leftarrow}(\mathbf{X}_i), \tag{4}$$

of independent and identically distributed random vectors $\mathbf{X}_i, i = 1, \dots, n$, where k is a non-random counting function, coincides with the class of processes presented on Chart 4.1 by triangle DEF.

Proof. Let $\mathbf{Y}$ be a random process that appears as a weak limit of a sequence (4). By Proposition 1 of E. Pancheva [6] $\mathbf{Y} \in SS(\mathcal{H})$, where $\mathcal{H}\{(\sigma_{\sqrt[\beta]{s}}(t), \mathbf{U}_{\sqrt[\beta]{s}}(\mathbf{x})) : s > 0\}$. By discussions in [7] the d.f. $\mathbb{P}(\mathbf{Y}(1) < \mathbf{x})$ is max-stable with respect to c.o.g.

$$\mathcal{L} = \{\mathbf{U}_{\sqrt[\beta]{s}}(\mathbf{x}) : s > 0\},$$

where $\beta > 0$ is such that $s^\beta = \lim_{n\to\infty} \frac{k(\tau_{[ns]}(1))}{k(\tau_n(1))}$.

We only have to show that $\lambda^{\leftarrow}(\lambda(t)s)\sigma_{\sqrt[\beta]{s}}(t)$.

For $t > 0$ and $\mathbf{x} \in \{\mathbf{y} \in \mathbb{R} : \mathbb{P}(\mathbf{Y}(1) < \mathbf{y}) \in (0,1)\}$ we have

$$\mathbb{P}(\mathbf{Y}(t) < \mathbf{x}) = \lim_{n\to\infty} F^{k(\tau_n(t))}(\mathbf{u}_n(\mathbf{x})) = \lim_{n\to\infty} (F^{k(\tau_n(1))}(\mathbf{u}_n(\mathbf{x})))^{\frac{k(\tau_n(t))}{k(\tau_n(1))}}.$$

By convergence of the sequence $F^{k_n(1)}(\mathbf{u}_n(\mathbf{x}))$ to $\mathbb{P}(\mathbf{Y}(1) < \mathbf{x}) \in (0,1)$, when $n \to \infty$, there exists a non-decreasing function $\lambda : [0,\infty) \to [0,\infty)$ such that

$$\frac{k(\tau_n(t))}{k(\tau_n(1))} \to \lambda(t), n \to \infty.$$

So we have

$$\mathbb{P}(\mathbf{Y}(t) < \mathbf{x}) = \mathbb{P}^{\lambda(t)}(\mathbf{Y}(1) < \mathbf{x}).$$

By self-similarity of $\mathbf{Y}$ (see [6]) it is stochastically continuous and then λ is a continuous function.

Last equality, self-similarity of $\mathbf{Y}$ and max-stability of $\mathbf{Y}(1)$ entails that for $\mathbf{x} \in \{\mathbf{y} \in \mathbb{R} : \mathbb{P}(\mathbf{Y}(1) < \mathbf{y}) \in (0,1)\}$ an for all $t > 0$ and $s > 0$

$$\mathbb{P}^{\lambda(\sigma_{\sqrt[\beta]{s}}(t))}(\mathbf{Y}(1) < \mathbf{x}) = \mathbb{P}(\mathbf{Y}(\sigma_{\sqrt[\beta]{s}}(t)) < \mathbf{x}) = \mathbb{P}(\mathbf{Y}(t) < \mathbf{U}^{\leftarrow}_{\sqrt[\beta]{s}}(\mathbf{x}))$$

$$= \mathbb{P}^{\lambda(t)}(\mathbf{Y}(1) < \mathbf{U}^{\leftarrow}_{\sqrt[\beta]{s}}(\mathbf{x})) = \mathbb{P}^{\lambda(t)s}(\mathbf{Y}(1) < \mathbf{x}) \in (0,1).$$

Consequently for all $t > 0$ and $s > 0$ $\lambda(\sigma_{\sqrt[\beta]{s}}(t)) = \lambda(t)s$. For $t = 1$ we obtain that $\lambda(\sigma_{\sqrt[\beta]{s}}(1)) = s$, i.e. $\lambda(t)$ is a strictly increasing function and $\lambda^{\leftarrow}(s) = \sigma_{\sqrt[\beta]{s}}(1)$.

Now we suppose that the (G,λ)-extremal process $\mathbf{Y}$ is max-stable with respect to c.o.g. $\mathcal{L} = \{\mathbf{L}_s(t) : s > 0\}$, the random vector $\mathbf{Y}(1)$ has d.f. G and $\mathbf{Y} \in SS(\mathcal{H})$, where $\mathcal{H}\{(\lambda^{\leftarrow}(\lambda(t)s), \mathbf{L}_s(\mathbf{x})) : s > 0\}$ and λ is a continuous and strictly increasing function. Then there exist a counting function $k(t) = [t]$, a regular sequence of time-space changes $(\tau_n(t), \mathbf{u}_n(\mathbf{x})) = (n\lambda(t), \mathbf{L}_n(t))$ and i.i.d. random vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots$ with d.f. G such that

$$\mathbb{P}(\bigvee_{i=1}^{k(\tau_n(t))} \mathbf{u}_n^{\leftarrow}(\mathbf{X}_i) < \mathbf{x})\mathbb{P}(\bigvee_{i=1}^{[n\lambda(t)]} \mathbf{L}_n^{\leftarrow}(\mathbf{X}_i) < \mathbf{x})G^{[n\lambda(t)]}(\mathbf{L}_n(\mathbf{x})) \to G^{\lambda(t)}(\mathbf{x}), \quad n \to \infty.$$

Analogously for all $0 < s < t$

$$\mathbb{P}(\bigvee_{i=k(\tau_n(s))+1}^{k(\tau_n(t))} \mathbf{u}_n^{\leftarrow}(\mathbf{X}_i) < \mathbf{x})\mathbb{P}(\bigvee_{i=[n\lambda(s)]+1}^{[n\lambda(t)]} \mathbf{L}_n^{\leftarrow}(\mathbf{X}_i) < \mathbf{x}) =$$

$$G^{[n\lambda(t)]-[n\lambda(s)]}(\mathbf{L}_n(\mathbf{x})) \to G^{\lambda(t)-\lambda(s)}(\mathbf{x}), \quad n \to \infty.$$

Consequently we have convergence of all f.d.ds. Because these processes have non-decreasing sample paths and the limiting process is stochastically continuous by Proposition 3 in N. Bingham [3]

$$\bigvee_{i=1}^{k(\tau_n(t))} \mathbf{u}_n^{\leftarrow}(\mathbf{X}_i) \overset{J_1}{\Longrightarrow} \mathbf{Y}.$$

□

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