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THE NUMBER OF PARTS OF GIVEN MULTIPLICITY IN A RANDOM INTEGER PARTITION

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Let $X_{m,n}$ denote the number of parts of multiplicity m in a random partition of the positive integer n. We study the asymptotic behaviour of the variance of $X_{m,n}$ as $n \to \infty$ and fixed m.

1. Introduction

Let n be a positive integer. By a *partition* ω of n we mean a representation

$$\omega : \ n = \sum_j j\mu_\omega(j),$$

where $\mu_\omega(j), j = 1, 2, \ldots$ are nonnegative integers. $\mu_\omega(j)$ is called the multiplicity of part j.

The generating function $g(x)$ of the number of all partitions of n, usually denoted by $p(n)$, is determined by Euler(see [1, Chap.1]):

$$g(x) = 1 + \sum_{n=1}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j}. \tag{1}$$

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We shall introduce the uniform probability measure P on the set of all partitions of n, assuming that the probability $1/p(n)$ is assigned to each partition. Thus, each characteristic of the parts can be regarded as random variable.

Let $X_{m,n}(\omega)$ be the number of parts of multiplicity m in a partition ω of the positive integer n, i.e. $X_{m,n}(\omega) = |\{j : \mu_\omega(j) = m\}|$. Corteel, Pittel, Savage, Wilf [2] found the generating function $H(x, y)$ of $X_{m,n}(\omega)$:

$$(2) H(x,y) = \sum_{n,j\geq 0} x^n y^j p(n) P(X_{m,n} = j) = g(x) \prod_{k=1}^{\infty} \left[1 + (y-1)x^{mk}(1-x^k)\right].$$

They use this generating function and found the following identity:

$$\mathbf{E}(X_{m,n}) = \sum_{j\geq 0} \frac{p(n-jm) - p(n-(m+1)j)}{p(n)}.$$

Then, they applied the asymptotic formula [3] for $p(n)$ and showed that for each fixed m the average number of parts of multiplicity m of a partition of n is

$$\mathbf{E}(X_{m,n}) \sim \frac{\sqrt{6n}}{\pi} \frac{1}{m(m+1)} = \mu_n(m), \qquad n \to \infty. \tag{3}$$

Our main goal in this paper is to continue their research determining the asymptotic of the variance of $X_{m,n}$. We apply the saddle point method.

The paper is organized as follows. In section 2 are given three lemmas related to the partition generating function and the number of integer partitions. Section 3 contains the theorem for the variance of $X_{m,n}$ and its proof.

2. Preliminary asymptotic

We need some auxiliary facts related to the asymptotic behaviour of generating function $g(x)$ and the numbers of partitions $p(n)$. The results are given in the next three lemmas.

Lemma 1. [4]. *If r_n satisfy*

$$r_n = 1 - \frac{\pi}{\sqrt{6n}} + \frac{\pi^2}{12n} + O(n^{-3/2}). \tag{4}$$

and

$$b(r) = \frac{\pi}{3(1-r)^3}, \; 0 < r < 1, \tag{5}$$

then, as $n \to \infty$, the partition generating function determined by (2) satisfies

$$g(r_n e^{i\theta})e^{-i\theta n} = g(r_n)\exp\left(\frac{-\theta^2 b(r_n)}{2}\right)\left[1 + O(1/\log n)\right],$$

uniformly for $|\theta| \le \delta_n$, where

(6) $$\delta_n = \frac{n^{-2/3}}{\log n}.$$

Lemma 2. [5]. *If r_n and δ_n satisfy (4) and (6) , respectively, then there exist two constants $c > 0$ and $n(c) > 0$ such that*

$$|g(r_n e^{i\theta})| \le g(r_n)\exp\left(\frac{-cn^{1/6}}{\log^2 n}\right)$$

uniformly for $\delta_n \le |\theta| \le \pi$ and $n \le n(c)$.

We shall also essentially use the asymptotic for of the numbers $p(n)$. It is given by Hardy and Ramanujan's formula [3], however, we need this result in a slightly different form as it is given in [5].

Lemma 3. *We have*

$$p(n) \sim \frac{g(r_n)r_n^{-n}}{\sqrt{2\pi b(r_n)}}$$

as $n \to \infty$, where r_n satisfies the equation

(7) $$\frac{r_n\, g'(r_n)}{g(r_n)} = n$$

for sufficiently large n and $b(r)$ is defined by (5).

3. Variance of $X_{m,n}$

Theorem 1. *Let $X_{m,n}$ is the number of parts of multiplicity m in a random integer partition. Then, for every fixed m,*

$$\mathrm{Var}(X_{m,n}) \sim \frac{\sqrt{6n}}{\pi}\frac{4m+1}{2m(2m+1)(m+1)},$$

as $n \to \infty$.

Proof. First, we point out that when taking logarithms we will consider the main branch of the logarithmic function assuming that $\log z < 0$ for $0 < z < 1$. It is easy to see that (2) yields

$$\frac{d}{dy}\log H(x,y) = \frac{\frac{d}{dy}H(x,y)}{H(x,y)} = \sum_{k=1}^{\infty} \frac{x^{mk}(1-x^k)}{1+(y-1)x^{mk}(1-x^k)}.$$

Then

$$\frac{d}{dy}H(x,y)\bigg|_{y=1} = H(x,1)\sum_{k=1}^{\infty} x^{mk}(1-x^k) = g(x)\sum_{k=1}^{\infty} x^{mk}(1-x^k).$$

Therefore, again by (2)

$$\sum_{n\geq 0} x^n p(n)\mathbf{E}(X_{m,n}) = g(x)\sum_{k=1}^{\infty} x^{mk}(1-x^k).$$

In the same way one can calculate

$$\frac{\partial^2}{\partial y^2}H(x,y)\bigg|_{y=1} = g(x)\left\{\left[\sum_{k=1}^{\infty} x^{mk}(1-x^k)\right]^2 - \sum_{k=1}^{\infty}\left[x^{mk}(1-x^k)\right]^2\right\}.$$

For the sake of convenience we denote the function in the curly brackets with $F(x)$:

$$F(x) = \left[\sum_{k=1}^{\infty} x^{mk}(1-x^k)\right]^2 - \sum_{k=1}^{\infty}\left[x^{mk}(1-x^k)\right]^2 \tag{8}$$

$$= \left[\frac{x^m}{1-x^m} - \frac{x^{m+1}}{1-x^{m+1}}\right]^2 - \frac{x^{2m}}{1-x^{2m}} + \frac{2\,x^{2m+1}}{1-x^{2m+1}} - \frac{x^{2m+2}}{1-x^{2m+2}}.$$

Therefore (2) again implies that

$$\sum_{n\geq 0} x^n p(n)\mathbf{E}[X_{m,n}(X_{m,n}-1)] = g(x)F(x). \tag{9}$$

We apply Cauchy's coefficient formula to (9) on the circle $x = r_n e^{i\theta}$ with r_n determined by (4). Thus we obtain

$$\mathbf{E}[X_{m,n}(X_{m,n}-1)] = \frac{r_n^{-n}}{2\pi\, p(n)}\int_{-\pi}^{\pi} g(r_n e^{i\theta})F(r_n e^{i\theta})e^{-i\theta n}d\theta = I_1 + I_2. \tag{10}$$

We break integral into two parts as follows:

$$I_1 = \frac{r_n^{-n}}{2\pi\, p(n)} \int_{-\delta_n}^{\delta_n} g(r_n e^{i\theta}) F(r_n e^{i\theta}) e^{-i\theta n} d\theta, \tag{11}$$

$$I_2 = \frac{r_n^{-n}}{2\pi\, p(n)} \int_{\delta_n \le |\theta| \le \pi} g(r_n e^{i\theta}) F(r_n e^{i\theta}) e^{-i\theta n} d\theta. \tag{12}$$

The asymptotic analysis of these integrals follows in the next two subsections.

3.1. An asymptotic estimate for $I_1(n)$

First, we will show that r_n defined by (4) satisfies condition (7). From (1) it follows that

$$\frac{r\, g'(r)}{g(r)} = r\, \frac{\partial}{\partial r} \log g(r) = \sum_{j=1}^{\infty} \frac{j\, r^j}{1 - r^j}. \tag{13}$$

We interpret this sum by a Riemann's one with the step size $y_n = -\log r_n$

$$\sum_{j=1}^{\infty} \frac{j\, r^j}{1 - r^j} \sim \frac{1}{y_n^2} \int_0^{\infty} \frac{y}{e^y - 1}\, dy. \tag{14}$$

Next, we use a well-known properties of Riemann zeta function (see e.g. [6, 1.7.8(II)])

$$\int_0^{\infty} \frac{y}{e^y - 1}\, dy = \zeta(2)\, \Gamma(2) = \frac{\pi^2}{6}. \tag{15}$$

Combining (13), (14), (15) and (4), we find that

$$\frac{r_n\, g'(r_n)}{g(r_n)} \sim \frac{\pi^2}{6\, \log^2 r_n} = \frac{1}{n^{-1} + O(n^{-3/2})},$$

which completes the proof of (7).

We apply lemmas 1 and 3 to (11) and find that

$$I_1 \sim \frac{b^{1/2}(r_n)}{\sqrt{2\,\pi}} \int_{-\delta_n}^{\delta_n} e^{-\frac{\theta^2 b(r_n)}{2}} F(r_n e^{i\theta}) d\theta. \tag{16}$$

Next, we expand $F(r_n e^{i\theta})$ around the point r_n by Taylor's formula:

$$\begin{aligned} F(r_n e^{i\theta}) &= F(r_n) + r_n (e^{i\theta} - 1) F'(r_n) + O\left(|\theta^2| F''(r_n)\right) \\ &= F(r_n) \left[1 + O\left(|\theta|\, \frac{F'(r_n)}{F(r_n)}\right)\right]. \end{aligned} \tag{17}$$

We will first estimate the error term in (17). From definition (4) of r_n it follows that

$$\frac{r_n^m}{1-r_n^m} = \frac{(1+O(n^{-1/2}))^m}{1-\left(1-\frac{\pi}{\sqrt{6n}}+O(n^{-1})\right)^m} = \frac{1+O(n^{-1/2})}{\frac{\pi m}{\sqrt{6n}}+O(n^{-1})} \sim \frac{\sqrt{6n}}{\pi}. \tag{18}$$

We use this asymptotic equivalence and equation (8) to estimate $F(r_n)$ as follows:

$$\begin{aligned} F(r_n) &= \frac{6n}{\pi^2}\left(\frac{1}{m}-\frac{1}{m+1}\right)^2 - \frac{\sqrt{6n}}{\pi}\left(\frac{1}{2m}-\frac{2}{2m+1}+\frac{1}{2m+2}\right) \\ &\sim \frac{6n}{\pi^2}\frac{1}{m^2(m+1)^2} - \frac{\sqrt{6n}}{\pi}\frac{1}{2m(2m+1)(m+1)}. \end{aligned} \tag{19}$$

Next, we use (8) and (18) again to calculate $F'(r_n)$:

$$\begin{aligned} F'(r_n) = 2&\left[\frac{r_n^m}{1-r_n^m}-\frac{r_n^{m+1}}{1-r_n^{m+1}}\right]\left[\frac{mr_n^{m-1}}{(1-r_n^m)^2}-\frac{(m+1)r_n^m}{(1-r_n^{m+1})^2}\right] \\ &-\frac{2mr_n^{2m-1}}{(1-r_n^{2m})^2}+\frac{2\,(2m+1)r_n^{2m}}{(1-r_n^{2m+1})^2}-\frac{(2m+2)r_n^{2m+1}}{(1-r_n^{2m+2})^2} \end{aligned} \tag{20}$$

$$\sim 2\frac{(6n)^{3/2}}{\pi^3}\left(\frac{1}{m}-\frac{1}{m+1}\right)^2 - \frac{6n}{\pi^2}\left(\frac{1}{2m}-\frac{2}{2m+1}+\frac{1}{2m+2}\right) = O(n^{3/2}).$$

From (19), (20) and (6) it follows that

$$O\left(|\theta|\,\frac{F'(r_n)}{F(r_n)}\right) = O\left(|\delta_n|\,\frac{n^{3/2}}{n}\right) = O(n^{-1/6}). \tag{21}$$

Substituting (17) and (21) in (16) we obtain

$$I_1 \sim \frac{b^{1/2}(r_n)F(r_n)}{\sqrt{2\,\pi}}\int_{-\delta_n}^{\delta_n} e^{-\frac{\theta^2 b(r_n)}{2}}\,d\theta = \frac{F(r_n)}{\sqrt{2\,\pi}}\int_{-\delta_n b^{1/2}(r_n)}^{\delta_n b^{1/2}(r_n)} e^{-\frac{t^2}{2}}\,dt. \tag{22}$$

In last integral we changed the variable of integration into $t = \theta\, b^{1/2}(r_n)$.Finally, (4) and (5) imply that

$$\begin{aligned} b(r_n) &= \frac{\pi^2}{3\left[\frac{\pi}{\sqrt{6n}}-\frac{\pi^2}{12\,n}+O(n^{-3/2})\right]^3} = \frac{(6\,n)^{3/2}\,\pi^2}{3\,\pi^3\left[1+O(n^{-1/2})\right]} \\ &= \frac{(6\,n)^{3/2}}{3\,\pi}\left[1+O(n^{-1/2})\right] = \frac{2\sqrt{6}}{\pi}n^{3/2}+O(n). \end{aligned} \tag{23}$$

If we combine this equation with the definition of δ_n given by (6), we find that

$$\delta_n\, b^{1/2}(r_n) \sim d\, n^{1/12}/\log n, \qquad d = (2/\pi)^{1/2} 6^{1/4}. \tag{24}$$

To complete the asymptotic analysis of I_1 we apply a well known property of Gaussian density to (22). So we get

$$I_1 \sim F(r_n). \tag{25}$$

3.2. An asymptotic estimate for $I_2(n)$

Now we will show that the integral I_2 is negligible. It is easy to see that

$$\left|\frac{r_n^m\, e^{i\theta m}}{1 - r_n^m\, e^{i\theta m}}\right| = \frac{r^m}{\sqrt{1 - 2\, r^m\, \cos\theta + r^{2m}}} \le \frac{r^m}{\sqrt{1 - 2\, r^m + r^{2m}}} \le \frac{1}{1 - r^m}$$

$$= \frac{1}{1 - \left(1 - \frac{\pi}{\sqrt{6n}} + O(n^{-1}\right)^m} = \frac{1}{\frac{m\,\pi}{\sqrt{6n}} + O(n^{-1})} = O\left(n^{1/2}\right). \tag{26}$$

From (8) and (26) it follows that

$$\left|F(r_n e^{i\theta})\right| \le \left[O\left(n^{1/2}\right) + O\left(n^{1/2}\right)\right]^2 + O\left(n^{1/2}\right) = O(n). \tag{27}$$

We apply lemmas 2 and 3 and inequality (27) to (12). Then, for $n \ge n(c)$, we have

$$\begin{aligned} |I_2(n)| &\sim \left|\frac{b^{1/2}(r_n)}{\sqrt{2\,\pi}\, g(r_n)} \int_{\delta_n \le |\theta| \le \pi} g(r_n e^{i\theta}) F(r_n e^{i\theta}) e^{-i\theta n} d\theta\right| \\ &\le \frac{b^{1/2}(r_n) \exp\left\{\frac{-cn^{1/6}}{\log^2 n}\right\}}{\sqrt{2\,\pi}} \int_{\delta_n \le |\theta| \le \pi} \left|F(r_n e^{i\theta}) e^{-i\theta n}\right| d\theta \\ &\le b^{1/2}(r_n) \exp\left\{\frac{-cn^{1/6}}{\log^2 n}\right\} O(n)\, \sqrt{2\,\pi}. \end{aligned} \tag{28}$$

If we combine (23) and (28), we find that

$$|I_2(n)| \le \exp\left\{\frac{-cn^{1/6}}{\log^2 n}\right\} O\left(n^{5/2}\right) = o(1), \tag{29}$$

as $n \to \infty$.

3.3. Formula about variance of $X_{m,n}$

Equations (10), (25) and (29) imply that

$$\mathbf{E}[X_{m,n}(X_{m,n}-1)] \sim F(r_n).$$

Substituting this, (19) and (3) in the well known formula

$$\mathrm{Var}(X_{m,n}) = \mathrm{E}[X_{m,n}(X_{m,n}-1)] + \mathrm{E}(X_{m,n}) - [\mathrm{E}(X_{m,n})]^2$$

after simple manipulations we obtain

$$\mathrm{Var}(X_{m,n}) \sim \frac{\sqrt{6n}}{\pi}\left(\frac{1}{m(m+1)} - \frac{1}{2m(2m+1)(m+1)}\right),$$

which completes the proof. □

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