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ON THE MOVING BOUNDARY HITTING PROBABILITY FOR THE BROWNIAN MOTION

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Consider the probability that the Brownian motion hits a moving two-sided boundary by a certain moment. In some special cases we find formulae for this probability.

1. Introduction

Let B_t be the Brownian motion (drift $= 0$, volatility $= 1$), $B_0 = 0$; $T > 0$; $upper(t)$ and $lower(t)$ be two functions defined at least for $t \in [0;T]$, $lower(t) < upper(t), \forall t \in [0;T]$, $lower(0) \leq 0 \leq upper(0)$; τ be the first hitting moment, i.e. $\tau = \inf\{t \in [0;T] : B_t = upper(t) \text{ or } B_t = lower(t)\}$ ($\tau = T$, if the set is empty).

We interpret T as a horizon and are interested in the probabilities of the events $\mathcal{U} = \{B_\tau = upper(\tau)\}$ and $\mathcal{L} = \{B_\tau = lower(\tau)\}$.

In 1960 T. W. Anderson [1] discovered the crossing probabilities for rectilinear boundaries with no horizon — two straight lines that are parallel or cross to the left of the origin. In 1964 A. V. Skorohod [2] found the probability that the motion will get out of the domain through a little "door" at the horizon; his formula holds for rectilinear boundaries. In 1967 L. A. Shepp [3] found a formula for the expectation of the first hitting time for a two-sided symmetric square-root boundary with no horizon. In 1971 A. A. Novikov [4] solved the same problem

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for a one-sided square-root boundary. In 1981 A. A. Novikov [5] published a formula for the probability that the motion will get out of the domain through the horizon; it holds for curvilinear boundaries that are close to each other. A little later in the same year A. V. Mel'nikov and D. I. Hadžiev [6] published a solution to a similar problem for Gaussian martingales. In 1999 A. Novikov, V. Frishling and N. Kordzakhia [7] found approximate formulae for the crossing probabilities both for a one-sided and a two-sided boundary with a horizon; they were able to derive an exact formula for a one-sided and a two-sided symmetric square-root boundary.

In the current paper we find formulae for $P(\mathcal{U})$ and $P(\mathcal{L})$ for parallel rectilinear boundaries as well as for arbitrary square-root boundaries.

2. Calculation of $P(\mathcal{U})$

We have a random process starting at time 0 at point 0. Consider a diffusion process starting at time t at point x instead and let $v(t,x) = P(\mathcal{U})$, $0 \le t \le T$, $lower(t) \le x \le upper(t)$. So we may safely omit the inequality $lower(0) \le 0 \le upper(0)$, keeping this one: $lower(t) < upper(t), \forall t \in [0;T]$.

Then (cf. [8], chapter 10) the function $v(t,x)$ satisfies the conditions:

$$(1) \qquad \left| \begin{array}{ll} \dfrac{\partial v}{\partial t} + \dfrac{1}{2} \cdot \dfrac{\partial^2 v}{\partial x^2} = 0 & \\ v(T,x) = 0, & \forall x \in (lower(T); upper(T)) \\ v(t, lower(t)) = 0, & \forall t \in [0;T] \\ v(t, upper(t)) = 1, & \forall t \in [0;T] \end{array} \right.$$

The equation is simple enough, but the boundary is too complicated. To get a rectangular boundary, set

$$h(t) = upper(t) - lower(t) > 0, \quad v(t,x) = v_1\left(t, \ \frac{x - lower(t)}{h(t)}\right).$$

Then the function $v_1(t,x)$ is a solution to the problem:

$$(2) \qquad \left| \begin{array}{ll} \dfrac{\partial v_1}{\partial t} - \dfrac{h'(t)\,.\,x + lower'(t)}{h(t)} \cdot \dfrac{\partial v_1}{\partial x} + \dfrac{1}{2} \cdot \dfrac{1}{h^2(t)} \cdot \dfrac{\partial^2 v_1}{\partial x^2} = 0 & \\ v_1(T,x) = 0, & \forall x \in (0;1) \\ v_1(t,0) = 0, & \forall t \in [0;T] \\ v_1(t,1) = 1, & \forall t \in [0;T] \end{array} \right.$$

2.1. Two "parallel" curves: $h(t) = c = const.$ $(c > 0)$

Therefore $h'(t) = 0$ and the equation takes this form:

$$\frac{\partial v_1}{\partial t} - \frac{lower'(t)}{c} \cdot \frac{\partial v_1}{\partial x} + \frac{1}{2c^2} \cdot \frac{\partial^2 v_1}{\partial x^2} = 0$$

2.1.1. Two parallel straight lines: $lower(t) = bt + c_1$

Then $upper(t) = bt + c_2$, $c_2 > c_1$, $c = c_2 - c_1 > 0$, $lower'(t) = b$ and the equation becomes:

$$\frac{\partial v_1}{\partial t} - \frac{b}{c} \cdot \frac{\partial v_1}{\partial x} + \frac{1}{2c^2} \cdot \frac{\partial^2 v_1}{\partial x^2} = 0$$

Let $\kappa = \dfrac{1}{2c^2} > 0$, $\lambda = \dfrac{b}{c}$. Then we have to solve the problem:

$$\left|\begin{array}{l} \dfrac{\partial v_1}{\partial t} - \lambda \cdot \dfrac{\partial v_1}{\partial x} + \kappa \cdot \dfrac{\partial^2 v_1}{\partial x^2} = 0 \\ v_1(T, x) = 0, \quad \forall x \in (0; 1) \\ v_1(t, 0) = 0, \quad \forall t \in [0; T] \\ v_1(t, 1) = 1, \quad \forall t \in [0; T] \end{array}\right.$$

We would prefer to have initial conditions rather than final ones, so we set

$$v_1(t, x) = v_2(T - t, x)$$

and reformulate the problem as follows:

$$\left|\begin{array}{l} -\dfrac{\partial v_2}{\partial t} - \lambda \cdot \dfrac{\partial v_2}{\partial x} + \kappa \cdot \dfrac{\partial^2 v_2}{\partial x^2} = 0 \\ v_2(0, x) = 0, \quad \forall x \in (0; 1) \\ v_2(t, 0) = 0, \quad \forall t \in [0; T] \\ v_2(t, 1) = 1, \quad \forall t \in [0; T] \end{array}\right.$$

Now the time horizon T takes part in the intervals only. We can get rid of it by means of the following argument: when the value of T changes, only the domain of the function $v_2(t, x)$ changes, not its values. Since T can be an arbitrary positive number, we may assume that the function $v_2(t, x)$ is defined at least for $t \in [0; +\infty)$:

$$\left|\begin{array}{l} \kappa \cdot \dfrac{\partial^2 v_2}{\partial x^2} - \lambda \cdot \dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_2}{\partial t} = 0 \\ v_2(0, x) = 0, \quad \forall x \in (0; 1) \\ v_2(t, 0) = 0, \quad \forall t \in [0; +\infty) \\ v_2(t, 1) = 1, \quad \forall t \in [0; +\infty) \end{array}\right.$$

After the Laplace transformation $V(p,x) = L[v_2(t,x)]$ we get the problem:

$$(3)\qquad \left|\begin{array}{l} \kappa\,.\,V'' - \lambda\,.\,V' - p\,.\,V = 0 \\ V(0) = 0 \\ V(1) = \dfrac{1}{p} \end{array}\right.$$

(V is considered a function of x, and p is just a parameter.)

The characteristic equation is

$$\kappa\nu^2 - \lambda\nu - p = 0$$

$D = \lambda^2 + 4\kappa p > 0$, because $p > 0$, so $\nu_{1,2} = \dfrac{\lambda \pm \sqrt{\lambda^2 + 4\kappa p}}{2\kappa}$ and

$$V(x) = \left(C_1\ .\ \cosh \frac{\sqrt{\lambda^2 + 4\kappa p}\ .\ x}{2\kappa} + C_2\ .\ \sinh \frac{\sqrt{\lambda^2 + 4\kappa p}\ .\ x}{2\kappa} \right) \cdot \exp\left(\frac{\lambda x}{2\kappa}\right)$$

From $V(0) = 0$ it follows that $C_1 = 0$. Then from $V(1) = \dfrac{1}{p}$ we get $C_2 = \dfrac{\exp\left(-\frac{\lambda}{2\kappa}\right)}{p\ .\ \sinh \frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}}$.

Finally, we obtain the equality:

$$V(x) = \frac{\sinh \frac{\sqrt{\lambda^2+4\kappa p}\ .\ x}{2\kappa}}{p\ .\ \sinh \frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}} \cdot \exp\left(\frac{\lambda(x-1)}{2\kappa}\right)$$

So we have just proved the following theorem (where L^{-1} stands for the reversed Laplace transformation):

Theorem 1. *If* $lower(t) = bt + c_1$, $upper(t) = bt + c_2$, $c = c_2 - c_1 > 0$, *then* $P(\mathcal{U}) = v(t,x)$, $v(t,x) = v_1\left(t,\ \dfrac{x - bt - c_1}{c}\right)$, $v_1(t,x) = v_2(T-t,x)$,

$v_2(t,x) = L^{-1}[V(p,x)]$, $\qquad V(p,x) = \dfrac{\sinh \frac{\sqrt{\lambda^2+4\kappa p}\ .\ x}{2\kappa}}{p\ .\ \sinh \frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}} \cdot \exp\left(\dfrac{\lambda(x-1)}{2\kappa}\right)$,

where $\kappa = \dfrac{1}{2c^2}$, $\lambda = \dfrac{b}{c}$.

Fortunately, the function $V(p,x)$ can be explicitly transformed to the function $v_2(t,x)$ and then back to $v(t,x)$. According to [9], we have:

$$v_2(t,x) = L^{-1}\left[V(p,x)\right] = \sum_{p_n} \operatorname*{res}_{p_n} V(p,x)\exp(pt) =$$

$$= \left(\frac{\sinh\frac{\lambda x}{2\kappa}}{\sinh\frac{\lambda}{2\kappa}} + 2\kappa\pi\sum_{n=1}^{\infty} \frac{(-1)^n.\,n.\,\sin(n\pi x).\exp\left\{-\left(\kappa n^2\pi^2+\frac{\lambda^2}{4\kappa}\right)t\right\}}{\kappa n^2\pi^2+\frac{\lambda^2}{4\kappa}} \right)\cdot\exp\left(\frac{\lambda(x-1)}{2\kappa}\right)$$

The value of the single addend must be considered equal to x, when $\lambda = 0$. This addend comes from the residuum at $p_0 = 0$. The n-th addend in the sum comes from the residuum at $p_n = -\kappa n^2\pi^2 - \frac{\lambda^2}{4\kappa}$, $n \in \mathbb{N}$. Then

$$v_1(t,x) = v_2(T-t,x) =$$

$$= \left(\frac{\sinh\frac{\lambda x}{2\kappa}}{\sinh\frac{\lambda}{2\kappa}} + 2\kappa\pi\sum_{n=1}^{\infty} \frac{(-1)^n.\,n.\,\sin(n\pi x).\exp\left\{\left(\kappa n^2\pi^2+\frac{\lambda^2}{4\kappa}\right)(t-T)\right\}}{\kappa n^2\pi^2+\frac{\lambda^2}{4\kappa}} \right)\cdot\exp\left(\frac{\lambda(x-1)}{2\kappa}\right)$$

Substituting κ and λ in the last expression, we get

$$\left(\frac{\sinh(bcx)}{\sinh(bc)} + 2\pi\sum_{n=1}^{\infty} \frac{(-1)^n.\,n.\,\sin(n\pi x).\exp\left\{\left(n^2\pi^2+b^2c^2\right)\frac{t-T}{2c^2}\right\}}{n^2\pi^2+b^2c^2} \right)\cdot\exp\left\{bc(x-1)\right\}$$

Finally, the probability we are looking for is equal to

$$P(\mathcal{U}) = v(t,x) = v_1\left(t,\ \frac{x-bt-c_1}{c}\right)$$

and can be found from the expression above.

Theorem 2. *If* $\quad lower(t) = bt + c_1\,,\ \ upper(t) = bt + c_2\,,\quad c = c_2 - c_1 > 0,$ *then* $P(\mathcal{U}) = v(t,x) = e^{b(x-bt-c_2)}\cdot \left(\frac{\sinh\left\{b(x-bt-c_1)\right\}}{\sinh(bc)} + \right.$

$$\left. + 2\pi\sum_{n=1}^{\infty} \frac{(-1)^n.\,n.\,\sin\left(n\pi\frac{x-bt-c_1}{c}\right).\exp\left\{\left(n^2\pi^2+b^2c^2\right)\frac{t-T}{2c^2}\right\}}{n^2\pi^2+b^2c^2} \right),$$

where the single addend is equal to $\frac{x-c_1}{c}$, *when* $b = 0$.

To check, let $T \to +\infty$; we can do this, because the series is convergent uniformly with respect to $T \in [t; +\infty)$; thus we obtain the formula for the case, when there is no horizon:

Corollary 1. *If* $lower(t) = bt + c_1$, $upper(t) = bt + c_2$, $c = c_2 - c_1 > 0$ *and there is no horizon, then*

$$P(\mathcal{U}) = v(t,x) = \begin{cases} \dfrac{e^{2b(x - bt - c_1)} - 1}{e^{2bc} - 1} & \text{for } b \neq 0 \\ \dfrac{x - c_1}{c} & \text{for } b = 0 \end{cases}$$

The formula for $b = 0$ is well known from the martingale theory. The formula for $b \neq 0$ can be found (in different denotation) in [1] as Theorem 4.1 on page 175.

Again, let $c_1 \to -\infty$ in Corollary 1; thus we get the solution for a one-sided boundary with no horizon:

Corollary 2. *If* $upper(t) = bt + c_2$ *and there is no lower boundary and no horizon, then*

$$P(\mathcal{U}) = v(t,x) = \begin{cases} e^{2b(x - bt - c_2)} & \text{for } b > 0 \\ 1 & \text{for } b \leq 0 \end{cases}$$

This result can be verified by means of Kendall's famous formula.

Unfortunately, this technique does not work for Theorem 2: the series is not uniformly convergent with respect to $c_1 \in (-\infty; x_0], \forall x_0 \leq x - bt$.

2.2. Square-root boundaries

Let $lower(t) = a\sqrt{t+\gamma} + c_0$, $upper(t) = b\sqrt{t+\gamma} + c_0$, $b > a$. Then the problem (2) takes the form:

$$\left| \begin{array}{l} \dfrac{\partial v_1}{\partial t} - \dfrac{1}{2(t+\gamma)} \cdot \left(x + \dfrac{a}{b-a} \right) \cdot \dfrac{\partial v_1}{\partial x} + \dfrac{1}{2} \cdot \dfrac{1}{(b-a)^2 (t+\gamma)} \cdot \dfrac{\partial^2 v_1}{\partial x^2} = 0 \\ v_1(T,x) = 0, \quad \forall x \in (0;1) \\ v_1(t,0) = 0, \quad \forall t \in [0;T] \\ v_1(t,1) = 1, \quad \forall t \in [0;T] \end{array} \right.$$

Remark: The number γ must be positive, because $v(t,x)$ is defined for $t \in [0;T]$ and $upper(t) > lower(t), \forall t \in [0;T]$. If the domain of the function changes, the domain of γ will change respectively.

It is essential that the multiplier $(t+\gamma)$ is raised to the same power in both denominators; this happens for square-root boundaries only. That is why, by multiplying the equation by $2(t+\gamma)$ we can ensure that the variable t takes part in only one coefficient:

$$2(t+\gamma)\cdot\frac{\partial v_1}{\partial t}-\left(x+\frac{a}{(b-a)}\right)\cdot\frac{\partial v_1}{\partial x}+\frac{1}{(b-a)^2}\cdot\frac{\partial^2 v_1}{\partial x^2}=0$$

We can get rid of the multiplier $(t+\gamma)$ by means of a suitable substitution. Let $v_1(t,x)=v_2\left(\frac{1}{2}\ln(t+\gamma),x\right)$; then the function $v_2(t,x)$ is a solution to the following problem:

$$\left|\begin{array}{ll}\frac{\partial v_2}{\partial t}-\left(x+\frac{a}{b-a}\right)\cdot\frac{\partial v_2}{\partial x}+\frac{1}{(b-a)^2}\cdot\frac{\partial^2 v_2}{\partial x^2}=0 & \\ v_2\left(\frac{1}{2}\ln(T+\gamma),x\right)=0, & \forall x\in(0;1)\\ v_2(t,0)=0, & \forall t\in\left[\frac{1}{2}\ln\gamma;\frac{1}{2}\ln(T+\gamma)\right]\\ v_2(t,1)=1, & \forall t\in\left[\frac{1}{2}\ln\gamma;\frac{1}{2}\ln(T+\gamma)\right]\end{array}\right.$$

Finally, let $v_2(t,x)=v_3\left(-t+\frac{1}{2}\ln(T+\gamma),x\right)$ in order to have an initial condition instead of a final one.

$$\left|\begin{array}{ll}-\frac{\partial v_3}{\partial t}-\left(x+\frac{a}{b-a}\right)\cdot\frac{\partial v_3}{\partial x}+\frac{1}{(b-a)^2}\cdot\frac{\partial^2 v_3}{\partial x^2}=0 & \\ v_3(0,x)=0, & \forall x\in(0;1)\\ v_3(t,0)=0, & \forall t\in\left[0;\frac{1}{2}\ln(T+\gamma)-\frac{1}{2}\ln\gamma\right]\\ v_3(t,1)=1, & \forall t\in\left[0;\frac{1}{2}\ln(T+\gamma)-\frac{1}{2}\ln\gamma\right]\end{array}\right.$$

Again, we would prefer to search for $v_3(t,x)$ defined for $t\in[0;+\infty)$:

$$\left|\begin{array}{ll}-\frac{\partial v_3}{\partial t}-\left(x+\frac{a}{b-a}\right)\cdot\frac{\partial v_3}{\partial x}+\frac{1}{(b-a)^2}\cdot\frac{\partial^2 v_3}{\partial x^2}=0 & \\ v_3(0,x)=0, & \forall x\in(0;1)\\ v_3(t,0)=0, & \forall t\in[0;+\infty)\\ v_3(t,1)=1, & \forall t\in[0;+\infty)\end{array}\right.$$

Apply the Laplace transformation: $V(p,x)=L[v_3(t,x)]$. Then

$$\left|\begin{array}{l}\frac{1}{(b-a)^2}\cdot V''-\left(x+\frac{a}{b-a}\right)\cdot V'-p\cdot V=0\\ V(0)=0\\ V(1)=\frac{1}{p}\end{array}\right.$$

(V is considered a function of x, and p is just a parameter.)

This boundary value problem has a unique solution; the solution is an analytical function: $V(x)=\sum_{n=0}^{\infty} c_n x^n$. The differential equation turns into the following equation for the coefficients of the series:

$$\frac{1}{(b-a)^2}(n+2)(n+1)c_{n+2}-\left(nc_n+\frac{a}{b-a}(n+1)c_{n+1}\right)-p\,c_n=0, n\in\mathbb{N}_0;$$

$$c_{n+2}=\frac{a(b-a)}{n+2}\,c_{n+1}+\frac{(b-a)^2(n+p)}{(n+2)(n+1)}\,c_n\ , \quad n\in\mathbb{N}_0$$

We have to find c_0 and c_1 in order to specify the sequence $(c_n)_{n=0}^{\infty}$. From $V(0)=0$ it follows that $c_0=0$. Let $c_1=c$, $c_n=c\alpha_n$, $n\in\mathbb{N}_0$. Then

$$\alpha_0=0,\quad \alpha_1=1,\quad \alpha_{n+2}=\frac{a(b-a)}{n+2}\,\alpha_{n+1}+\frac{(b-a)^2(n+p)}{(n+2)(n+1)}\,\alpha_n\ ,\ \forall n\in\mathbb{N}_0;$$

$V(x)=c\,.\sum_{n=0}^{\infty}\alpha_n x^n=c\,.\sum_{n=1}^{\infty}\alpha_n x^n$; the unknown constant c can be found from the boundary condition $V(1)=\frac{1}{p}$; it follows that $c=\frac{1}{p\,.\sum_{n=1}^{\infty}\alpha_n}\cdot$ We have just proved the following theorem:

Theorem 3. *If* $lower(t)=a\sqrt{t+\gamma}+c_0$, $upper(t)=b\sqrt{t+\gamma}+c_0$, $b>a$, *then* $P(\mathcal{U})=v(t,x)=v_1\left(t,\frac{x-c_0-a\sqrt{t+\gamma}}{(b-a)\sqrt{t+\gamma}}\right)$, $v_1(t,x)=v_2\left(\frac{1}{2}\ln(t+\gamma),\,x\right)$, $v_2(t,x)=v_3\left(-t+\frac{1}{2}\ln(T+\gamma),\,x\right)$, $v_3(t,x)=L^{-1}\left[V(p,x)\right]$, *where* L^{-1} *stands for the reversed Laplace transformation,* $V(p,x)=c\,.\sum_{n=1}^{\infty}\alpha_n x^n$,

$$\alpha_0=0,\quad \alpha_1=1,\quad \alpha_{n+2}=\frac{a(b-a)}{n+2}\,\alpha_{n+1}+\frac{(b-a)^2(n+p)}{(n+2)(n+1)}\,\alpha_n\ ,\ \forall n\in\mathbb{N}_0;$$

$$c=\frac{1}{p\,.\sum_{n=1}^{\infty}\alpha_n}\cdot$$

When $a=0$, we can obtain an explicit formula for $(\alpha_n)_{n=0}^{\infty}$. Indeed, $\alpha_0=0$, $\alpha_1=1, \alpha_{n+2}=\frac{b^2(n+p)}{(n+2)(n+1)}\,\alpha_n, \forall n\in\mathbb{N}_0$; so $\alpha_{2m}=0, \forall m\in\mathbb{N}_0$.

Let $\beta_m = \alpha_{2m+1}, \forall m \in \mathbb{N}_0$; then $\beta_0 = 1,\ \beta_{m+1} = \dfrac{b^2(2m+1+p)}{(2m+3)(2m+2)}\ \beta_m$,
i.e. $\beta_m = \dfrac{b^2(2m-1+p)}{(2m+1)(2m)}\ \beta_{m-1}$, hence $\beta_m = \dfrac{b^{2m}}{(2m+1)!}\prod\limits_{\widetilde{m}=1}^{m}(2\widetilde{m}-1+p)$.
That is why, the following statement holds:

Corollary 3. *If* $lower(t) = c_0$, $upper(t) = b\sqrt{t+\gamma}+c_0$, $b>0$, *then* $P(\mathcal{U}) = v(t,x)$, $v(t,x) = v_1\left(t, \dfrac{x-c_0}{b\sqrt{t+\gamma}}\right)$, $v_1(t,x) = v_2\left(\frac{1}{2}\ln(t+\gamma), x\right)$, $v_2(t,x) = v_3\left(-t+\frac{1}{2}\ln(T+\gamma),\ x\right)$, $v_3(t,x) = L^{-1}\left[V(p,x)\right]$, *where* L^{-1} *stands for the reversed Laplace transformation,* $V(p,x) = c\,.\sum\limits_{m=0}^{\infty}\beta_m x^{2m+1}$,

$$\beta_0 = 1,\ \ \beta_m = \frac{b^{2m}}{(2m+1)!}\prod_{\widetilde{m}=1}^{m}(2\widetilde{m}-1+p),\ \forall m\in\mathbb{N};\quad c = \frac{1}{p\,.\sum\limits_{m=0}^{\infty}\beta_m}\,.$$

3. Calculation of $P(\mathcal{L})$

The only difference is that 0 and 1 change places in the boundary conditions of (1). This change propagates through the sequence of substitutions.

3.1. Two parallel straight lines

Now (3) changes in this way:

$$\left|\begin{array}{l} \kappa\,.\,V'' - \lambda\,.\,V' - p\,.\,V = 0 \\ V(0) = \dfrac{1}{p} \\ V(1) = 0 \end{array}\right.$$

After the substitution $V(x) = W(1-x)$ we get the problem:

$$\left|\begin{array}{l} \kappa\,.\,W'' + \lambda\,.\,W' - p\,.\,W = 0 \\ W(0) = 0 \\ W(1) = \dfrac{1}{p} \end{array}\right.$$

This is the same problem as (3), only λ has changed its sign. Then

$$W(x) = \frac{\sinh \frac{\sqrt{\lambda^2+4\kappa p}\,.\,x}{2\kappa}}{p \cdot \sinh \frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}} \cdot \exp\left(\frac{\lambda(1-x)}{2\kappa}\right),$$

$$V(x) = \frac{\sinh \frac{\sqrt{\lambda^2+4\kappa p}\,.\,(1-x)}{2\kappa}}{p \cdot \sinh \frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}} \cdot \exp\left(\frac{\lambda x}{2\kappa}\right)$$

and the following statement holds:

Theorem 4. *If* $lower(t) = bt + c_1$, $upper(t) = bt + c_2$, $c = c_2 - c_1 > 0$, *then* $P(\mathcal{L}) = v(t,x)$, $v(t,x) = v_1\left(t,\ \frac{x - bt - c_1}{c}\right)$, $v_1(t,x) = v_2(T-t,x)$,
$v_2(t,x) = L^{-1}\left[V(p,x)\right]$, $V(p,x) = \frac{\sinh \frac{\sqrt{\lambda^2+4\kappa p}\,.\,(1-x)}{2\kappa}}{p \cdot \sinh \frac{\sqrt{\lambda^2+4\kappa p}}{2\kappa}} \cdot \exp\left(\frac{\lambda x}{2\kappa}\right)$,
where L^{-1} *stands for the reversed Laplace transformation,* $\kappa = \frac{1}{2c^2}$, $\lambda = \frac{b}{c}$.

Theorem 5 can be deduced from Theorem 4 as Theorem 2 was deduced from Theorem 1. Or we may notice that the changes in the formulae are equivalent to swapping the lower and the upper boundary: we replace $x - lower(t)$ with $upper(t) - x$ and vice versa as well as b with $-b$ in Theorem 2.

Theorem 5. *If* $lower(t) = bt + c_1$, $upper(t) = bt + c_2$, $c = c_2 - c_1 > 0$, *then*

$$P(\mathcal{L}) = v(t,x) = e^{b(x-bt-c_1)} \cdot \left(\frac{\sinh\left\{b(bt+c_2-x)\right\}}{\sinh(bc)} + {} \right.$$
$$\left. + 2\pi \sum_{n=1}^{\infty} \frac{(-1)^n .\ n .\ \sin\left(n\pi \frac{bt+c_2-x}{c}\right) .\ \exp\left\{\left(n^2\pi^2 + b^2c^2\right)\frac{t-T}{2c^2}\right\}}{n^2\pi^2 + b^2c^2} \right),$$

where the single addend is equal to $\frac{c_2 - x}{c}$, *when* $b = 0$.

Corollary 4. *If* $lower(t) = bt + c_1$, $upper(t) = bt + c_2$, $c = c_2 - c_1 > 0$ *and there is no horizon, then*

$$P(\mathcal{L}) = v(t,x) = \begin{cases} \dfrac{e^{-2b(bt+c_2-x)} - 1}{e^{-2bc} - 1} & \text{for } b \neq 0 \\[2ex] \dfrac{c_2 - x}{c} & \text{for } b = 0 \end{cases}$$

Corollary 5. *If* $lower(t) = bt + c_1$ *and there is no upper boundary and no horizon, then*

$$P(\mathcal{L}) = v(t,x) = \begin{cases} e^{2b(x-bt-c_1)} & \text{for } b < 0 \\ 1 & \text{for } b \geq 0 \end{cases}$$

3.2. Square-root boundaries

Theorem 6. *If* $lower(t) = a\sqrt{t+\gamma} + c_0$, $upper(t) = b\sqrt{t+\gamma} + c_0$, $b > a$, *then* $P(\mathcal{L}) = v(t,x) = v_1\left(t, \frac{x-c_0-a\sqrt{t+\gamma}}{(b-a)\sqrt{t+\gamma}}\right)$, $v_1(t,x) = v_2\left(\frac{1}{2}\ln(t+\gamma), x\right)$, $v_2(t,x) = v_3\left(-t+\frac{1}{2}\ln(T+\gamma), x\right)$, $v_3(t,x) = L^{-1}[V(p,x)]$, *where* $V(p,x) = W(p, 1-x)$, $W(p,x) = c \,.\, \sum_{n=1}^{\infty} \alpha_n x^n$, $c = \left(p \,.\, \sum_{n=1}^{\infty} \alpha_n\right)^{-1}$,

$$\alpha_0 = 0,\ \alpha_1 = 1,\ \alpha_{n+2} = \frac{-b(b-a)}{n+2}\alpha_{n+1} + \frac{(b-a)^2(n+p)}{(n+2)(n+1)}\alpha_n\ ,\ \forall n \in \mathbb{N}_0.$$

Corollary 6. *If* $lower(t) = a\sqrt{t+\gamma} + c_0$, $upper(t) = c_0$, $a < 0$, *then* $P(\mathcal{L}) = v(t,x)$, $v(t,x) = v_1\left(t, \frac{x-c_0-a\sqrt{t+\gamma}}{-a\sqrt{t+\gamma}}\right)$, $v_1(t,x) = v_2\left(\frac{1}{2}\ln(t+\gamma), x\right)$, $v_2(t,x) = v_3\left(-t+\frac{1}{2}\ln(T+\gamma), x\right)$, $v_3(t,x) = L^{-1}[V(p,x)]$, *where* $V(p,x) = W(p, 1-x)$, $W(p,x) = c \,.\, \sum_{m=0}^{\infty} \beta_m x^{2m+1}$, $c = \left(p \,.\, \sum_{m=0}^{\infty} \beta_m\right)^{-1}$,

$$\beta_0 = 1,\ \beta_m = \frac{(-a)^{2m}}{(2m+1)!}\prod_{\widetilde{m}=1}^{m}(2\widetilde{m}-1+p)\ ,\ \forall m \in \mathbb{N}.$$

4. Numerical experiments

In order to return to the original formulation of the problem ($B_0 = 0$, i.e. starting moment $= 0$, initial position $= 0$), one has to put $t = 0$ and $x = 0$ in our formulae; for square-root boundaries γ must be positive.

The formulae were programmed and tabulated. The results were compared with the values of the crossing probabilities calculated by means of the Monte Carlo method and dynamical programming. The idea of the last method is to calculate the values of $v(t,x)$, beginning from the horizon and moving to the starting moment step by step; for each t an array of values of $v(t,x)$ is calculated using the array of the preceding step.

The three results agree with one another. So we have a numerical confirmation of our formulae besides the theoretical one. The algorithm that uses the formulae is the fastest one.

5. Conclusion

The propositions above give a comprehensive answer to the question about the crossing probabilities in two special cases: rectilinear parallel boundaries and square-root boundaries with a time horizon. The obtained formulae are suitable for programming: numerical calculations based on them are fast enough.

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