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SOME PROBABILISTIC RESULTS IN A BISEXUAL BRANCHING PROCESS WITH IMMIGRATION

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A bisexual branching process with immigration of females and males is introduced. It is allowed, in each generation, that the mating function and the probability distributions associated to the offspring and the immigration may change depending on the number of progenitor couples. Relationships among the probability generating functions involved in the model and some transition and stochastic monotony properties are established.

1. Introduction

From the bisexual process investigated in [1], new discrete time two–sex branching models have been developed. We refer the reader to [4], or [3], for surveys about these processes. Recently, a general continuous time bisexual process has been also introduced in [7]. However, in order to describe suitably the probabilistic evolution of populations where females and males coexist and form couples (female–male mating units), the range of bisexual models investigated is not large enough. For example, in some populations it is reasonable to assume an individual's mating behaviour dependent on the number of their progenitor couples. It might seem conceivable that by environmental, social, or other factors, the same number of females and males gives rise to different number of couples in different

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generations. By similar reasons, the offspring law may be also influenced for the number of couples in the population. In an attempt to improve the probabilistic modelling in such populations, in this work we introduce and investigate a discrete time bisexual branching process which considers, in each generation, the possibility of females and males immigration. Moreover, the function governing the mating and the offspring and immigration distributions may change depending on the number of couples at the population. The paper is structured as follows: In Section 2, we introduce the probability model, we provide its mathematical formal description, its intuitive interpretation and an illustrative example. Also, we establish and discuss some working assumptions. Section 3 is devoted to determining transition properties about the Markov chains associated to the process. In Section 4 we investigate some relationships among the probability generating functions involved in the model. As a consequence, some recursive expressions for the main moments are derived. Finally, Section 5 deals with the study of stochastic monotony properties.

2. The probability model

For $N \in \mathbb{Z}^+$, let $\{(f_{n,i}(N), m_{n,i}(N))\}_{n\geq 0; i\geq 1}$ and $\{(f^I_{n+1}(N), m^I_{n+1}(N))\}_{n\geq 0}$ be independent sequences. Each one of these sequences is formed by independent, identically distributed, non–negative and integer–valued random vectors. Let $\{L_N\}_{N\geq 0}$ be a sequence of functions, where each $L_N : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be monotonic non–decreasing in each argument, integer–valued on the integers, and such that $L_N(0, y) = L_N(x, 0) = 0,\ x, y \in \mathbb{R}^+$, where $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the non–negative integer and real numbers respectively. Under these conditions we define, recursively for $n \in \mathbb{Z}^+$, the following bisexual branching process with females and males immigration:

$$(F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{n,i}(Z_n), m_{n,i}(Z_n)) + (f^I_{n+1}(Z_n), m^I_{n+1}(Z_n)),$$

$$Z_0 = N_0 \geq 1, \quad Z_{n+1} = L_{Z_n}(F_{n+1}, M_{n+1})\ . \tag{1}$$

The process starts with N_0 couples and from an intuitive viewpoint, if $Z_n = N$ for some $n \geq 1$, then $(f_{n,i}(N), m_{n,i}(N))$ represents the number of females and males descending from the ith couple of the nth generation. Obviously $P(f_{n,i}(0) = 0, m_{n,i}(0) = 0) = 1$. The vector $(f^I_{n+1}(N), m^I_{n+1}(N)$ denotes the number of immigrant females and males in the $(n+1)$th generation. The probability laws associated to $(f_{0,1}(N), m_{0,1}(N))$ and $(f^I_1(N), m^I_1(N))$ will be referred as the offspring and the immigration distributions, respectively. It follows that (F_{n+1}, M_{n+1}) is the number of females and males in the $(n+1)$th generation, which form Z_{n+1} couples according to the mating function L_N. It is easy to verify that $\{(Z_{n-1}, F_n, M_n)\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 0}$ are homogeneous Markov chains. It is worth pointing out that in addition to its theoretical interest, the bisexual process introduced in this work has several practical implications in population dynamics. In particular, in phenomena concerning to inhabit or re–inhabit environments with animal species which have sexual reproduction, the probabilistic evolution of the number of females, males, and couples in the population may be described in term of this model. Indeed, the motivation behind the process introduced in (1) is the interest in developing mathematical models to describe such situations.

Remark 2.1. *Notice that the process (1) includes several models studied in the bisexual branching process literature; see for example [1], [2], [5], [6], or [8].*

Example 2.1. *Given that $Z_n = z$, let us consider a process (1) with offspring law the trinomial distribution $M(3;\ p_1(z),\ p_2(z))$, where*

$$p_1(z) = 0.35(z+1)(z+4)^{-1} \quad \textit{and} \quad p_2(z) = 0.35(z+1)(z+6)^{-1},\ z = 1, 2, \ldots$$

and immigration law, the product of two independent Poisson distributions with means $\lambda_f(z)$ and $\lambda_m(z)$ given by:

$$\lambda_f(z) = 2.4(z+1)(z+10)^{-1} \quad \textit{and} \quad \lambda_m(z) = 2.5(z+1)(z+10)^{-1},\ z \in \mathbb{Z}^+.$$

As mating function we have considered:

$$L_z(x, y) = \min\{x, \lfloor 3yz(1+z)^{-1} \rfloor\},\ z \in \mathbb{Z}^+\ , \tag{2}$$

where $\lfloor z \rfloor$ denotes the integer part of z. Assuming that the process starts with $Z_0 = 8$ couples, we have simulated 50 generations (see Figures 1 and 2).

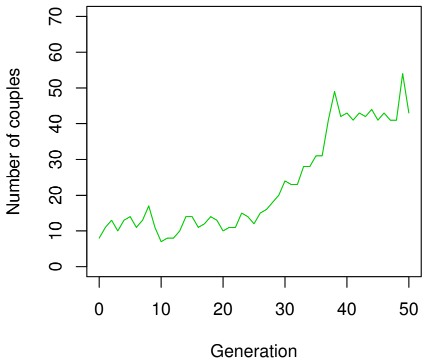

Figure 1. Evolution of the number of couples.

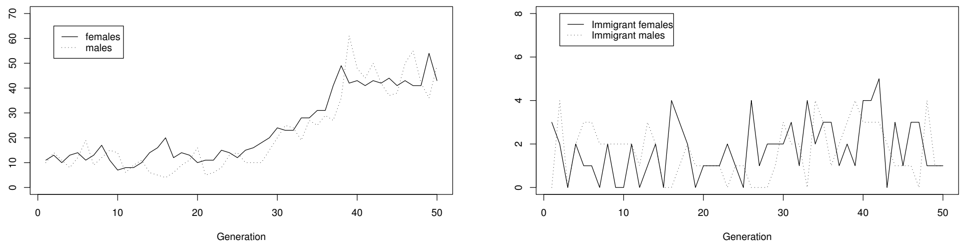

Figure 2. Evolution of the number of females and males (on the left) and immigrant females and males (on the right).

With the objective to derive probabilistic results about (1) we introduce the following working assumptions on the sequence of mating functions and the offspring and immigration distributions.

(A1): $\{L_N\}_{N\geq 0}$ is such that each L_N is a superadditive function, namely, for $x_i, y_i \in \mathbb{R}^+,\ i = 1, 2, \quad L_N(x_1 + x_2, y_1 + y_2) \geq L_N(x_1, y_1) + L_N(x_2, y_2)$.

(A2): For $x, y \in \mathbb{R}^+$ fixed, $\{L_N(x, y)\}_{N\geq 0}$ is non–decreasing.

(A3): $\{f_{0,1}(N)\}_{N\geq 0}$, $\{m_{0,1}(N)\}_{N\geq 0}$, $\{f_1^I(N)\}_{N\geq 0}$ and $\{m_1^I(N)\}_{N\geq 0}$ are non–decreasing sequences.

Remark 2.2. *Assumption* $(A1)$ *expresses the following intuitive behaviour:* x_1+x_2 *females and* y_1+y_2 *males coexisting together will form a number of couples greater than or equal to the total number of couples produced from* x_1 *females and* y_1 *males, and from* x_2 *females and* y_2 *males, living separately. Superadditivity is not a serious restriction, most of the mating functions considered in the bisexual branching process theory are superadditive. Assumption* $(A2)$ *represents the usual fact in many biological situations which the number of matings in certain generation depends on the number of couples in the previous one in such a way that the mating is promoted when the number of progenitor couples grows. Some sequences* $\{L_N\}_{N\geq 0}$ *verifying assumptions* $(A1)$ *and* $(A2)$ *are for example: (a)* $L_N(x,y) = x\min\{N,y\}$*; (b)* $L_N(x,y)=\min\{x,Ny\}$*; or (c)* $L_N(x,y)=\min\{x,y\}$ *if* $N\leq k_0$ *or* $x\min\{N,y\}$ *if* $N>k_0$ *(restricted to non–negative integers) where* k_0 *is a fixed positive integer. Finally, assumption* $(A3)$ *consider reproduction and immigration conducts such that the offspring per couple and the females and males immigration are encouraged when the number of couples grows.*

3. Transition properties

Next, we study some transition properties about the Markov chains $\{Z_n\}_{n\geq 0}$ and $\{(Z_{n-1},F_n,M_n))\}_{n\geq 1}$. By simplicity, for $N,h,j,k,l\in\mathbb{Z}^+$, we will denote by $p_{h,j}(N)=P(f_{0,1}(N)=h, m_{0,1}(N)=j)$ and $q_{k,l}(N)=P(f_1^I(N)=k, m_1^I(N)=l)$ the offspring and immigration distributions respectively. We assume that $q_{k,l}(0)>0$ for some $(k,l)\neq(0,0)$ and that, given $h,j,k,l\in\mathbb{Z}^+$, if for some $N_1,N_2\in\mathbb{Z}^+$, $p_{h,j}(N_1)>0$ and $q_{k,l}(N_2)>0$ then, $p_{h,j}(N_1')>0$ and $q_{k,l}(N_2')>0$, $N_i'>N_i$, $i=1,2$. Also, let us write by S and T the state spaces of $\{Z_n\}_{n\geq 0}$ and $\{(Z_{n-1},F_n,M_n))\}_{n\geq 1}$ respectively, and for $k\in\mathbb{Z}^+$, $S_k=\{j\in S: P(Z_{n+m}=j\mid Z_n=k)>0 \text{ for some } m\geq 1\}$.

Proposition 1. *Assume* $(A3)$ *and that* L_N *is increasing in any of its arguments,* $N\in\mathbb{Z}^+$. *If there exists* $h,j>0$ *such that* $p_{h,j}(1)>0$ *and* $k,l\in\mathbb{Z}^+$ *such that* $q_{k,l}(0)L_0(k,l)>0$ *then, for any* $(N,f,m)\in T$*, there exists* $(N^*,f^*,m^*)\in T$ *with* $L_{N^*}(f^*,m^*)\geq L_N(f,m)$ *such that* $(0,k,l)$ *leads to* (N^*,f^*,m^*).

P r o o f. Since $L_0(k,l)>0$ one deduces that $k,l>0$. Let us define the sequences $\{(m_n,h_n,j_n)\}_{n\geq 0}$ and $\{\kappa_n\}_{n\geq 0}$ as follows:

$$(m_0,h_0,j_0)=(0,k,l),\quad \kappa_0=L_{m_0}(h_0,j_0)=L_0(k,l)\geq 1$$

$$(m_{n+1},h_{n+1},j_{n+1})=\kappa_n(0,h,j)+(\kappa_n,k,l)\ ,\ \kappa_{n+1}=L_{\kappa_n}(h_{n+1},j_{n+1}),\ n\in\mathbb{Z}^+.$$

The sequence $\{\kappa_n\}_{n\geq 0}$ is increasing. Indeed, by using the induction procedure and considering $(A3)$

$$\kappa_1 = L_{\kappa_0}(h_1, j_1) = L_{\kappa_0}(\kappa_0 h + k, \kappa_0 j + l) > L_{\kappa_0}(k,l) \geq L_0(k,l) = \kappa_0.$$

Suppose that $\kappa_n > \kappa_{n-1}$, then using again $(A3)$

$$\begin{aligned} \kappa_{n+1} &= L_{\kappa_n}(\kappa_n h + k, \kappa_n j + l) > L_{\kappa_n}(\kappa_{n-1} h + k, \kappa_{n-1} j + l) \\ &\geq L_{\kappa_{n-1}}(\kappa_{n-1} h + k, \kappa_{n-1} j + l) = \kappa_n. \end{aligned}$$

In consequence, $\{\kappa_n\}_{n\geq 0}$ tends to infinity and, given $(N, f, m) \in T$, there exists n such that $\kappa_n \geq L_N(f,m)$. To complete the proof it is sufficient to prove that $(0,k,l)$ leads to (m_n, h_n, j_n) for any n. Now,

$$\begin{aligned} & P\left((Z_{n-1}, F_n, M_n) = (m_n, h_n, j_n) \mid (Z_0, F_1, M_1) = (0,k,l)\right) \\ \geq & \prod_{i=1}^{n-1} P\left((Z_i, F_{i+1}, M_{i+1}) = (m_{i+1}, h_{i+1}, j_{i+1}) \mid (Z_{i-1}, F_i, M_i) = (m_i, h_i, j_i)\right) \\ \geq & \prod_{i=1}^{n-1} (p_{h,j}(\kappa_i))^{\kappa_i} q_{k,l}(\kappa_i) > 0. \end{aligned}$$

□

Proposition 2. *Assume conditions in Proposition 1 Then, given $N \in S$, there exists $N^* > N$, $N^* \in S$ such that $N^* \in S_{\kappa_0}$ where $\kappa_0 = L_0(k,l)$.*

Proof. Consider the sequences $\{(m_n, h_n, j_n)\}_{n\geq 0}$ and $\{\kappa_n\}_{n\geq 0}$ defined in Proposition 1 I was proved that $\lim_{n\nearrow\infty} \kappa_n = \infty$. Consequently, given $N \in S$, there exists n such that $\kappa_n = L_{\kappa_{n-1}}(h_n, j_n) > N$. It is sufficient to verify that $\kappa_0 = L_0(k,l)$ leads to κ_n for any n. In fact

$$P\left(Z_{\delta+n} = \kappa_n \mid Z_\delta = \kappa_0\right) \geq \prod_{i=1}^{n} (p_{h,j}(\kappa_{i-1}))^{\kappa_{i-1}} q_{k,l}(\kappa_{i-1}) > 0.$$

□

Proposition 3. *Assume conditions in Proposition 1 If $p_{0,0}(1) > 0$, then any state $(N, f, m) \in T$ such that $L_N(f,m) \geq 1$ leads to the state (N^*, k, l) in one step, where $N^* = L_N(f,m)$.*

P r o o f. The result follows from the fact that

$$\begin{aligned} & P\left((Z_n, F_{n+1}, M_{n+1}) = (N^*, k, l) \mid (Z_{n-1}, F_n, M_n) = (N, f, m)\right) \\ = \ & P\left(\sum_{i=1}^{N^*} f_{n,i}(N^*) + f_{n+1}^I(N^*) = k, \sum_{i=1}^{N^*} m_{n,i}(N^*) + m_{n+1}^I(N^*) = l\right) \\ \geq \ & q_{k,l}(N^*)(p_{0,0}(N^*))^{N^*} > 0. \end{aligned}$$

□

4. Probability generating functions and moments

In this section we determine some relationships among the probability generating functions associated to the variables involved in the model (1). As consequence, we derive some recursive expressions for the main moments. For $n, N \in \mathbb{Z}^+$ and $s, t \in [0, 1]$ let us denote by

$$h_{n+1}(s,t) = E[s^{F_{n+1}} t^{M_{n+1}}], \quad \varphi_N(s,t) = E[s^{f_{0,1}(N)} t^{m_{0,1}(N)}] \quad (\varphi_0(s,t) = 1)$$

and $\varphi_N^I(s,t) = E[s^{f_1^I(N)} t^{m_1^I(N)}]$.

Proposition 4. *For $n \in \mathbb{Z}^+$ and $s, t \in [0, 1]$*

$$h_{n+1}(s,t) = E\left[(\varphi_{Z_n}(s,t))^{Z_n} \varphi_{Z_n}^I(s,t)\right].$$

P r o o f.

$$\begin{aligned} h_{n+1}(s,t) &= E\left[s^{F_{n+1}} t^{M_{n+1}}\right] E\left[\ E[s^{F_{n+1}} t^{M_{n+1}} \mid Z_n]\ \right] \\ &= \sum_{N=0}^{\infty} E\left[s^{\sum_{i=1}^{N} f_{n,i}(N) + f_{N+1}^I(N)} t^{\sum_{i=1}^{N} m_{n,i}(N) + m_{N+1}^I(N)} \right] P(Z_n = N) \\ &= \sum_{N=0}^{\infty} \left(E\left[s^{f_{0,1}(N)} t^{m_{0,1}(N)}\right]\right)^N E\left[s^{f_1^I(N)} t^{m_1^I(N)}\right] P(Z_n = N) \\ &= E\left[(\varphi_{Z_n}(s,t))^{Z_n} \varphi_{Z_n}^I(s,t)\right], \quad n \in \mathbb{Z}^+. \end{aligned}$$

□

Let us write, for $N, n \in \mathbb{Z}^+$

$$\mu(N) = E\left[(f_{0,1}(N), m_{0,1}(N))\right], \quad \Sigma(N) = Cov[(f_{0,1}(N), m_{0,1}(N))],$$

$$\mu^I(N) = E\left[(f_1^I(N), m_1^I(N))\right], \quad \Sigma^I(N) = Cov[(f_1^I(N), m_1^I(N))],$$

$$\mu_{n+1} = E\left[(F_{n+1}, M_{n+1})\right], \quad \Sigma_{n+1} = Cov[(F_{n+1}, M_{n+1})].$$

As a direct consequence of Proposition 4, one deduces for $n \in \mathbb{Z}^+$

$$\mu_{n+1} = E\left[Z_n\mu(Z_n) + \mu^I(Z_n)\right] \quad \text{and}$$

$$\Sigma_{n+1} = E\left[Z_n\Sigma(Z_n) + Var\left[Z_n\mu(Z_n)^t\mu(Z_n)\right] + E[\Sigma^I(Z_n)]\right].$$

Let us consider, for $N, n \in \mathbb{Z}^+$ and $s, t \in [0,1]$, the probability generating functions $\phi_N(s) = E[s^{L_{0,1}(N)}]$, $\phi_N^I(s) = E[s^{L_1^I(N)}]$, $g_n(s) = E[s^{Z_n}]$ and $g_n^*(s,t) = E[s^{Z_n^*}t^{Z_n}]$ where $L_{0,1}(N) = L_N(f_{0,1}(N), m_{0,1}(N))$, $L_1^I(N) = L_N(f_1^I(N), m_1^I(N))$ and $Z_n^* = \sum_{k=0}^{n} Z_k$. Clearly $g_0(s) = s^{N_0}$ and $g_0^*(s,t) = (st)^{N_0}$, $s, t \in [0,1]$.

Proposition 5. *Assume (A1). Then, for $s, t \in [0,1]$ and $n \in \mathbb{Z}^+$,*

(i) $g_{n+1}(s) \leq E\left[(\phi_{Z_n}(s))^{Z_n}\, \phi_{Z_n}^I(s)\right]$,

(ii) $g_{n+1}^*(s,t) \leq E\left[s^{Z_n^*}(\phi_{Z_n}(st))^{Z_n}\phi_{Z_n}^I(st)\right]$.

Proof. Taking into account $(A1)$ one has for $n \in \mathbb{Z}^+$ and $s \in [0,1]$

$$\begin{aligned} g_{n+1}(s) &= \sum_{N=0}^{\infty} E\left[s^{L_N\left(\sum_{i=1}^{N} f_{n,i}(N) + f_{n+1}^I(N),\, \sum_{i=1}^{N} m_{n,i}(N) + m_{n+1}^I(N)\right)}\right] P(Z_n = N) \\ &\leq \sum_{N=0}^{\infty} E\left[s^{\sum_{i=1}^{N} L_{n,i}(N) + L_{n+1}^I(N)}\right] P(Z_n = N) \\ &= \sum_{N=0}^{\infty} (\phi_N(s))^N\, \phi_N^I(s) P(Z_n = N) = E\left[(\phi_{Z_n}(s))^{Z_n}\phi_{Z_n}^I(s)\right] \end{aligned}$$

and therefore (i) holds.

Using again $(A1)$, one deduces for $n \in \mathbb{Z}^+$ and $s, t \in [0,1]$

$$E\left[(st)^{Z_{n+1}} \mid Z_n\right] \leq (\phi_{Z_n}(st))^{Z_n}\, \phi_{Z_n}^I(st) \quad a.s.$$

Hence, denoting by $\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n)$, one obtains

$$\begin{aligned} g_{n+1}^*(s,t) &= E\left[s^{Z_{n+1}^*}t^{Z_{n+1}}\right] E\left[E\left[s^{Z_n^*}(st)^{Z_{n+1}}\middle|\ \mathcal{F}_n\right]\right] \\ &= E\left[s^{Z_n^*}E[(st)^{Z_{n+1}}|\ Z_n]\right] \leq E\left[s^{Z_n^*}\,(\phi_{Z_n}(st))^{Z_n}\, \phi_{Z_n}^I(st)\right]. \end{aligned}$$

which completes the proof. □

Note that, from Proposition 5, one derives for $n \in \mathbb{Z}^+$

$$E[Z_{n+1}] \geq E\left[\ Z_n\lambda(Z_n) + \lambda^I(Z_n)\ \right], \tag{3}$$

$$E[Z^*_{n+1}] \geq \sum_{k=0}^{n+1} E\left[\ Z_k\lambda(Z_k) + \lambda^I(Z_k)\ \right], \tag{4}$$

where $\lambda(N) = E[L_{0,1}(N)]$ and $\lambda^I(N) = E[L_1^I(N)]$, $N \in \mathbb{Z}^+$.

Proposition 6. *Assume (A1), $L_{0,1}(N) \geq L_{0,1}(1)$ and $L_1^I(N) \geq L_1^I(0)$, $N = 1, 2, \ldots$ Then, for $s, t \in [0,1]$ and $n \in \mathbb{Z}^+$*

(i) $g_{n+1}(s) \leq g_n(\phi_1(s))\phi_0^I(s)$,

(ii) $g^*_{n+1}(s,t) \leq g^*_n(s, \phi_1(st))\phi_0^I(st)$.

Proof. In order to verify (i), notice that, from the hypotheses, one deduces that $\phi_N(s) \leq \phi_1(s)$ and $\phi_N^I(s) \leq \phi_0^I(s)$, $s \in [0,1]$. Hence, taking into account Proposition 5 (i) and the fact that $\phi_0(s) = 1$, one obtains for $n \in \mathbb{Z}^+$ and $s, t \in [0,1]$

$$\begin{aligned} g_{n+1}(s) &\leq \sum_{N=0}^{\infty} (\phi_N(s))^N \phi_N^I(s) P(Z_n = N) \\ &= \sum_{N=1}^{\infty} (\phi_N(s))^N \phi_N^I(s) P(Z_n = N) + \phi_0^I(s) P(Z_n = 0) \\ &\leq \phi_0^I(s) \sum_{N=0}^{\infty} (\phi_1(s))^N P(Z_n = N) \phi_0^I(s) g_n(\phi_1(s)). \end{aligned}$$

(ii) is proved in a similar manner using the fact that

$$g^*_{n+1}(s,t) \leq \sum_{N=0}^{\infty} s^{(Z^*_{n-1}+N)} (\phi_N(st))^N \phi_N^I(st) P(Z_n = N).$$

□

As a consequence of Proposition 6, (3) and (4), it is matter of straightforward calculation to deduce the following inequalities:

$$E\left[Z_{n+1}\right] \geq (N_0 + \lambda^I(0)(n+1))\mathbf{1}_{\{\lambda(1)=1\}} + B_n(\lambda(1))\mathbf{1}_{\{\lambda(1)\neq 1\}}$$

with $B_n(a) = (1-a)^{-1}\left[a^{n+1}(N_0(1-a) - \lambda^I(0)) + \lambda^I(0)\right]$ and

$$E\left[Z^*_{n+1}\right] \geq ((n+2)[N_0 + \lambda^I(0)(n+1)2^{-1}])\mathbf{1}_{\{\lambda(1)=1\}} + C_n(\lambda(1))\mathbf{1}_{\{\lambda(1)\neq 1\}}$$

with

$$C_n(a) = (1-a)^{-2}\left[(1-a^{n+2})(N_0(1-a) - \lambda^I(0)) + \lambda^I(0)(1-a)(n+2)\right],$$

where $\mathbf{1}_A$ denotes the indicator function of A.

5. Stochastic monotony

We now provide some results about the stochastic monotony of the sequences $\{Z_n\}_{n\geq 0}$, $\{F_n\}_{n\geq 1}$, and $\{M_n\}_{n\geq 1}$. The first result establishes that $\{Z_n\}_{n\geq 0}$ is a stochastically monotone sequence.

Proposition 7. *Assume (A2) and (A3). Then, given $N_1, N_2 \in \mathbb{Z}^+$ with $N_1 < N_2$,*

$$P\left(Z_{n+1} \leq y \mid Z_n = N_2\right) \leq P\left(Z_{n+1} \leq y \mid Z_n = N_1\right), \quad y \in \mathbb{R}, \quad n \in \mathbb{Z}^+ .$$

Proof. Taking into account (*A*2), (*A*3) and the fact that the mating functions are non–decreasing in each argument, one has

$$\begin{aligned} & P(Z_{n+1} > y \mid Z_n = N_2) \\ = \; & P\left(L_{N_2}\left(\sum_{i=1}^{N_2} f_{n,i}(N_2) + f^I_{n+1}(N_2), \sum_{i=1}^{N_2} m_{n,i}(N_2) + m^I_{n+1}(N_2)\right) > y\right) \\ \geq \; & P\left(L_{N_1}\left(\sum_{i=1}^{N_2} f_{n,i}(N_2) + f^I_{n+1}(N_2), \sum_{i=1}^{N_2} m_{n,i}(N_2) + m^I_{n+1}(N_2)\right) > y\right) \\ \geq \; & P\left(L_{N_1}\left(\sum_{i=1}^{N_1} f_{n,i}(N_1) + f^I_{n+1}(N_1), \sum_{i=1}^{N_1} m_{n,i}(N_1) + m^I_{n+1}(N_1)\right) > y\right) \\ = \; & P(Z_{n+1} > y \mid Z_n = N_1). \end{aligned}$$

□

Let $\left\{\{(F_n^{(i)}, M_n^{(i)})\}_{n\geq 1}\right\}_{i\geq 1}$ and $\left\{\{Z_n^{(i)}\}_{n\geq 0}\right\}_{i\geq 1}$ independent versions of $\{(F_n, M_n)\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 0}$ respectively with $Z_0^{(i)} = 1,\ i = 1, 2, \ldots$.

Proposition 8. *Assume (A1)–(A3). If* $f_1^I(\sum_{j=1}^l N_j) \geq \sum_{j=1}^l f_1^I(N_j)$ *and* $m_1^I(\sum_{j=1}^l N_j) \geq \sum_{j=1}^l m_1^I(N_j)$, $N_j \in \mathbb{Z}^+$, $l = 1, 2, \ldots$ *then, for* $n, k \in \mathbb{Z}^+$ *and* $y \in \mathbb{R}$,

$$\text{(i)}\ P(Z_{k+n+1} \leq y) \leq P\left(\sum_{i=1}^{Z_k} Z_{n+1}^{(i)} \leq y\right),$$

$$\text{(ii)}\ P(F_{k+n+1} \leq y) \leq P\left(\sum_{i=1}^{Z_k} F_{n+1}^{(i)} \leq y\right),$$

$$\text{(iii)}\ P(M_{k+n+1} \leq y) \leq P\left(\sum_{i=1}^{Z_k} M_{n+1}^{(i)} \leq y\right).$$

Proof. First, we prove that for $n, k, N \in \mathbb{Z}^+$ and $y \in \mathbb{R}$

$$(5) \qquad P\left(Z_{k+n+1} \leq y \mid Z_{n+k} = N\right) \leq P\left(\sum_{i=1}^{Z_k} Z_{n+1}^{(i)} \leq y \;\middle|\; \sum_{i=1}^{Z_k} Z_n^{(i)} = N\right).$$

Now, by using the simplified notation $F_n^{(i)}(N) = \sum_{j=1}^{N} f_{n,j}^{(i)}(N) + f_{n+1}^{(i)\,I}(N)$ and $M_n^{(i)}(N) = \sum_{j=1}^{N} m_{n,j}^{(i)}(N) + m_{n+1}^{(i)\,I}(N)$, one deduces

$$\begin{aligned}
& P\left(\sum_{i=1}^{Z_k} Z_{n+1}^{(i)} \leq y \;\middle|\; \sum_{i=1}^{Z_k} Z_n^{(i)} = N\right) \\
= \;& P\left(\sum_{i=1}^{Z_k} \left(L_{Z_n^{(i)}}\left(F_n^{(i)}(Z_n^{(i)}), M_n^{(i)}(Z_n^{(i)})\right)\right) \leq y \;\middle|\; \sum_{i=1}^{Z_k} Z_n^{(i)} = N\right) \\
\geq \;& P\left(\sum_{i=1}^{Z_k} \left(L_{\sum_{i=1}^{Z_k} Z_n^{(i)}}\left(F_n^{(i)}(Z_n^{(i)}), M_n^{(i)}(Z_n^{(i)})\right)\right) \leq y \;\middle|\; \sum_{i=1}^{Z_k} Z_n^{(i)} = N\right) \\
\geq \;& P(Z_{k+n+1} \leq y \mid Z_{n+k} = N).
\end{aligned}$$

We now prove (i) by induction on n. Note that, for $n = 0$, using that $Z_0^{(i)} = 1$ one obtains that $P(Z_k \leq y) = P(\sum_{i=1}^{Z_k} Z_0^{(i)} \leq y)$. Suppose that $P(Z_{n+k} \leq y) \leq P(\sum_{i=1}^{Z_k} Z_n^{(i)} \leq y)$. Then, takin into account that $\{P(Z_{n+k+1} \leq y \mid Z_{n+k} =$

$N)\}_{N\geq 0}$ is non–increasing, the induction hypothesis, (5), and Lemma A1 (see Appendix), one obtains

$$\begin{aligned} P(Z_{n+k+1}\leq y) &= \sum_{N=0}^{\infty} P(Z_{n+k+1}\leq y \mid Z_{n+k}=N)P\left(Z_{n+k}=N\right) \\ &\leq \sum_{N=0}^{\infty} P(Z_{n+k+1}\leq y \mid Z_{n+k}=N)P\left(\sum_{i=1}^{Z_k} Z_n^{(i)}=N\right) \\ &\leq \sum_{N=0}^{\infty} P\left(\sum_{i=1}^{Z_k} Z_{n+1}^{(i)}\leq y \,\middle|\, \sum_{i=1}^{Z_k} Z_n^{(i)}=N\right)P\left(\sum_{i=1}^{Z_k} Z_n^{(i)}=N\right) \\ &= P\left(\sum_{i=1}^{Z_k} Z_{n+1}^{(i)}\leq y\right). \end{aligned}$$

To prove (ii), note that for $n,k,N\in\mathbb{Z}^+$ and $y\in\mathbb{R}$,

$$P\left(F_{k+n+1}\leq y \mid Z_{n+k}=N\right)\leq P\left(\sum_{i=1}^{Z_k} F_{n+1}^{(i)}\leq y \,\middle|\, \sum_{i=1}^{Z_k} Z_n^{(i)}=N\right). \tag{6}$$

In fact, from the requirements in Proposition,

$$\begin{aligned} & P\left(\sum_{i=1}^{Z_k} F_{n+1}^{(i)}\leq y \,\middle|\, \sum_{i=1}^{Z_k} Z_n^{(i)}=N\right) \\ \geq\; & P\left(\sum_{i=1}^{Z_k}\sum_{j=1}^{Z_n^{(i)}} f_{n,j}(Z_n^{(i)})+f_{n+1}^{I}\Big(\sum_{i=1}^{Z_k} Z_n^{(i)}\Big)\leq y \,\middle|\, \sum_{i=1}^{Z_k} Z_n^{(i)}=N\right) \\ =\; & P\left(\sum_{i=1}^{N} f_{n,i}(Z_n^{(1)})+f_{n+1}^{I}(N)\leq y\right)\geq P\left(\sum_{i=1}^{N} f_{n,i}(N)+f_{n+1}^{I}(N)\leq y\right) \\ =\; & P\left(\sum_{i=1}^{N} f_{k+n,i}(N)+f_{k+n+1}^{I}(N)\leq y\right)=P(F_{k+n+1}\leq y \mid Z_{n+k}=N). \end{aligned}$$

Now, considering (i), the fact that the sequence $\{P(F_{k+n+1}\leq y \mid Z_{n+k}=N)\}_{N\geq 0}$ is non–increasing, Lemma A1, and (6),

$$\begin{aligned}
P(F_{k+n+1} \le y) &= \sum_{N=0}^{\infty} P(F_{k+n+1} \le y \mid Z_{n+k} = N) P(Z_{n+k} = N) \\
&\le \sum_{N=0}^{\infty} P(F_{k+n+1} \le y \mid Z_{n+k} = N) P(\sum_{i=1}^{Z_k} Z_n^{(i)} = N) \\
&\le \sum_{N=0}^{\infty} P\left(\sum_{i=1}^{Z_k} F_{n+1}^{(i)} \le y \,\middle|\, \sum_{i=1}^{Z_k} Z_n^{(i)} = N \right) P\left(\sum_{i=1}^{Z_k} Z_n^{(i)} = N \right) \\
&= P\left(\sum_{i=1}^{Z_k} F_{n+1}^{(i)} \le y \right).
\end{aligned}$$

Considering a similar reasoning is proved (iii). □

Appendix

Lemma A1. Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n), (u_1, \ldots, u_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i, \ k = 1, \ldots, n$ and $u_1 \ge \ldots \ge u_n \ge 0$. Then $\sum_{i=1}^{n} u_i x_i \le \sum_{i=1}^{n} u_i y_i$.

Proof. Let $t_i = \sum_{j=1}^{i} x_j$ and $s_i = \sum_{j=1}^{i} y_j$, $i = 1, \ldots, n$. It is clear that $t_i \le s_i, \ i = 1, \ldots, n$. It is sufficient to verify that

$$\sum_{i=1}^{n-1} (u_i - u_{i+1}) t_i + u_n t_n \le \sum_{i=1}^{n-1} (u_i - u_{i+1}) s_i + u_n s_n.$$

Now, this inequality holds because $u_i - u_{i+1} \ge 0, \ i = 1, \ldots, n$ and $u_n \ge 0$. □

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