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LIMIT THEOREMS FOR MAXIMA OF HEAVY–TAILED TERMS WITH RANDOM DEPENDENT WEIGHTS

Stilian Stoev Murad S. Taqqu [1]

Let $U_j,\ j \in \mathbb{N}$ be independent identically distributed random variables with heavy, regularly varying tails. The theory about the limit behavior of the maxima $\vee_{j=1}^{n} U_j$, as $n \to \infty$ is well developed. Here, we consider a sequence of non–negative weights $W_j,\ j \in \mathbb{N}$ and focus on the weighted maxima

$$M_n(t) := \begin{cases} \bigvee_{j=1}^{[nt]} W_j U_j & , \ \text{if } 1/n \le t, \\ \\ W_1 U_1 & , \ \text{if } 0 \le t < 1/n, \end{cases}$$

where the sequences $\{U_j\}_{j\in\mathbb{N}}$ and $\{W_j\}_{j\in\mathbb{N}}$ are independent. We study the general case when the weights $W_j,\ j \in \mathbb{N}$ can be dependent and in particular long–range dependent. Under mild tail and convergence conditions on the weights W_js, we establish limit theorems for scaled versions of the process $\{M_n(t)\}_{t\ge 0}$, as $n \to \infty$. The limit processes are mixtures of extremal Fréchet processes. The results are valid when the laws of the U_j's belong to the normal domain of attraction of a Fréchet distribution or to a sub–class of the general domain of attraction of a Fréchet law.

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1. Introduction

Consider a sequence of independent and identically distributed random variables $U_j,\ j \in \mathbb{N}$ and let

$$M_n(t) = \begin{cases} \bigvee_{j=1}^{[nt]} U_j & ,\ 1/n \le t \\ U_1 & ,\ 0 \le t < 1/n \end{cases}$$

where $a \vee b$ denotes $\max\{a,b\}$ and $[x]$ denotes the integer part of $x \in \mathbb{R}$. The theory on the limit behavior of the cumulative maxima process $\{M_n(t),\ t \ge 0\}$, under appropriate centering and normalization, is well–developed (see, Fréchet [6], Fisher and Tippett [5], Gnedenko [8] and e.g. the monographs Cramer and Leadbetter [4], Galambos [7], and Resnick [10]).

A stochastic process $\mathcal{E} = \{\mathcal{E}(t)\}_{t\ge 0}$ is said to have independent max–increments, if for all $0 = t_0 < t_1 < \cdots < t_k,\ k \in \mathbb{N}$,

$$\{\mathcal{E}(t_j)\}_{j=1}^k \stackrel{d}{=} \Big\{ \bigvee_{i=1}^{j} \mathcal{E}(t_{i-1}, t_i) \Big\}_{j=1}^k, \tag{1}$$

where $\mathcal{E}(t_{j-1}, t_j),\ j = 1, \ldots, k$ are independent random variables, and where $\stackrel{d}{=}$ denotes equality in distribution. The random variables $\mathcal{E}(s,t),\ 0 \le s < t$ can be viewed as the max–increments of the process $\mathcal{E}$. When the max–increments $\mathcal{E}(s,t)$ are also homogeneous, that is, they have cumulative distribution functions (c.d.f.) $\mathbb{P}\{\mathcal{E}(s,t) \le x\} = G(x)^{t-s}$, for some c.d.f. $G(x),\ x \in \mathbb{R}$, then the process $\mathcal{E}$ becomes an extremal process with distribution function G. It is also called $G-$*extremal* process (see, e.g. Ch. 4.3 in Resnick [11] and the references therein). The extremal processes can be viewed as a counterpart to the processes with independent and stationary increments, when the addition is operation replaced by the maximum.

Consider now the maxima:

$$M_n(t) = \begin{cases} \bigvee_{j=1}^{[nt]} W_j U_j & ,\ 1/n \le t \\ W_1 U_1 & ,\ 0 \le t \le 1/n, \end{cases} \tag{2}$$

where the random variables U_j satisfy the condition:

$$\mathbb{P}\{U_1 > x\} = (c + o(1)) L(x) x^{-\alpha}, \quad \text{as } x \to \infty,\ \alpha, c > 0. \tag{3}$$

We suppose that the random variables $W_j \geq 0,\ j \in \mathbb{N}$, are *non–negative* and view them as *random weights.* Our goal is to establish sufficient conditions for the convergence of the maxima $M_n(t)$, as $n \to \infty$, to a non–trivial stochastic process, under appropriate normalization.

Chuprunov [3], addresses the case of weighted maxima with random and independent weights. Here, we focus on the situation when the weights $W_j,\ j \in \mathbb{N}$ can have a general dependence structure, in particular, *long–range dependence.* Our framework is similar to that of Stoev and Taqqu [13] which studied the behavior of weighted sums $\sum_{j=1}^{[nt]} W_j(U_j - \mu)$.

We suppose that the sequence $\{W_j\}$ is independent of the sequence $\{U_j\}$. To be able to view the W_j's in Relation (2) as random weights to the U_j's and not vice versa, we impose the following conditions on $\{W_j\}$. A uniform negligibility condition:

$$\max_{1\leq j\leq n} \frac{W_j^\alpha}{n} \xrightarrow{\mathrm{P}} 0 \tag{4}$$

and a convergence condition of the type:

$$\frac{1}{n}\sum_{j=1}^{n} W_j^\alpha \xrightarrow{\mathrm{P}} \xi^\alpha, \qquad \text{as } n \to \infty, \tag{5}$$

where ξ^α denotes a non–degenerate random variable or a constant.

For convenience, we formulate next a simple result, which follows from Theorem 9.1 in Gnedenko and Kolmogorov [9]. It shows that the above conditions are in fact quite mild.

Lemma 1. *(a) If the sequence* $\{W_j\}_{j\in\mathbb{N}}$ *is strictly stationary and* $\mathbb{E}W_1^\alpha < \infty$, *then Conditions* (4) and (5) hold.

(b) Condition (4) holds, if $\sup_{j\in\mathbb{N}} \mathbb{E}\varphi(W_j^\alpha) < \infty$, for some positive, non–decreasing function $\varphi(x),\ x \geq 0$, such that $x = o(\varphi(x)),\ x \to \infty$.

Example. If $\sup_{j\in\mathbb{N}} \mathbb{E}W_j^{\alpha(1+\delta)} < \infty$, for some $\delta > 0$, then Condition (4) holds. This follows from Lemma 1 *(b)* with $\varphi(x) := x^{1+\delta},\ \delta > 0$.

The paper is structured as follows. In Section 2., we consider the case when the U_j's belong to the *normal domain of max–attraction*, that is, when $L(\cdot) \equiv 1$. We show that, as $n \to \infty$, the process $\{n^{-1/\alpha} M_n(t)\}_{t\geq 0}$ converges weakly in the Skorokhod J_1-topology to a non–trivial limit, which is a *scale* mixture of extremal Fréchet processes (Theorem 1). Section 3. covers the case when the U_j's belong to the *domain of max–attraction.* We show that (Theorem 2) a similar limit result holds, under certain conditions relating the tail behavior of the weights W_j and the slowly varying function $L(\cdot)$ in (3).

2. Convergence of weighted maxima

Let $\mathcal{D}[0,\infty)$ denote the space of right–continuous functions with limits to the left, defined on the unbounded interval $[0,\infty)$. The classical Skorokhod [12] topologies can be extended to functions defined on unbounded intervals (see e.g. Bingham [1] and Whitt [14, 15]). We denote by $\Rightarrow^{\mathrm{J}_1}$ the weak convergence for processes with paths in $\mathcal{D}[0,\infty)$, with respect to the Skorokhod J_1–topology.

We need the following key result of Bingham [1], which we state here for convenience.

Theorem 3 in Bingham [1]: *Let* $\{X_n : n \in \mathbb{N}\}$ *be a sequence of stochastic processes whose path–functions lie in* $\mathcal{D}[0,\infty)$. *If*

i) *The finite–dimensional distributions of* X_n *converge as* $n \to \infty$ *to those of* X_∞.

ii) *The process* X_∞ *is continuous in probability.*

iii) *The processes* X_n *have monotone path–functions,*

then $X_n \Rightarrow^{\mathrm{J}_1} X_\infty$, *as* $n \to \infty$.

Theorem 1. *Assume that* (3) holds with $L(\cdot) \equiv 1$. Then the conditions (4) and (5) imply that

$$\left\{\frac{1}{n^{1/\alpha}} M_n(t)\right\}_{t\geq 0} \stackrel{\mathrm{J}_1}{\Longrightarrow} \{\xi \mathcal{E}_\alpha(t)\}_{t\geq 0}, \quad \text{as } n \to \infty, \tag{6}$$

where ξ is the random variable in (5) and $\mathcal{E}_\alpha = \{\mathcal{E}_\alpha(t)\}_{t\geq 0}$, is an independent of ξ extremal process with Fréchet distributions. That is, $\mathcal{E}_\alpha(0) := 0$ and, for all $t > 0$,

$$\mathbb{P}\{\mathcal{E}_\alpha(t) \leq x\} = \Phi_\alpha(x)^t = \Phi_\alpha(x/t^{1/\alpha}) := \begin{cases} \exp\{-tc_1 x^{-\alpha}\} & , \ x \geq 0, \\ 0 & , \ x < 0. \end{cases} \tag{7}$$

Proof. In view of Bingham's Theorem above, since the processes $\{M_n(t)\}_{t\geq 0}$ have non–decreasing paths, it is enough to prove that the limit in (6) holds in the sense of the finite–dimensional distributions.

We will first prove the convergence of the marginal distributions. It suffices to show that, for all $t > 0$ and $x \in \mathbb{R},\ x \neq 0$,

$$(8)\quad \lim_{n\to\infty} \mathbb{P}\Big\{\frac{1}{n^{1/\alpha}} M_n(t) \le x\Big\} = \mathbb{E}\Phi_\alpha(x/\xi)^t = \begin{cases} \mathbb{E}\exp\{-t\xi^\alpha c x^{-\alpha}\} & ,\ x > 0 \\ 0 & ,\ x < 0 \end{cases}$$

(see Lemma 2). Here $\Phi_\alpha(x/\xi)^t$ is interpreted as $1 = \Phi_\alpha(\infty)^t$ when $\xi = 0$ and $x \ge 0$ and as $0 = \Phi_\alpha(-\infty)^t$, when $\xi = 0$ and $x < 0$.

Let first $x < 0$ and observe that

$$(9)\qquad \mathbb{P}\Big\{\frac{1}{n^{1/\alpha}} M_n(t) \le x\Big\} \le \mathbb{P}\Big(\bigcap_{j=1}^{[nt]} \{U_j < 0\}\Big) \le F(0)^{[nt]} \longrightarrow 0,$$

as $n \to \infty$, since $\mathbb{P}\{U_1 \le 0\} = F(0) < 1$ and because $\max_{1\le j\le [nt]} W_j U_j < 0$ only if, $U_j < 0$, for all $j = 1, \ldots, [nt]$ (recall that $W_j \ge 0$). This implies (8).

Let now $x > 0$ be arbitrary. By the independence of the sequences $\{W_j\}_{j\in\mathbb{N}}$ and $\{U_j\}_{j\in\mathbb{N}}$,

$$\begin{aligned} \mathbb{P}\Big\{\bigvee_{j=1}^{[nt]} \frac{W_j U_j}{n^{1/\alpha}} \le x\Big\} &= \mathbb{E}\prod_{j=1}^{[nt]} \mathbb{P}\Big\{W_j U_j \le x n^{1/\alpha} \Big| \mathcal{F}_W\Big\} \\ (10)\qquad &= \mathbb{E}\Big(\prod_{j=1}^{[nt]} (1 - \overline{F}(x n^{1/\alpha}/W_j))\Big) =: \mathbb{E}\eta_n, \end{aligned}$$

where $\overline{F}(y) := 1 - F(y) = \mathbb{P}\{U_1 > y\},\ y \in \mathbb{R},\ \mathcal{F}_W = \sigma\{W_j,\ j \in \mathbb{N}\}$ is the $\sigma-$algebra generated by the random variables W_j and where the term $\overline{F}(xn^{1/\alpha}/W_j)$ is interpreted as $\overline{F}(\infty) = 0$, if $W_j = 0$. Observe that if $W_j = 0$, $\mathbb{P}\{W_j U_j \le x n^{1/\alpha}\} = 1$, because $x > 0$.

Since the random variables $\eta_n = \prod_{j=1}^{[nt]} (1 - \overline{F}(xn^{1/\alpha}/W_j)) \in [0, 1]$, in (10) are uniformly bounded in $n \in \mathbb{N}$, the convergence

$$(11)\qquad \eta_n \xrightarrow{\mathrm{P}} \eta, \quad \text{as } n \to \infty,$$

implies the convergence $\mathbb{E}\eta_n \to \mathbb{E}\eta,\ n \to \infty$. We will show that (11) holds with $\eta = \exp\{-t\xi^\alpha c x^{-\alpha}\}$, where ξ is as in (5).

We will first argue that the convergence in (11) follows from the convergence

$$(12)\qquad \sum_{j=1}^{[nt]} \overline{F}(x n^{1/\alpha}/W_j) \xrightarrow{\mathrm{d}} \zeta, \quad \text{as } n \to \infty,$$

where $\eta = \exp\{-\zeta\}$, and then we will establish that $\zeta = t\xi^\alpha cx^{-\alpha}$. Suppose then that (12) holds and let

$$\mu_n := \max_{1\le j\le [nt]} \overline{F}\Big(\frac{xn^{1/\alpha}}{W_j}\Big) \quad \text{and} \quad B_n := \{\omega \,:\, \mu_n(\omega) \le 1/2\}.$$

For all $\omega \in B_n$, we have that $\overline{F}(xn^{1/\alpha}/W_j(\omega)) \le \mu_n(\omega) \le 1/2$, and therefore $\ln(1-\overline{F}(xn^{1/\alpha}/W_j(\omega)))$ is well–defined. Thus, in view of (10), by taking a log and by applying the inequality $|\ln(1-y)+y| \le Cy^2,\ |y| \le 1/2$, valid for some $C > 0$, we obtain, for all $\omega \in B_n$,

$$\Big|\ln(\eta_n(\omega)) + \sum_{j=1}^{[nt]} \overline{F}\Big(\frac{xn^{1/\alpha}}{W_j(\omega)}\Big)\Big| \le C\sum_{j=1}^{[nt]} \overline{F}\Big(\frac{xn^{1/\alpha}}{W_j(\omega)}\Big)^2 \le C\mu_n(\omega)\sum_{j=1}^{[nt]} \overline{F}\Big(\frac{xn^{1/\alpha}}{W_j(\omega)}\Big). \tag{13}$$

By (4), and the fact that $\overline{F}(y) \downarrow 0,\ y \to \infty$, it follows that $\mu_n \to^{\mathrm{P}} 0$ and consequently $\mathbb{P}\{\mu_n \le 1/2\} = \mathbb{P}(B_n) \to 1$, as $n \to \infty$. These facts and Relation (12) imply that the left–hand side of (13) converges to 0 in probability over the event B_n and hence $\ln(\eta_n)1_{B_n} \to^{\mathrm{P}} -\zeta$, as $n \to \infty$, because of (13). By the continuity of the function $z \mapsto \exp(z),\ z \in \mathbb{R}$, the last relation implies that

$$\exp\{\ln(\eta_n)1_{B_n}\} = \eta_n + (1-\eta_n)1_{B_n^c} \xrightarrow{\mathrm{P}} \exp\{-\zeta\}, \quad \text{as } n \to \infty.$$

However, since $\mathbb{P}(B_n^c) \to 0,\ n \to \infty$, we have that $(1-\eta_n)1_{B_n^c} \to^{\mathrm{P}} 0,\ n \to \infty$, and therefore, the last relation implies (11). We have thus shown that (11) follows from (12).

We shall now obtain (12) with $\zeta = t\xi^\alpha cx^{-\alpha}$. Relation (3) with $L(\cdot) \equiv 1$ implies that for all $y > 0$,

$$|\overline{F}(y) - cy^{-\alpha}| \le g(y)cy^{-\alpha},$$

where $g(y) := \sup_{z\ge y} |\overline{F}(z)/cz^{-\alpha} - 1| \to 0$, as $y \to \infty$. The triangle inequality and the monotonicity of the function g, imply

$$\Big|\sum_{j=1}^{[nt]} \overline{F}\Big(\frac{xn^{1/\alpha}}{W_j}\Big) - \sum_{j=1}^{[nt]} \frac{cx^{-\alpha}W_j^\alpha}{n}\Big| \le \sum_{j=1}^{[nt]} cg\Big(\frac{xn^{1/\alpha}}{W_j}\Big)\Big(\frac{xn^{1/\alpha}}{W_j}\Big)^{-\alpha} \le g\Big(x\min_{1\le j\le [nt]}\frac{n^{1/\alpha}}{W_j}\Big)\sum_{j=1}^{[nt]} \frac{cx^{-\alpha}W_j^\alpha}{n}. \tag{14}$$

However, since $g(y) \downarrow 0,\ y \to \infty$, by (4) and (5), we have

$$g\Big(x \min_{1\le j\le [nt]} \frac{n^{1/\alpha}}{W_j}\Big) \xrightarrow{\mathrm{P}} 0, \quad \text{and} \quad \sum_{j=1}^{[nt]} \frac{cx^{-\alpha}W_j^{\alpha}}{n} \xrightarrow{\mathrm{d}} t\xi^{\alpha}cx^{-\alpha},$$

as $n \to \infty$. This, in view of Relation (14), implies that (12) holds with $\zeta = t\xi^{\alpha}cx^{-\alpha}$ and consequently (11) holds with $\eta = \exp\{-t\xi^{\alpha}cx^{-\alpha}\}$, which completes the proof of the convergence of the marginal distributions.

We now prove the convergence of the finite–dimensional distributions. Let $0 = t_0 < t_1 < t_2 < \cdots < t_k$, and $x_j \in \mathbb{R},\ x_j \neq 0,\ j = 1,\dots,k,\ k \in \mathbb{N}$. Introduce the random variables,

$$M_n(t,s) := \max_{[nt]+1\le j\le [ns]} W_jU_j,\ 0 \le t < s,$$

where max over an empty set is interpreted as $-\infty$. Observe that

$$\{M_n(t_j)\}_{j=1}^k = \Big\{\bigvee_{i=1}^{j} M_n(t_{i-1},t_i)\Big\}_{j=1}^k. \tag{15}$$

We will establish that

$$\Big\{\frac{1}{n^{1/\alpha}} M_n(t_{j-1},t_j)\Big\}_{j=1}^k \xrightarrow{\mathrm{d}} \{\xi\mathcal{E}_\alpha(t_{j-1},t_j)\}_{j=1}^k, \tag{16}$$

where ξ and $\mathcal{E}_\alpha(t_{j-1},t_j),\ j = 1,\dots,k$ are independent random variables, such that $\mathcal{E}_\alpha(t,s),\ 0 \le t < s$ has a c.d.f. $\Phi_\alpha(x/(s-t)^{1/\alpha}) = \Phi_\alpha(x)^{s-t}$ (see (7)). In view of (1) and (15), Relation (16) would imply $\{n^{-1/\alpha}M_n(t_j)\}_{j=1}^k \to^{\mathrm{d}} \{\xi\mathcal{E}_\alpha(t_j)\}_{j=1}^k$, $n \to \infty$, which would complete the proof of the finite–dimensional distributions.

By the independence of the sequences $\{W_j\}_{j\in\mathbb{N}}$ and $\{U_j\}_{j\in\mathbb{N}}$, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\frac{1}{n^{1/\alpha}} M_n(t_{j-1},t_j) \le x_j,\ j = 1,\dots,k\Big\} \\
&= \mathbb{E}\prod_{j=1}^{k}\ \prod_{i=[nt_{j-1}]+1}^{[nt_j]} \mathbb{P}\{W_iU_i \le x_j n^{1/\alpha} | \mathcal{F}_W\} \\
&=: \mathbb{E}\prod_{j=1}^{k} \eta_{n,j},
\end{aligned} \tag{17}$$

where $\eta_{n,j} := \prod_{i=[nt_{j-1}]+1}^{[nt_j]}(1 - \overline{F}(x_jn^{1/\alpha}/W_i)),\quad j = 1,\dots,n$. As in the proof of the convergence of the marginal distributions, we have that $\eta_{n,j} \to^{\mathrm{P}} \exp\{-(t_j -$

$t_{j-1})\xi^{\alpha}cx_j^{-\alpha}\}$, $x_j > 0$, and $\eta_{n,j} \to^{\mathrm{P}} 0$, $x_j < 0$, as $n \to \infty$. This, since the rv's $\eta_{n,j} \in [0,1]$, $n \in \mathbb{N}$, $j = 1, \dots, k$ are uniformly bounded, implies

$$\mathbb{E} \prod_{j=1}^{k} \eta_{n,j} \longrightarrow \mathbb{E}\Big(\prod_{j=1}^{k} \Phi_\alpha(x_j/\xi)^{t_j - t_{j-1}} \Big), \quad \text{as } n \to \infty,$$

where $\Phi_\alpha(x_j/\xi)^{t_j-t_{j-1}}$ is interpreted as 1 if $\xi = 0$ and $x_j > 0$ and as 0, if $\xi = 0$ and $x_j < 0$. By the last convergence and Relation (17), we obtain (16), which completes the proof of the theorem. $\square$

Remarks

1. The limit laws in (6) are scale–mixtures of Fréchet distributions. The case of independent and identically distributed (iid) weights was studied by Chuprunov [3]. This was done in the more general setting of triangular arrays, that is, when the distributions of the weights W_j depend also on n. In that case, the limits of the weighted maxima can have other limit distributions.

2. The proof of the convergence of the marginal distributions in (6) remains valid if in Condition (5) one requires the weaker mode of convergence $n^{-1} \sum_{j=1}^{n} W_j^{\alpha} \to^{\mathrm{d}} \xi^{\alpha}$, as $n \to \infty$. The stronger mode of convergence appearing in (5) is used in the proof of the convergence of the finite–dimensional distributions in (6).

The following lemma is used in the proof of Theorem 1.

Lemma 2. *Let X and X_n be rv's with c.d.f.'s F and F_n, $n \in \mathbb{N}$, respectively. Let $C(F)$ denote the set of continuity points of the function $F(x)$, $x \in \mathbb{R}$. Let also $D \subset C(F)$ be an everywhere dense subset of $\mathbb{R}$. If, for all $x \in D$, $F_n(x) \to F(x)$, as $n \to \infty$, then $X_n \to^{\mathrm{d}} X$, as $n \to \infty$.*

P r o o f. We need to show that $F_n(x) \to F(x)$, $n \to \infty$, for any $x \in C(F)$. Let $x \in C(F)$. Since the set D is dense in $\mathbb{R}$, for all $\epsilon > 0$, there exist $x_1(\epsilon) \in D \cap (x - \epsilon, x)$ and $x_2(\epsilon) \in D \cap (x, x+\epsilon)$. Thus,

$$F(x_1(\epsilon)) = \lim_{n\to\infty} F_n(x_1(\epsilon)) \le \lim_{n\to\infty} F_n(x_2(\epsilon)) = F(x_2(\epsilon)).$$

Since $x_1(\epsilon) < x < x_2(\epsilon)$, we have that $F_n(x_1(\epsilon)) \le F_n(x) \le F_n(x_2(\epsilon))$, $n \in \mathbb{N}$ and thus

$$F(x_1(\epsilon)) \le \liminf_{n\to\infty} F_n(x) \le \limsup_{n\to\infty} F_n(x) \le F(x_2(\epsilon)). \tag{18}$$

However, by using the continuity of the function F at x, we obtain $F(x_1(\epsilon)) \to F(x)$ and $F(x_2(\epsilon)) \to F(x)$, as $\epsilon \to 0$. Therefore, by letting $\epsilon \to 0$ in (18), we obtain that $\lim_{n\to\infty} F_n(x) = F(x)$. $\square$

When the sequence of weights $\{W_j\}_{j\in\mathbb{N}}$ has some additional structure, the assumptions of Theorem 1 can be simplified. Indeed, suppose that $\{W_j\}_{j\in\mathbb{N}}$ is a strictly stationary sequence. If $\mathbb{E}W_j^\alpha < \infty$, then Lemma 1 *(a)* implies the Conditions (4) and (5).

The following result concerns the case of strictly stationary weights $\{W_j\}_{j\in\mathbb{N}}$. It also provides necessary and sufficient conditions for the convergence in (6) to hold when the sequence of weights is ergodic.

Corollary 1. *Let the sequence of weights* $\{W_j\}_{j\in\mathbb{N}}$ *be strictly stationary.*
(a) If $\mathbb{E}W_1^\alpha < \infty$*, then* (6) holds.
(b) If the sequence $\{W_j\}_{j\in\mathbb{N}}$ is ergodic, then (6) holds if and only if, $\mathbb{E}W_1^\alpha < \infty$.

P r o o f. Lemma 1 *(a)* and Theorem 1 imply part *(a)*. We now prove part *(b)*. If $\mathbb{E}W_1^\alpha < \infty$, then the result follows from part *(a)*. Suppose, on the other hand, that $\mathbb{E}W_1^\alpha = \infty$. We will show that the convergence in (6) is then impossible. Consider the "truncated" random variables $W_j^{(\gamma)} = \min\{\gamma, W_j\} = W_j 1_{\{W_j\le\gamma\}} + \gamma 1_{\{W_j>\gamma\}}$, for some constant $\gamma > 0$. Since $\mathbb{E}(W_j^{(\gamma)})^\alpha < \infty$ and since the sequence $\{W_j^{(\gamma)}\}_{j\in\mathbb{N}}$ is stationary, by part *(a)* it follows that the convergence in (6) holds, when W_j are replaced by $W_j^{(\gamma)}$, for any fixed $\gamma > 0$.

If $U_j \ge 0$, for some $j = 1, \dots, [nt]$, then $U_jW_j \ge U_jW_j^{(\gamma)}$ and hence $M_n(t) \ge M_n^{(\gamma)}(t)$, for all $\gamma > 0$, where $M_n^{(\gamma)}(t)$ denotes $\max_{1\le j\le n} W_j^{(\gamma)}U_j$. Since, as $n \to \infty$, $\mathbb{P}(U_j < 0,\ j = 1, \dots, n) \le F(0)^n \to 0$, we can ignore the event $\{U_j < 0,\ j = 1, \dots, n\}$, as $n \to \infty$, and for all $\gamma > 0$, $t > 0$ and $x > 0$, we obtain

$$
\begin{aligned}
(19)\qquad \limsup_{n\to\infty} \mathbb{P}\{\frac{1}{n^{1/\alpha}} M_n(t) \le x\} &\le \lim_{n\to\infty} \mathbb{P}\{\frac{1}{n^{1/\alpha}} M_n^{(\gamma)}(t) \le x\} \\
&= \mathbb{E}\exp\{-t(\xi^{(\gamma)})^\alpha c x^{-\alpha}\},
\end{aligned}
$$

where $(\xi^{(\gamma)})^\alpha = \mathrm{plim}_{n\to\infty} n^{-1} \sum_{j=1}^n (W_j^{(\gamma)})^\alpha$.

By ergodicity, the Birkhoff theorem implies that the random variable $\xi^{(\gamma)}$ is constant. More precisely, for all $\gamma > 0$,

$$
\frac{1}{n}\sum_{j=1}^n (W_j^{(\gamma)})^\alpha \xrightarrow{\text{a.s.}} (\xi^{(\gamma)})^\alpha \equiv \mathbb{E}(W_1^{(\gamma)})^\alpha,
$$

as $n \to \infty$. Since $\mathbb{E}W_1^\alpha = \infty$, we have that $(\xi^{(\gamma)})^\alpha \to \infty$, as $\gamma \to \infty$. Therefore, the right–hand side of (19) converges to 0, as $\gamma \to \infty$. Since the left–hand side of (19) does not depend on γ, it follows that $\mathbb{P}\{n^{-1/\alpha} M_n(t) \le x\} \to 0$, as $n \to \infty$, for all $x > 0$. This shows that the convergence in (6) does not hold. $\square$

The following remarks are similar to those in Stoev and Taqqu [13].

Remarks

1. If $W_j \ge 0,\ j \in \mathbb{N}$ are iid, then the Conditions (4) and (5) are equivalent to $\mathbb{E}W_1^\alpha < \infty$. Indeed, by Lemma 1 *(a)*, the fact that $\mathbb{E}W_1^\alpha < \infty$ implies (4) and (5). If $\mathbb{E}W_1^\alpha = \infty$, then by the (converse) Strong Law of Large Numbers, $n^{-1} \sum_{j=1}^{n} W_j^\alpha$ does not converges in probability (and almost surely).

   This fact and Corollary 1 *(b)* show that in the case when the random weights W_j's are *iid* Conditions (4) and (5) are, in fact, *necessary and sufficient* for the convergence in (6) to hold.

   Here our main focus is on the case of *dependent* weights W_j. The assumptions of Theorem 1, namely Conditions (4) and (5), are relatively mild since they are necessary and sufficient in the case of iid weights.

2. Corollary 1 *(b)* implies that the convergence in (6) does not hold, when $\mathbb{E}W_1^\alpha = \infty$. More precisely, as shown in the proof of Corollary 1, under the normalization used in (6), the weighted maxima $M_n(t)$ converge to infinity, as $n \to \infty$. Other normalizations of the $M_n(t)$'s may yield a non–trivial limit. Suppose, for example that the W_js are independent and such that $\mathbb{P}\{W_j > x\} \sim \text{const}\, x^{-\beta}$, for some $0 < \beta < \alpha$. In this case the random variables U_j and W_j switch roles and the U_js can be viewed as weights to the W_js because they have lighter tails. If $U_j \ge 0$, for example, Theorem 1 implies that

   $$\left\{ \frac{1}{n^{1/\beta}} M_n(t) \right\}_{t \ge 0} \overset{\mathrm{J}_1}{\Longrightarrow} \{\eta \mathcal{E}_\beta(t)\}_{t\ge 0}, \quad \text{as } n \to \infty,$$

   where $\mathcal{E}_\beta(t)$ denotes a Fréchet–extremal process with parameter β (see (7)) and where $\eta = (\mathbb{E}U_1^\beta)^{1/\beta}$.

3. The assumption of ergodicity in Corollary 1 *(b)* is essential. If the sequence $\{W_j\}$ is not ergodic, then the convergence in (6) may continue to hold, even when $\mathbb{E}W_1^\alpha = \infty$. Indeed, suppose for example, that $W_j := \xi,\ j \in \mathbb{N}$, where ξ is an arbitrary non–constant random variable. The sequence $\{W_j\}_{j\in\mathbb{N}}$ is

then non–ergodic and it trivially satisfies Conditions (4) and (5). Therefore Theorem 1 applies and the convergence (6) holds.

3. The case of the general domain of attraction

In this section, we extend the result of Theorem 1 to the case when the U_j's belong to a certain sub–class of the *domain of max–attraction* of a Fréchet distribution. This class is defined in terms of the tail behavior of the weights $W_j,\ j \in \mathbb{N}$.

Let the U_j's belong to the domain of max–attraction of a Fréchet law. Namely, suppose that Relation (3) holds with $c > 0$ and with some *non–trivial* slowly varying function $L(x)$ defined for all $x \geq 0$, that is, not asymptotically equivalent to a constant. For any $\alpha > 0$, there is a unique, up to asymptotic equivalence, slowly varying function $\ell(x)$, such that

$$\ell^{\alpha}(x) \sim L(x^{1/\alpha}\ell(x)), \quad \text{as } x \to \infty, \tag{20}$$

where $a(x) \sim b(x),\ x \to \infty$ means $a(x)/b(x) \to 1,\ x \to \infty$. Indeed, for any slowly varying function $L_1(x)$, there is a slowly varying function $L_1^{\#}(x)$, called its *de Bruijn conjugate*, such that

$$L_1^{\#}(x) \sim L_1(xL_1^{\#}(x)), \quad \text{as } x \to \infty. \tag{21}$$

The function $L_1^{\#}(x)$ is unique up to asymptotic equivalence (see e.g. Theorem 1.5.13 in Bingham *et al.* [2]). Relation (21) becomes (20) if we set $L_1(x) := L(x^{1/\alpha})$ and $\ell(x) := (L_1^{\#}(x))^{1/\alpha}$. The function $\ell(x)$ in (20) will be used in the normalizations of the maxima, considered in the sequel.

We now suppose that the W_j's have moments of order slightly more than α, through:

Condition 1. *Let $\varphi(x)$ be a positive and non–decreasing function such that $x = o(\varphi(x)),\ x \to \infty$. Suppose that, for all $c > 0$,*

$$\sup_{j\in\mathbb{N}} \mathbb{E}\varphi(cW_j^{\alpha}) < \infty. \tag{22}$$

The next condition concerns the asymptotic behavior of the slowly varying function $L(x)$ which appears in the distribution of the U_j's (see (3)). It makes use of the representation:

$$L(x) = c(x)\exp\Big\{\int_1^x \frac{\epsilon(u)}{u}du\Big\}, \quad x > 0 \tag{23}$$

where $c(x) \to c_0 > 0,\ x \to \infty$ and $\epsilon(x) \to 0,\ x \to \infty$ (see e.g. Theorem 1.3.1 in Bingham *et al.* [2]). Note that since $L(x)$ is a *non–trivial* slowly varying function, that is, not asymptotically equivalent to a constant, we have that $\overline{\epsilon}(x) := (\sup_{y\geq x} |\epsilon(y)|) > 0$, for all $x > 0$.

The following technical condition relates the slowly varying function L with the function φ in Condition 1.

Condition 2. *Let φ be a positive and non–decreasing function such that $x = o(\varphi(x)),\ x \to \infty$. Suppose also that there exists a sequence $G_n \to \infty,\ n \to \infty$, such that*

$$\frac{n}{\varphi(G_n^\alpha)} \longrightarrow 0,\ n \to \infty \tag{24}$$

and

$$\ln(G_n)\overline{\epsilon}\Big(\frac{n^{1/\alpha}\ell(n)}{G_n}\Big) \longrightarrow 0,\ n \to \infty, \tag{25}$$

where $\overline{\epsilon}(x) = (\sup_{y\geq x} |\epsilon(y)|)$, $\epsilon(\cdot)$ is as in (23) and where $\ell(\cdot)$ is as in (20).

The next result, proved in Stoev and Taqqu [13], provides a practical way to check the above conditions.

Proposition 1. *Conditions 1 and 2 are satisfied with the same function φ, in either one of the following two cases:*

(a) If, for some $\delta > 0$,

$$\sup_{j\in\mathbb{N}} \mathbb{E}W_j^{\alpha+\delta} < \infty \tag{26}$$

and the slowly varying function L is such that

$$\overline{\epsilon}(x) = o\Big(\frac{1}{\ln(x)}\Big), \quad \text{as } x \to \infty, \tag{27}$$

where $\overline{\epsilon}(x)$ is as in Condition 2.

(b) If, for some $\delta > 0$, we have that

$$\sup_{j\in\mathbb{N}} \mathbb{E}\exp\{cW_j^\delta\} < \infty, \quad \text{for all } c > 0, \tag{28}$$

and the slowly varying function L is such that

$$\overline{\epsilon}(x) = o\Big(\frac{1}{\ln(\ln(x))}\Big), \quad \text{as } x \to \infty. \tag{29}$$

The following lemmas from Stoev and Taqqu [13] are used in the proof of the next theorem.

Lemma 3. *Let $W_j \geq 0$, $j \in \mathbb{N}$ and let $\alpha > 0$. Then, for all $K > 0$, $n \in \mathbb{N}$ and $r = 1, \dots, n$,*

$$D_{r,n}(K) := \tag{30}$$

$$\Big|\frac{1}{n}\sum_{j=1}^{r} W_j^\alpha 1_{\{W_j\leq K\}}\ell^{-\alpha}(n)L\Big(\frac{n^{1/\alpha}\ell(n)}{W_j 1_{\{W_j\leq K\}}}\Big) - \frac{1}{n}\sum_{j=1}^{r} W_j^\alpha 1_{\{W_j\leq K\}}\Big|$$

$$\leq h_n \frac{rK^\alpha}{n},$$

where $h_n \to 0$, $n \to \infty$ does not depend neither on r, $1 \leq r \leq n$, nor on the W_j's. Here $\ell(\cdot)$ and $L(\cdot)$ are as in (20) and the term $W_j^\alpha 1_{\{W_j\leq K\}}\ell^{-\alpha}(n)L(n^{1/\alpha}\ell(n)/W_j 1_{\{W_j\leq K\}})$ is interpreted as 0 if $W_j > K$.

Lemma 4. *Assume that the sequence G_n, $n \in \mathbb{N}$ satisfies Condition 2. Then, for all $K > 0$,*

$$\sup_{W\in[K,G_n]}\Big|\ell(n)^{-\alpha}L\Big(\frac{n^{1/\alpha}\ell(n)}{W}\Big) - 1\Big| \longrightarrow 0, \quad \textit{as } n \to \infty. \tag{31}$$

The next result extends Theorem 1.

Theorem 2. *Let $\alpha > 0$ and assume that* (3) holds with $c > 0$ and some *non–trivial* slowly varying function L. Assume that the W_j's are non–negative and satisfy (5).

Then, the convergence

$$\Big\{\frac{1}{n^{1/\alpha}\ell(n)}M_n(t)\Big\}_{t\geq 0} \stackrel{\mathrm{J_1}}{\Longrightarrow} \{\xi\mathcal{E}_\alpha(t)\}_{t\geq 0}, \quad \text{as } n \to \infty, \tag{32}$$

follows from either one of the following two conditions:

(a) For some $K > 0$, $\sup_{j\in\mathbb{N}} W_j \leq K$, almost surely.

(b) The W_j's satisfy Conditions 1 and 2, with the same sequence function φ. Here ξ and $\mathcal{E}_\alpha = \{\mathcal{E}_\alpha(t)\}_{t\geq 0}$ are as in Theorem 1 and $\ell(\cdot)$ satisfies (20).

P r o o f. Again, by Bingham's Theorem (see Section 2., above), it is enough to prove convergence of the finite–dimensional distributions.

Following the proof of Theorem 1, we will first prove the convergence of the marginal distributions in (32), by showing that (8) holds, where now the normalization $n^{1/\alpha}$ is replaced by $n^{1/\alpha}\ell(n)$. The case of $x < 0$ can be dealt with as in the proof of Theorem 1. Let now $x > 0$ and observe that as in (10), we have that

$$\mathbb{P}\Big\{\frac{1}{d(n)}\bigvee_{j=1}^{[nt]} W_j U_j \le x\Big\} \tag{33}$$

$$= \mathbb{E}\prod_{j=1}^{[nt]}(1 - \overline{F}(xd(n)/W_j)) =: \mathbb{E}\eta_n,$$

where $d(n) := n^{1/\alpha}\ell(n)$ and where $\overline{F}(y) = \mathbb{P}\{U_1 > y\}$. To prove the convergence of the marginal distributions in (32), it suffices to show that (11) holds, where now the η_n's are defined in (33). As in the proof of Theorem 1, we will first show that the convergence $\eta_n \to^{\mathrm{P}} \eta,\ n \to \infty$ follows from the convergence

$$\sum_{j=1}^{[nt]} \overline{F}(xd(n)/W_j) \stackrel{\mathrm{d}}{\longrightarrow} \zeta, \quad n \to \infty, \tag{34}$$

with $\eta = \exp(-\zeta)$.

In case *(a)*, since the random variables W_j are uniformly bounded by a constant, one can essentially repeat the argument in the proof of Theorem 1 with $n^{1/\alpha}$ replaced by $d(n)$, where one uses the fact that $d(n) = n^{1/\alpha}\ell(n) \to \infty$, as $n \to \infty$.

Consider now case *(b)* and let

$$\mu_n := \overline{F}\Big(\frac{xd(n)}{G_n}\Big)$$

and

$$B_n := \{\omega : \max_{1\le j\le [nt]} W_j(\omega) \le xG_n\},$$

where $G_n,\ n \in \mathbb{N}$ is the sequence in Condition 2. Since L is a non–trivial slowly varying function, the function $\overline{\epsilon}(x),\ x > 0$ involved in (25) is strictly positive.

Therefore, the fact that $\ln(G_n) \to \infty,\ n \to \infty$ and Relation (25) imply that $n^{1/\alpha}\ell(n)/G_n = d(n)/G_n \to \infty$, as $n \to \infty$. Thus, since $\overline{F}(y) \to 0,\ y \to \infty$ and since $d(n)/G_n \to \infty,\ n \to \infty$, we have that $\mu_n \to 0$, as $n \to \infty$ and as in the proof of Theorem 1 (see Relation (13)), we have that for all $\omega \in B_n$ and for all sufficiently large n,

$$\Big| \ln \eta_n(\omega) + \sum_{j=1}^{[nt]} \overline{F}\Big(\frac{xd(n)}{W_j(\omega)}\Big)\Big| \le C\mu_n \sum_{j=1}^{[nt]} \overline{F}\Big(\frac{xd(n)}{W_j(\omega)}\Big), \tag{35}$$

for some constant $C > 0$.

We have that

$$\frac{1}{G_n} \max_{1\le j\le [nt]} W_j \xrightarrow{\mathrm{P}} 0, \quad \text{as } n \to \infty. \tag{36}$$

Indeed, for all $\epsilon > 0$,

$$\begin{aligned} &\mathbb{P}\Big\{\frac{1}{G_n} \max_{1\le j\le n} W_j > \epsilon\Big\} \\ &\le \sum_{j=1}^{n} \mathbb{P}\{\varphi(\epsilon^{-\alpha}W_j^\alpha) \ge \varphi(G_n^\alpha)\} \le \frac{n}{\varphi(G_n^\alpha)} \sup_{j\in\mathbb{N}} \mathbb{E}\varphi(\epsilon^{-\alpha}W_j^\alpha), \end{aligned} \tag{37}$$

where in the first inequality above we used the monotonicity of the function φ and in the second, the Markov's inequality. Relations (22) and (24) imply that the right–hand side of the last expression converges to 0, as $n \to \infty$, and hence (36) holds.

Relation (36) shows that $\mathbb{P}(B_n) \to 1$, as $n \to \infty$. Thus, by (35) and the facts that $\mu_n \to 0$ and $\mathbb{P}(B_n) \to 1$, as $n \to \infty$, as in the proof of Theorem 1, we get that the convergence $\eta_n \to^{\mathrm{P}} \eta,\ n \to \infty$ follows from the convergence in (34) with $\eta = \exp(-\zeta)$.

Now, we will complete the proof of the convergence of the marginal distributions, by showing that (34) holds with $\zeta = t\xi^\alpha c x^{-\alpha}$. Observe that, since $d(n)/G_n = n^{1/\alpha}\ell(n)/G_n \to \infty,\ n \to \infty$, Relations (3) and (20) imply that

$$\overline{F}\Big(\frac{xd(n)}{W}\Big) \sim \frac{c}{n} x^{-\alpha} W^\alpha \ell^{-\alpha}(n) L\Big(\frac{xn^{1/\alpha}\ell(n)}{W}\Big), \quad \text{as } n \to \infty,$$

uniformly in $W \in (0, xG_n]$. Therefore, for all $\omega \in B_n$, as $n \to \infty$,

$$\sum_{j=1}^{[nt]} \overline{F}\Big(\frac{xd(n)}{W_j(\omega)}\Big) \sim \frac{1}{n} \sum_{j=1}^{[nt]} c x^{-\alpha} W_j^\alpha(\omega) \ell^{-\alpha}(n) L\Big(\frac{xn^{1/\alpha}\ell(n)}{W_j(\omega)}\Big), \tag{38}$$

uniformly in $w \in B_n$. Since $\mathbb{P}(B_n) \to 1$, as $n \to \infty$, to prove (34), in view of (38), it suffices to focus on the right–hand side of (38) in place of the left–hand side of (34).

In case *(a)*, since $0 \le W_j/x \le K/x,\ j \in \mathbb{N}$, Lemma 3 implies that, for all sufficiently large n,

$$D_{t,n} := \Big|\frac{1}{n}\sum_{j=1}^{[nt]} cx^{-\alpha}W_j^{\alpha}\ell^{-\alpha}(n)L\Big(\frac{xn^{1/\alpha}\ell(n)}{W_j}\Big) - \frac{1}{n}\sum_{j=1}^{[nt]} cx^{-\alpha}W_j^{\alpha}\Big| \le h_n\frac{[nt]K^{\alpha}}{n}, \tag{39}$$

where $h_n \to 0,\ n \to \infty$ does not depend on t and $W_j,\ j \in \mathbb{N}$. The fact that, the right–hand side of the last inequality converges to zero, as $n \to \infty$ and the first convergence in Relation (5) imply that (34) holds with $\zeta = t\xi^{\alpha}cx^{-\alpha}$.

In case *(b)*, we apply Lemma 3 to the terms in (39), for which $W_j/x < K$. Observe that for all $\omega \in B_n$, we have $W_j(\omega)/x \le G_n$. Therefore, for all $\omega \in B_n$, we apply Lemma 4, to the terms where $K \le W_j/x$, and obtain

$$D_{t,n} \le h_n \times \Big(\frac{[nt]K^{\alpha}}{n} + \frac{1}{n}\sum_{j=1}^{[nt]} cx^{-\alpha}W_j^{\alpha}1_{\{K \le W_j/x\}}\Big), \tag{40}$$

where $h_n \to 0,\ n \to \infty$ does not depend on t and $W_j,\ 1 \le j \le n$.

The convergence in (5) and the fact $h_n \to 0,\ n \to \infty$, imply that the right–hand side of (40) converges to zero in probability. Thus, we can drop the terms involving slowly varying functions in Relation (38). By using Relation (5) again, and the fact that $\mathbb{P}(B_n) \to 1$, as $n \to \infty$, we obtain (34) where $\zeta = t\xi^{\alpha}cx^{-\alpha}$.

We have thus completed the proof of the convergence of the marginal distributions. The convergence of the finite–dimensional distributions can be established by following the corresponding argument in the proof of Theorem 1. □

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