PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. Bulgar. **17** (2005), 13–26

PLISKA
STUDIA MATHEMATICA
BULGARICA

CHARACTERIZATION OF SCHRÖDINGER PROCESSES WITH UNBOUNDED POTENTIALS

A. Benchettah

This work is concerned with a class of Schrödinger process with unbounded potentials : a variant of Jamison's theorem is given without the assumption of continuity and of everywhere strict positivity of q. It associates with Jamison data (q, P_a, P_b), the Csiszar's projection Q^* of a reference measure R^* on a set $E(P_a, P_b)$ of probability measures with marginals P_a, P_b. Existence of a solution to the corresponding Schrödinger's system, construction of the Schrödinger's bridge and variational characterisation of Schrödinger process are established.

1. Introduction

In his paper Schrödinger[10] (1931) has solved the problem: "knowing the position of a Brownian particle in an Euclidean space at times a and $b > a$; what is the probability for this particle to have passed through some prescribed domain of the space at some intermediate time?"

A generalization of this problem by prescribing probability distribution at the initial and terminal time has led to the concept of Schrödinger bridge which has been approached from different points of view:

– the theory of reciprocal processes: Jamison[6];

– Information theory and statistics with the concept of entropy: Kullback and Csiszar[2];

2000 *Mathematics Subject Classification:* 49L20, 60J60, 93E20

Key words: Schrödinger process,minimum entropy distance, stochastic optimal control, Schrödinger's system, variational characterisation.

– the theory of large deviations: Föllmer[5], Wakolbinger[12];

– the theory of stochastic optimal control: Benchettah[1], Nagasawa[8] and Wakolbinger[12].

The most concise formulation of the convex optimization problem is probably the one of Csiszar[2]: let E be a given convex set of probability measures on some measure space $(\Omega, \mathcal{F})$, a reference measure μ, we wishe to find a probability measure $\nu^* \in E$, whenever it exists, for which the entropy distance

$$H(\nu, \mu) = \int \log \frac{d\nu}{d\mu} d\nu, \text{ if } \nu \ll \mu \text{ and } +\infty \text{ otherwise}, \nu \in E.$$

is minimum, i.e., $H(\nu^*, \mu) = \min\limits_{\nu \in E} H(\nu, \mu)$.

Shrödinger has treated the case: $\Omega = C^0([0, T]; \Re)$, $\mathcal{F}$: the $\sigma-$field of Borels sets of C^0, μ : the Wiener measure and $\mathbf{E}$ the set of probability measures $\mathbf{P}$ on $(\Omega, \mathcal{F})$ with given marginal $\mathbf{P}_0$ and $\mathbf{P}_T$ which represent the end conditions. In other words, the reference measure μ is associated with the Brownian transition probability density, but Shrödinger's problem can be formulated for a given not necessarily complete transition density $p(s, x; t, y)$ as well Existence of a solution to Shrödinger's problem has been reduced by Shrödinger to existence conditions for a solution to a system of two integral equations (7), *the Shrödinger system.* Beurling-Jamson's condition for existence of a solution to the corresponding Shrödinger's system is that the function $q(x, y) \overset{\Delta}{=} p(o, x; T, y)$ be *strictly positive and jointly continuous* at x, y. Note that, since in this more general framework the transition density p needs not be complete, the definition of a probability measure μ associated with p requires a normalization. At this point we are faced with two directions: we can discuss existence in terms of p, on the the basis of Beurling-Jamison's work in the area of reciprocal processes, or in terms of μ thus entering Csiszar's geometric approach. Of course the two methods are in correspondence to one another.

In this work, we give a variant of Beurling-Jamison's without the assumptions of continuity and everywhere strict positivity of q. Our Theorem 1 links Beurling-Jamison's statement and Csiszar's geometric point of view together. In particular it associates with Beurling-Jamison's data $(q, \mathbb{P}_a, \mathbb{P}_b)$ the Csiszar's projection $\mathbb{Q}^*$ of a reference measure $\mathbb{R}^*$ on a set $\mathbb{E}(\mathbb{P}_a, \mathbb{P}_b)$ of probability measures with marginal $\mathbb{P}_a$ and $\mathbb{P}_b$. Theorem 2 extends this result by associating with this $\mathbb{Q}^*$ a set $\{R\}$ containing $\mathbb{R}^*$ of reference measures with the same projection. Theorem 3 is concerned with a function q, given explicitly. We obtain sufficient conditions for existence of a solution to the Schrödinger's system. In most of paragraph 3 , we suppose $M = \Re^n$. By relying on arguments of Föllmer[5] and Wakolbinger[12], we pass to the construction of a Schrödinger bridge with creation and killing. The main result is enclosed in relations (13) and (14) which will be given later.

Paragraph 4 extends a characterization of Schrödinger processes given by Wakolbinger[12] to the larger class of potential functions c considered in this paper. With Theorems 1 and 3 in hand the proof is similar to the proof of Wakolbinger's Theorem[12].

2. Shrödinger System for a process with birth and death

Theorem 1. *Let M be a $\sigma-$compact metric space and $\mathbb{P}_a$, $\mathbb{P}_b$ two probability measures on Σ,($\sigma-$field of borel sets).*
Let $\mathbb{E} = \{\mathbb{P} \ / \Sigma \otimes \Sigma$: $\mathbb{P}\left(. \times M\right) = \mathbb{P}_a\left(.\right), \mathbb{P}\left(M \times .\right) = \mathbb{P}_b\left(.\right)\}$ and $q : M \times M \to \mathfrak{R}$ borel bounded away from zero below$(\mathbb{P}_a \otimes \mathbb{P}_b) -a.s.$ and $(\mathbb{P}_a \otimes \mathbb{P}_b) -$integrable. Then $\exists!$ pair of measures$(\mathbb{Q}^*, \pi)$on $\Sigma \otimes \Sigma$ for which:
(a) $\mathbb{Q}^* \in \mathbb{E}$ and π is a finite product measure;
(b) $\dfrac{d\mathbb{Q}^*}{d\pi} = q$;
(c) $H(\mathbb{Q}^*; \mathbb{R}^*) \leq H\left(\mathbb{P} \ ; \mathbb{R}^*\right) \ \forall \mathbb{P} \in \mathbb{E}$ where

$$d\mathbb{R}^* = qd\left(\mathbb{P}_a \otimes \mathbb{P}_b\right) / \int qd\left(\mathbb{P}_a \otimes \mathbb{P}_b\right); \tag{1}$$

$$(d) \quad d\pi = \phi\gamma d\left(\mathbb{P}_a \otimes \mathbb{P}_b\right) / \int qd\left(\mathbb{P}_a \otimes \mathbb{P}_b\right) \tag{2}$$

with $\log\phi \in L^1\left(\mathbb{P}_a\right)$ and $\log\gamma \in L^1\left(\mathbb{P}_b\right)$;
(e) If $\mathbb{P}_a \ll \lambda_a$ and $\mathbb{P}_b \ll \lambda_b$, λ_a, λ_b two $\sigma-$finites measures and $log\dfrac{d\mathbb{P}_a}{d\lambda_a} \in L^1\left(\mathbb{P}_a\right)$ and $log\dfrac{d\mathbb{P}_b}{d\lambda_b} \in L^1\left(\mathbb{P}_b\right)$, then

$$d\mathbb{Q}^* = \varphi_a \ q \ \varphi_b d\lambda_a d\lambda_b, \tag{3}$$

where $\log\varphi_a \in L^1\left(\mathbb{P}_a\right), \log\varphi_b \in L^1\left(\mathbb{P}_b\right)$.

To prove this theorem we need the following result.

Lemma 2. *If $\mathbb{R}$ is a probability. on $\Sigma\otimes\Sigma$ for which $\exists\overline{\mathbb{P}} \in \mathbb{E}$ with $H\left(\overline{\mathbb{P}} \ ; \mathbb{R}\right) < \infty$. Then $\exists!\mathbb{Q} \in \mathbb{E}$ such that*

$$H(\mathbb{Q}; \mathbb{R}) \leq H\left(\mathbb{P} \ ; \mathbb{R}\right), \forall\mathbb{P} \in \mathbb{E}.$$

Furthermore if $\overline{\mathbb{P}} \sim \alpha \otimes \beta$ *and* $\mathbb{R} \sim \alpha \otimes \beta$ *on* $\Sigma \otimes \Sigma$, *then* $\dfrac{d\mathbb{Q}}{d\mathbb{R}}$ *is such that*

$$\frac{d\mathbb{Q}}{d\mathbb{R}}(x,y) = \phi(x)\gamma(y), \textit{with } 0 \leq \phi(x) < \infty \ \alpha - a.s, 0 \leq \gamma(y) < \infty \ \beta - a.s.$$

P r o o f. The first statement follows from Theorem 2-1 of Csiszar[2]. Now M being a $\sigma-$compact and metric space, it is generated by a countable class of sets, that is

$$\mathbb{E} = \left\{ \mathbb{P}/\Sigma \otimes \Sigma : \int f_i d\mathbb{P} = \int f_i d\mathbb{P}_a, \int g_i d\mathbb{P} = \int g_i d\mathbb{P}_b, i = 1, 2, ... \right\}$$

where the $f_i's$ and $g_i's$ are bounded measurable real valued functions, only depending on one of the two arguments $x, y \in M$, respectively. We consider then the sequence

$$\mathbb{E}_n = \left\{ \mathbb{P}/\Sigma \times \Sigma : \int f_i d\mathbb{P} = \int f_i d\mathbb{P}_a, \int g_i d\mathbb{P} = \int g_i d\mathbb{P}_b, i = 1, ..., n \right\}$$

Using Föllmer's argument[5], we have $\mathbb{E}_n \downarrow \mathbb{E}$ and, for each n, $\exists \mathbb{Q}$ Csiszar's projection of $\mathbb{R}$ on $\mathbb{E}_n$ which converges in variation to $\mathbb{Q}$ as $n \to \infty$. According to corollary 3-1 of Csiszar[2], the $\mathbb{Q}_n's$ have densities with respect to $\mathbb{R}$ of the form $\dfrac{d\mathbb{Q}_n}{d\mathbb{R}} = \mathbb{Q}_n \gamma_n$, where $\mathbb{Q}_n$ and γ_n are bounded strictly positive functions of x and y, respectively, except possibly for a subset N_n of $M \times M$ where $\dfrac{d\mathbb{Q}_n}{d\mathbb{R}}$ vanishes and $\mathbb{P}_n(N_n) = 0 \ \forall \mathbb{P}_n \in \mathbb{E}_n$ with $H(\mathbb{P}_n; \mathbb{R}) < \infty$.

Then, $\forall \mathbb{P}_n$ we have $0 < \dfrac{d\mathbb{Q}_n}{d\mathbb{R}}(x,y) = \phi_n(x)\gamma_n(y) \quad \mathbb{P}_n - a.s.$

Furthermore

$$\int \left| \frac{d\mathbb{Q}}{d\mathbb{R}} - \frac{d\mathbb{Q}_n}{d\mathbb{R}} \right| d\mathbb{R} \to 0.$$

So, $\exists \left\{ \dfrac{d\mathbb{Q}_n}{d\mathbb{R}} \right\}$ a sub-sequence for which

$$\lim_{n_q} \left| \frac{d\mathbb{Q}}{d\mathbb{R}} - \frac{d\mathbb{Q}_n}{d\mathbb{R}} \right| = 0 \quad , \mathbb{R} - a.s \qquad \text{with } 0 \leq \frac{d\mathbb{Q}}{d\mathbb{R}} < \infty \qquad \text{a.s.}$$

Since $\mathbb{P} \in \mathbb{E} \Rightarrow \mathbb{P} \in \mathbb{E}_n$, $n = 1, 2, ...$, we show that $\dfrac{d\mathbb{Q}}{d\mathbb{R}}(x,y) = \phi(x)\gamma(y)$ with $0 \leq \phi(x) < \infty \ \alpha - a.s.$; $0 \leq \gamma(y) < \infty \ \beta - a.s.$ □

Proof of Theorem 1.

Let $\overline{\mathbb{P}} = \mathbb{P}_a \otimes \mathbb{P}_b$. Then $\overline{\mathbb{P}} \in \mathbb{E}$, and from the assumption of Theorem 1 and the definition of $\mathbb{R}^*$ it follows that

$$0 \leq H\left(\overline{\mathbb{P}}; \mathbb{R}^*\right) \leq \left|\log \int q d\left(\mathbb{P}_a \times \mathbb{P}_b\right)\right| + \int |\log q| \, d\left(\mathbb{P}_a \otimes \mathbb{P}_b\right) < \infty.$$

Then $\exists! \mathbb{Q}^* \in \mathbb{E}$ satisfying:

$$H\left(\mathbb{Q}^*; \mathbb{R}^*\right) \leq H\left(\mathbb{P}; \mathbb{R}^*\right) \;\; \forall \mathbb{P} \in \mathbb{E}$$

From lemma 2.1 of Csisar [2], it follows that:

$$0 \leq \int \log \frac{d\mathbb{Q}^*}{d\mathbb{R}^*} d\mathbb{P} < \infty, \; \forall \mathbb{P} \in \mathbb{E} \text{ such that } H\left(\mathbb{P}; \mathbb{R}^*\right) < \infty.$$

In particular:

$$0 \leq \int \log \frac{d\mathbb{Q}^*}{d\mathbb{R}^*} d\left(\mathbb{P}_a \times \mathbb{P}_b\right) < \infty.$$

Let $m = \dfrac{q}{\int q d\left(\mathbb{P}_a \otimes \mathbb{P}_b\right)}$, then $d\mathbb{R}^* = m \; d\left(\mathbb{P}_a \times \mathbb{P}_b\right)$. Since $m > 0 \; \left(\mathbb{P}_a \times \mathbb{P}_b\right)$ a.s., $\mathbb{R}^*$ is equivalent to $\mathbb{P}_a \otimes \mathbb{P}_b$. We deduce from Lemma 1 that:

$$\frac{d\mathbb{Q}^*}{d\mathbb{R}^*}(x, y) = \phi(x) \gamma(y), \quad x, y \in M$$

with

$$0 \leq \phi(x) < \infty \quad \mathbb{P}_a - a.s., \; 0 \leq \gamma(y) < \infty \quad \mathbb{P}_b - a.s.,$$

but, since $\left(\mathbb{P}_a \otimes \mathbb{P}_b\right) \left\{(x, y) : \dfrac{d\mathbb{Q}^*}{d\mathbb{R}^*}(x, y) = 0\right\}$, we have in fact

$$\frac{d\mathbb{Q}^*}{d\mathbb{R}^*}(x, y) = \phi(x) \gamma(y), \quad x, y \in M$$

with $0 < \phi(x) < \infty \;\; \mathbb{P}_a - a.s., \; 0 < \gamma(y) < \infty \;\; \mathbb{P}_b - a.s.$
It is easy to find that: $\log \phi \in L^1\left(\mathbb{P}_a\right), \log \gamma \in L^1\left(\mathbb{P}_b\right).$
Also, from the definition of $\mathbb{R}^*$, we get $\dfrac{d\mathbb{Q}^*}{d\pi} = q$, where π is the product measure defined by

$$d\pi = \frac{\phi \gamma d\left(\mathbb{P}_a \otimes \mathbb{P}_b\right)}{\int q d\left(\mathbb{P}_a \otimes \mathbb{P}_b\right)},$$

which is finite since $d\pi = \dfrac{d\mathbb{Q}^*}{q}$ with $\dfrac{1}{q}$ bounded above $\left(\mathbb{P}_a \otimes \mathbb{P}_b\right) - a.s.$

Finally, from the assumptions of (e), we have

$$d\mathbb{Q}^* = \varphi_a q \varphi_b d\lambda_a d\lambda_b,$$

where

$$\begin{aligned}\log\varphi_a &= const + \log\phi + \log\frac{d\mathbb{P}_a}{d\lambda_a} \in L^1(\mathbb{P}_a)\\ \log\varphi_b &= const + \log\gamma + \log\frac{d\mathbb{P}_b}{d\lambda_b} \in L^1(\mathbb{P}_b).\end{aligned}$$

Theorem 3. *Suppose that* $\mathbb{P}_a$ *and* $\mathbb{P}_b$ *satisfy the assumptions (e) of theorem1 and let the function* q *be as in Theorem1. Then* $\mathbb{Q}^*$ *given by:*

$$d\mathbb{Q}^* = \varphi_a\ q\ \varphi_b d\lambda_a d\lambda_b \tag{4}$$

satisfies $H(\mathbb{Q}^*;\mathbb{R}) \leq H(\mathbb{P}\ ;\mathbb{R}), \forall\mathbb{P}\in\mathbb{E}$ *and* $\forall\mathbb{R}$ *given by:*

$$d\mathbb{R}(x,y) = f(x)\,g(y)\,q(x,y)\,d\lambda_a(x)\,d\lambda_b(y) \tag{5}$$

with $\log f \in L^1(\mathbb{P}_a)$ *and* $\log g \in L^1(\mathbb{P}_b)$.

Proof. Since $\frac{d\mathbb{Q}^*}{d\mathbb{R}}(x,y) = \frac{\varphi_a(x)\,\varphi_b(y)}{f(x)\,g(y)}$ $x,y\in M$. $\mathbb{P}_a\otimes\mathbb{P}_b - a.s.$ and $\log\varphi_a,\ \log f \in L^1(\mathbb{P}_a)$ and $\log\varphi_b,\ \log g \in L^1(\mathbb{P}_b)$ .Corollary 3.1 of Csiszar[2] $\Rightarrow H(\mathbb{Q}^*;\mathbb{R}) \leq H(\mathbb{P};\mathbb{R})\ \ \forall\mathbb{P}\in\mathbb{E}$. □

Now we suppose that the space M is a complete $\sigma-$compact metric space (then separable), Σ its σ-field borel sets and $\{\xi(t), a\leq t<\infty\}$ is a (M,Σ) valued continuous Markov process on a probability space $(\Omega,\mathcal{F},\mathbf{P})$ with a transition probability:

$$P(s,x;t,B) = \mathbf{P}(\xi(t)\in B \mid \xi(s)=x) = \mathbf{P}_{s,x}(\xi(t)\in B),$$

and initial distribution $\mathbb{P}_a$. We suppose that:

$$P(s,x;t,B) = \int_B p(s,x;t,y)\,\lambda(dy), a\leq s<t<\infty, B\in\Sigma, x\in M.$$

where λ is a $\sigma-$finite measure on Σ.

Let Θ be a not empty open relatively compact subset of M. For a terminal time $b>a$, the part $\{\eta(t), a\leq t\leq b\}$ of the process $\{\xi(t), a\leq t\leq b\}$ on the set Θ has transition density $\widetilde{p}(s,x;t,y)$ defined by:

$$\widetilde{p}(s,x;t,y) = \mathbf{E}_{s,x}\left[\mathbb{I}_{t\,<\,\tau_s} \mid \eta(t)=y\right] p(s,x;t,y)\,; a\leq\ s<t\ \leq b$$

$$\text{where } \tau_s = \begin{cases} \inf\{t > s : \xi(t) \in M - \Theta\} & \text{if it exists,} \\ +\infty & \text{otherwise.} \end{cases}$$

Let the function $c : [a,b] \times M \to \overline{\mathfrak{R}}$ measurable and $D \triangleq \{(s,x) \in [a,b] \times M : |c(s,x)| < \infty\}$.

Assume:

(H$_1$) $\dfrac{d\mathbb{P}_a}{d\lambda} = \Phi_a$ and $\dfrac{d\mathbb{P}_b}{d\lambda} = \Phi_b$ are continuous with compact support K_a and K_b respectively such that $K_a \cup K_b \subset \Theta$;

(H$_2$) $p(a,.;b,.)$ is C^0 on $\Theta \otimes \Theta$ and strictly positive on $K_a \otimes K_b$;

(H$_3$) c is finite and continuous on $[a,b] \times \overline{\Theta}$;

(H_4) $\int \left[\int \exp\left(\int_a^b c(r,\xi(r))\,dr \right) \mathbb{I}_{b<T} d\mathbf{P}_{ax} \right] \Phi_a(x)\,\lambda(dx) < \infty$,

$$\text{where } T = \begin{cases} \inf\{t > a : |c(t,\xi(t))| = \infty\} & \text{if it exists} \\ +\infty & \text{otherwise,} \end{cases} \text{ is measurable.}$$

Theorem 4. *Let (H$_1$)-(H$_4$) hold. Then the function q given by*

$$q(x,y) = \mathbf{E}_{ax}\left[\exp\left(\int_a^b c(r,\xi(r))\,dr \right) \mathbb{I}_{b<T} \mid \xi(b) = y \right] p(a,x;b,y), \tag{6}$$

$x, y \in M$, *satisfies the assumptions of Theorem1*

Proof. By assumption $(H4)$, we have

$$\begin{aligned} &\int \left[\int q(x,y)\,\lambda(dy) \right] \Phi_a(x)\,\lambda(dx) \\ = &\int \left[\int \exp\left(\int_a^b c(s,\xi(s))\,ds \right) \chi_{b<T} d\mathbf{P}_{ax} \right] \Phi_a(x)\,\lambda(dx) < \infty; \end{aligned}$$

therefore

$$\begin{aligned} \int q d(\mathbb{P}_a \times \mathbb{P}_b) &= \int q(x,y)\,\Phi_a(x)\,\Phi_b(y)\,\lambda(dx)\,\lambda(dy) \\ &\le \sup_{y \in K_b} \Phi_b(y) \int q(x,y)\,\Phi_a(x)\,\lambda(dx)\,\lambda(dy) < \infty. \end{aligned}$$

Let

$$\tilde{q}(x,y) = \mathbf{E}_{ax}[\exp\left(\int_a^b c(s,\xi(s))\,ds\right)\chi_{b<\tau}/\xi(b) = y]p(a,x;b,y).$$

Since the process $\xi(t)$, $a \le t \le b$, is continuous, we have $\tau \le T$. It follows that

$$\tilde{q}(x,y) \le q(x,y), \; x,y \in M.$$

By assumption (H_3), c is bounded on $[a,b] \times \bar{\Theta}$. Therefore, $e^{\left(\int_a^b c(s,\xi(s))ds\right)}\chi_{b<\tau} \geqslant m'\chi_{b<\tau}$ for some $m' > 0$, thus $\tilde{q}(x,y) \geqslant m'\mathbf{E}_{ax}[\chi_{b<\tau}/\xi(b) = y]\,p(a,x;b,y)$, $x,y \in M$. Furthermore, by assumption (H_2),

$$\mathbf{E}_{ax}[\chi_{b<\tau}/\xi(b) = y]\,p(a,x;b,y) \geqslant m"$$

for some $m" > 0$ $(\mathbb{P}_a \otimes \mathbb{P}_b) - a.s.$. Therefore

$$q(x,y) \geqslant \tilde{q}(x,y) \geqslant m'm" > 0 \quad (\mathbb{P}_a \otimes \mathbb{P}_b) - a.s.$$

□

Corollary 5. *Let (H₁)-(H₄) hold. Then there exists a unique (up to multiplicative constants) nonnegative solution* (φ_a, φ_b) *for the Schrödinger's system*

$$\text{(7)} \qquad \begin{cases} \Phi_a(x) = \varphi_a(x)\int q(x,y)\,\varphi_b(y)\,dy \\ \Phi_b(y) = \varphi_b(y)\int q(x,y)\,\varphi_a(x)\,dx \end{cases}$$

with q as in Theorem 3.

Let M be a $\sigma-$compact complete metric space; $c' : [a,\infty[\times M \to \overline{\Re}$ measurable;
$D' = \{(s,x) \in [a,\infty[\times M : |c'(s,x)| < \infty\}$;
$\zeta_s = \begin{cases} \inf\{t > s : |c'(t,\xi(t))| = \infty\} & \text{if it exists} \\ \infty & \text{otherwise;} \end{cases}$,
I_t^s : σ-field borel sets of $[s,t]$ and $\mathcal{N}_t^s = \sigma(\xi(u), a \le s \le u \le t < \infty)$
Consider the following hypotheses:
(h1): $\{\omega, \zeta_s(\omega) > t\} \in \mathcal{N}_t^s, a \le s < t < \infty$,(verified if $\partial D'$ is smooth)
(h₂): c' is continuous on D';
(h₃):$\mathbf{E}_{s,x}\left[\exp(\int_s^t c'(r,\xi(r))\,dr)\;\mathbb{I}_{\zeta_s>t}\right] < \infty, \; a \le s < t < \infty, x \in M;$
(h₄): $c'(s,x) = c(s,x)$, $(s,x) \in [a,b] \times M$.
Using Dynkin[4], we obtain.

Proposition 6. *Under assumptions (h$_1$)-(h$_4$), the function*

$$\begin{aligned} \mathbf{q}(s,x;t,y) &= \mathbf{E}_{s,x}\left[\exp(\int_s^t c(r,\xi(r))\,dr)\ \mathbb{I}_{T_s>t} \mid \xi(t)=y\right]\times p(s,x;t,y), \\ a &\le s<t<\infty;\ x,\ y\in M;\ \text{where} \\ T_s &= \begin{cases} \inf\{t>s : |c(t,\xi(t))|=\infty\} & \text{if it exists} \\ +\infty & \text{otherwise,}\end{cases} \end{aligned} \tag{1}$$

is a quasi-transition density .

3. Schrödinger bridge over process with birth and death

Let $\mathcal{F}=\mathcal{N}\,(=\sigma(\xi(t),a\le t\le b))$ and define the probability $\mathbf{R}$ on $(\Omega,\mathcal{N})$ by

$$d\mathbf{R}=\frac{\exp\left(\int_a^b c(r,\xi(r)\,dr\right)\mathbb{I}_{b<T}d\mathbf{P}_a}{\int\exp\left(\int_a^b c(r,\xi(r)\,dr\right)\mathbb{I}_{b<T}d\mathbf{P}_a}, \tag{9}$$

where $\mathbf{P}_a(.)=\displaystyle\int\mathbf{P}_{ax}(.)\,\Phi_a(x)\,dx$ is the probability on $\mathcal{N}$.

Therefore, the joint end-points distributions measure of the process $\xi(t)$ relatively to $\mathbf{R}$, is $\mathbb{R}$, given by:

$$d\mathbb{R}=\frac{\Phi_a(x)\,q(x,y)\,dxdy}{\int\exp\left(\int_a^b c(r,\xi(r)\,dr\right)\mathbb{I}_{b<T}d\mathbf{P}_a}, \tag{10}$$

with q given by (6). And if $\mathbb{P}_a$ and $\mathbb{P}_b\ll\lambda$, Lebesgue's measure in $\mathfrak{R}^n$, and $\log\frac{d\mathbb{P}_a}{dx}\in L^1(\mathbb{P}_a)$, $\log\frac{d\mathbb{P}_b}{dy}\in L^1(\mathbb{P}_b)$ then $\mathbb{Q}^*(x,y)$ given by

$$\mathbb{Q}^*(x,y)=\varphi_a(x)\,q(x,y)\,\varphi_b(y)\,dxdy \tag{11}$$

is the Csiszar's projection of $\mathbb{R}$ on

$$\mathbb{E}\triangleq\{\ \mathbb{P}\text{on }\mathcal{B}\times\mathcal{B}:\mathbb{P}(.\times\mathfrak{R}^n)=\mathbb{P}_a(.),\ \mathbb{P}(\mathfrak{R}^n\times.)=\mathbb{P}_b(.)\},$$

i.e.,

$$H(\mathbb{Q}^*;\mathbb{R})\le H(\mathbb{P};\mathbb{R}),\ \forall\mathbb{P}\in\mathbb{E}.$$

Consider now the case where $M = \mathfrak{R}^n$. Let $\Omega_0 = C^0([a,b];\mathfrak{R}^n)$, $\mathcal{M}$ its σ-field of Borels subsets of Ω_0 and the process $X(t)$ defined by $X(t, x(.)) = x(t)$, $x(.) \in \Omega_0$, $t \in [a,b]$. Let μ_{ax}^{ξ} and μ_a^{ξ} the distribution measures on the path space of the process ξ with respect to $\mathbf{P}_{ax}$ and $\mathbf{P}_a$ respectively, i.e., $\mu_{ax}^{\xi}(M) = \mathbf{P}_{ax}\{\omega : \xi(.,\omega) \in M\}$ and $\mu_a^{\xi}(M) = \mathbf{P}_a\{\omega : \xi(.,\omega) \in M\}$, $M \in \mathcal{M}$. Thus the distribution measure on $(\Omega_0, \mathcal{M})$ of ξ with respect to $\mathbf{R}$ is given by:

$$(9') \qquad d\mu = \frac{\exp\left(\int_a^b c(r, X(r))\,dr\right)\mathbb{I}_{b\,<T} d\mu_a^{\xi}}{\int \exp\left(\int_a^b c(r, X(r))\,dr\right)\mathbb{I}_{b\,<T} d\mu_a^{\xi}}.$$

The problem is the following:

Find the probability measure ν^* on M, which minimizes the relative entropy $H(\nu;\mu)$ on the set

$$\mathbf{E} = \{\nu \text{ on } \mathcal{M} \text{: } \nu[X(a) \in .] = \mathbb{P}_a(.),\ \nu[X(b) \in .] = \mathbb{P}_b(.)\}.$$

Using the multiplication formula:

$$(12) \qquad \frac{d\nu}{d\mu}(X) = \frac{d\mathbb{P}}{d\mathbb{R}}(X(a), X(b))\,\frac{d\nu_{X(a)}^{X(b)}}{d\mu_{X(a)}^{X(b)}}(X), \mu - p.s.,$$

where

$$\begin{aligned} \mathbb{P}(A \times B) &= \nu[X(a) \in A, X(b) \in B], \quad A, B \in \mathcal{B}, \\ \mathbb{R}(A \times B) &= \mu[X(a) \in A, X(b) \in B], \quad A, B \in \mathcal{B}, \end{aligned}$$

$$\begin{aligned} \nu_x^y(.) &= \nu[. \mid X(a) = x, X(b) = y], \quad x, y \in \mathfrak{R}^n, \\ \mu_x^y(.) &= \mu[. \mid X(a) = x, X(b) = y], \quad x, y \in \mathfrak{R}^n. \end{aligned}$$

The problem is reduced to the one studied above, since we have

$$H(\nu;\mu) = \mathbf{E}_\nu\left[\log \frac{d\mathbb{P}}{d\mathbb{R}}(X(a), X(b))\right] + \mathbf{E}_\nu\left[\log \frac{d\nu_{X(a)}^{X(b)}}{d\mu_{X(a)}^{X(b)}}\right]$$

$$
\begin{aligned}
&= \int \log \frac{d\mathbb{P}}{d\mathbb{R}} d\mathbb{P} + \int \int \int \log \frac{d\nu_x^y}{d\mu_x^y} (.) \, d\nu_x^y (.) \, d\mathbb{P}(x,y) \\
&= H(\mathbb{P}; \mathbb{R}) + \int H(\nu_x^y; \mu_x^y) \, d\mathbb{P}(x,y), \quad \text{si } \nu \ll \mu.
\end{aligned}
$$

The right hand side of this relation is minimum (zero) iff $\nu_x^y = \mu_x^y, \mathbb{P}-a.a.$, $x, y \in \mathfrak{R}^n$. Thus, the problem becomes

$$
\min_{\mathbb{P} \in \mathbb{E}} H(\mathbb{P}; \mathbb{R}).
$$

Thus the minimizing one is given by:

$$
\nu^* (.) = \int \mu_x^y (.) \, d\mathbb{Q}^* (x,y).
$$

We find

$$
\begin{aligned}
\frac{d\nu^*}{d\mu} (.) &= \frac{d\mathbb{Q}^*}{d\mathbb{R}} (X(a), X(b)) \\
&= \frac{\varphi_a (X(a)) \, \varphi_b (X(b))}{\Phi_a (X(a))} \int \exp \left(\int_a^b c(r, \xi(r) \, dr \right) \mathbb{I}_{b\, <T} d\mathbf{P}_a,
\end{aligned}
$$

from which

$$
d\nu^* (.) = \frac{\varphi_b (X(b))}{\varphi (a, X(a))} \exp \left(\int_a^b c(r, X(r)) \, dr \right) \mathbb{I}_{b\, <T} d\mu_a^\xi, \tag{13}
$$

with $\varphi(a, X(a)) = \Phi_a (X(a)) / \varphi_a (X(a))$.
Note that $\varphi_a (X(a)) > 0, \mathbb{P}_a - a.s.$, and then $\mu_a^\xi - a.s.$, since $\log \varphi_a \in L^1 (\mathbb{P}_a)$. Furthermore, letting

$$
d\nu_{ax}^* (.) = \frac{\varphi_b (X(b))}{\varphi (a, X(a))} \exp \left(\int_a^b c(r, X(r)) \, dr \right) \mathbb{I}_{b\, <T} d\mu_{ax}^\xi, \tag{14}
$$

where

$$
\mu_a^\xi (.) = \int \mu_{ax}^\xi (.) \, \Phi_a (x) \, dx,
$$

we have

$$
\nu^* (.) = \int \nu_{ax}^* (.) \, \Phi_a (x) \, dx. \tag{15}
$$

4. Variational Characterization for the case c unbounded

Assume that the Markov process $\xi(t)$ is a $n-$dimensional Wiener process $W(t)$, $t \in [a,b]$, with initial distribution $\mathbb{P}_a$. Thus μ_a^{ξ} and μ_{ax}^{ξ} will be replaced by μ_a^w and μ_{ax}^w and we shall take the same notations for μ, $\nu^*$and ν_{ax}^*. Let's recall the result of Girsanov's transformation.

Lemma 7. *Let* $(\Omega_0, \mathcal{M}, \nu)$ *probability space,* $\mathcal{N}_t = \sigma(X(s),\ a \le s \le t)$ *a non decreasing family of sub-*σ*-algebras of* $\mathcal{M}$*. Assume that* $\nu \ll \mu_a^w$*, then, on* $(\Omega_0, \mathcal{M}, \nu)$, $\exists$ *a Wiener* $w = (w(t), \mathcal{N}_t)$, $t \in [a,b]$ *and a nonanticipatif process* $\upsilon = (\upsilon(t), \mathcal{N}_t)$ *such that:*

$$X(t) = X(a) + \int_a^t \upsilon(r)\,dr + w(t),\ t \in [a,b], \nu - a.s.$$

$$\nu\left[\int_a^b \upsilon^2(t)\,dt < \infty\right] = 1,$$

$$\frac{d\nu}{d\mu_a^w}(.) = \exp\left(\int_a^b \upsilon(t)\,dX(t) - \frac{1}{2}\int_a^b \upsilon^2(t)\,dt\right), \nu - a.s.$$

Thus, according to (13) with μ_a^{ξ} replaced by μ_a^w, we have $\nu^* \ll \mu_a^w$ and then $\exists\,(w^*(t), \upsilon^*(t))$, such that:

$$X(t) = X(a) + \int_a^t \upsilon^*(r)\,dr + w^*(t),\ t \in [a,b],\ \nu^* - a.s., \tag{16}$$

$$\nu^*\left[\int_a^b (\upsilon^*(t))^2\,dt < \infty\right] = 1, \tag{17}$$

$$\frac{d\nu^*}{d\mu_a^w}(.) = \exp\left(\int_a^b \upsilon^*(t)\,dX(t) - \frac{1}{2}\int_a^b (\upsilon^*(t))^2\,dt\right)\quad \nu^* - a.s.. \tag{18}$$

Now we are ready to extend a Theorem given by Wakolbinger[12] to a large class of potentials c.

Let $\mathcal{A}$ be the class of n-dimensional non anticipatif stochastic processes $\upsilon(t)$ with values in $\mathfrak{R}^n$ relatively $(\Omega_0, \mathcal{M}, \mathcal{N}_t, \nu)$, satisfying
(H_5): (i) $\nu\{b < T\} = 1$;

(ii) the marginal of ν at times a and b are $\mathbb{P}_a$ and $\mathbb{P}_b$ with $\log \dfrac{d\mathbb{P}_a}{dx} \in L^1(\mathbb{P}_a)$ and $\log \dfrac{d\mathbb{P}_b}{dy} \in L^1(\mathbb{P}_b)$;

(iii) $\mathbf{E}_\nu \left[\int_a^b |\upsilon(t)|^2 \, dt\right] < \infty$;

(iv) $X(t) - X(a) - \int_a^t \upsilon(r)\, dr$, $a \le t \le b$, is a standard Brownian motion on $[a, b]$ with respect to ν;

(v) $J(a, b, \upsilon) = \mathbf{E}_\nu \left\{\int_a^b \left[\frac{1}{2}\|\upsilon(r)\|^2 - c(r, X(r))\right] dr\right\}$ is defined.

(H_6): $\mathbf{E}_{\nu^*}\left[\int_a^b c(r, X(r))\, dr\right] < \infty$. (This condition is satisfied if c is bounded.)

Theorem 8. *Let (H₁)-(H₆) hold. Then* $\exists \vartheta^* = (\upsilon^*(t), \nu^*) \in \mathcal{A}$, $t \in [a, b]$, *such that:*

$$-\infty < J(a, b, \upsilon^*) = \min_{\upsilon \in \mathcal{A}} J(a, b, \upsilon) < \infty;$$

where ν^* *is the Csiszar's projection of* μ *given by*

$$d\mu = \frac{\exp\left(\int_a^b c(r, X(r))\, dr\right) \mathbb{I}_{b\, <T} d\, \mu_a^w}{\int \exp\left(\int_a^b c(r, X(r))\, dr\right) \mathbb{I}_{b\, <T} d\, \mu_a^w}, \quad on$$

$$\mathbf{E} = \{\nu \;/\; \mathcal{M} : \nu[X(a) \in .] = \mathbb{P}_a(.)\,,\ \nu[X(b) \in .] = \mathbb{P}_b(.)\}$$

that is, its joint end-points distributions measure is $\mathbb{Q}^*$ *given by Theorem 1 with* q *given by (6) (Theorem 4).*

REFERENCES

[1] Benchettah, A. Commande Optimale Stochastique pour l'Equation de Fokker-Planck it MMR **Vol. 11** N°1, (2002).

[2] Csiszar, I. I-Divergence Geometry of Probability Distribution and Minimization Problems *The Annals of Probability* **Vol. 3** N°1, (1975), 146–158. Sanov Property, Generalized I-Projection and a Conditional Limit Theorem. *The Annals of Probability* **Vol. 12** N°3, (1984), 768–793.

[3] Dawson, D., L. Gorostiza and A. Wakolbinger Schrödinger Processes and Large Deviation *J. Math. Phys.* **31(10)** (1990), 2385–2388.

[4] Dynkin, E.B. Theorie des Processus Markoviens. Dunod, Paris 1963. Markov Process **Vol. 1 et 2** Springer-Verlag, Berlin, Heidelberg, 1965.

[5] Föllmer, H. Random Fields and Diffusion Processes. in Ecole d'été de Saint Flour XV-XVII 1985-1987, Lecture Note in Mathemetics 1362, Springer-Verlag, Berlin, 1988.

[6] Jamison, B. Reciprocal Processes. Z. *Wahrscheinlichkeitstheorie ver. Gebiele* **30 65** (1974).

[7] Lipster, R. S., A. N. Shiryayev Statistics of Random Processes I General theory. Springer-Verlag, Berlin, Heidelberg, 1977.

[8] Nagazawa, M. Stochastic Variational Principle of Schrödinger Processes. in Seminar on Stochastic Processes 1989, Birkhäuser, Basel, 1990.

[9] Roelley, S., M. Thieullen A Characterisation of Reciprocal Processes via an Integration by Parts Formula on the Path Space. *Probability Theory and Related Fields* **Vol. 123** N°1, (2002).

[10] Schrödinger, E. Über die Umkehrung der naturgesetze. Sitzungber. *Preuss. Akad. Wiss. Phys. Math.* **K1** (1931), 144.

[11] Strook, D. W., S. R. S. Varadhan Multidimensional Diffusion Processes. Springer-Verlag, Berlin, Heidelberg, 1979.

[12] Wakolbinger, A. A Simplified Variational Characterisation of Schrödinger Processes. *J. Math. Phys.* **30** (1989), 2943.

A. Benchettah
Université d'Annaba,
Faculté des sciences,
Département de Mathématiques,
BP 12, Annaba 23000
e-mail: abenchettah@hotmail.com