PLISKA
STUDIA MATHEMATICA
BULGARICA

ПЛИСКА
БЪЛГАРСКИ
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. Bulgar. **17** (2005), 201–220

PLISKA
STUDIA MATHEMATICA
BULGARICA

SPLIT-ARCH

Biljana Č. Popović, Vladica S. Stojanović [1]

We supplied the GARCH Zoo with the new model and introduce it in this paper. We named it *Split–ARCH*. It was empirically motivated by means of the real data set on soybean meal price on the Product exchange. Split–ARCH is the superstructure of the previously known models of GARCH type. We defined volatility exchange to follow sudden and great changes of the price, and volatility also. As far as the log returns of the price are defined as $X_n = \sigma_n \varepsilon_n$, we set the volatility to be

$$\sigma_n^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{n-j}^2 + \sum_{k=1}^{q} f_k(\sigma_{n-k}^2) I\left(\varepsilon_{n-k}^2 > c\right) \quad , \quad n \geq 0$$

with the threshold $c > 0$. Under the stationarity conditions and specified f, we discus the possibilities of estimating parameters in this paper also.

1. Introduction. Conditional Heteroscedasticity

The stochastic analysis of financial sequences is commonly based on the time series modelling of data set which will be able to describe the distribution or behavior of a real data. It has been shown empirically that the most of financial series exhibit nonlinear changes in the dynamics which obviously will imply nonlinearity of the corresponding stochastic models.

[1]Partly supported by grant 1834 MSEP Republic of Serbia
2000 *Mathematics Subject Classification:* 62M10
Key words: conditional heteroscedasticity, conditional least squares

Further on, we shall suppose that there exists the probability space $(\Omega, \mathcal{F}, P)$ and a *filter* of nondecreasing σ-algebras $F = (\mathcal{F}_n)$,

$$\mathcal{F}_n \subseteq \mathcal{F}_m \subseteq \mathcal{F}, \quad \forall n \leq m$$

which consists of all the information available to any trader on the market in a certain moment of time.

Further on we shall use the well known representation of the financial series (X_n) described by the quasi–Gaussian distribution:

$$X_n = \sigma_n \varepsilon_n, \quad n \in D \tag{1}$$

where (σ_n) is a sequence of $\mathcal{F}_{n-1}$ – measurable random variables (the volatility sequence), and (ε_n) is the sequence of independent $\mathcal{F}_n$ – measurable random variables with $\mathcal{N}(0,1)$ distribution (so called 'white noise'). If the filter $(\mathcal{F}_n)$ is generated by $\varepsilon_1, \ldots, \varepsilon_n$, then according to (1), the sequence (X_n) is the sequence of uncorrelated random variables with the unconditional mean and variance

$$E\left(X_n\right) = E\left[E(X_n | \mathcal{F}_{n-1})\right] = 0,$$

$$Var(X_n) = E\left(X_n^2\right) = E\left[E(X_n^2 | \mathcal{F}_{n-1})\right] = E\left(\sigma_n^2\right)$$

respectively.

Robert Engle [5] introduced the recursive representation of the volatility in 1982. He named it *autoregressive conditional heteroscedastic (ARCH) model.* Tim Bolerslev [1] spread this idea in 1986. He introduced *generalized autoregressive conditional heteroscedastic (GARCH) model* where the volatility sequence was described by

$$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2. \tag{2}$$

These two models were able to explain a number of the properties of financial indexes (first of all heavy tails and clustering). Meanwhile, the lack of information about the increasing or decreasing direction of changes in the volatility sequence (σ_n) in both of these two models is evident. Many of the empirical data sets indicate outstanding nonlinearity of the empirical volatility:

$$\hat{\sigma}_n = \sqrt{\frac{1}{n-1} \sum_{k=0}^{n} \left(X_k - \overline{X}_k\right)^2}, \tag{3}$$

which can be manifested in the various manners. (We illustrate one of such situations in Fig.1, where the sharp growth of the price occurs in a relatively short time interval, and, consequently, empirical volatility growths extremely sharp.)

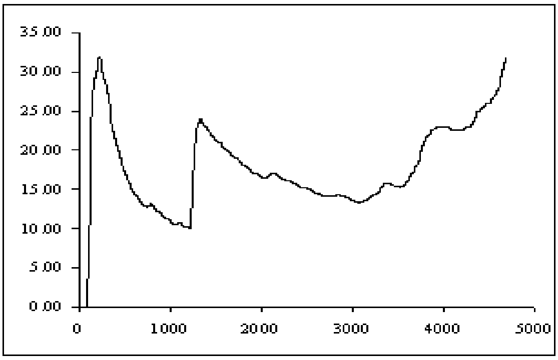

Figure 1: Empirical volatility of oil price. (Sorce: WTI Posted, Wall Street Journal)

In order to solve these problems, many generalizations of standard ARCH and GARCH models were done. So, Zakoian [9], following some linear models, defined some threshold (TGARCH) models. From the other side, Fornari and Mele [6] introduced so called *switching ARCH-model* following the increasing or decreasing volatility value by the sign of the elements of the sequence (X_n), or the same, the elements of the sequence (ε_n).

These models describe the asymmetric reaction of the conditional variance caused by sudden changes of price. But, they can not determine the values which will properly correspond to the changes of values of (X_n) and, also, the volatility sequence (σ_n^2). That's why the application to the real data set can cause the significant distinction between the empirical and the modelled values.

In this paper, we followed the idea of designing the new model of the conditional heteroscedasticity. The volatility sequence of this model will follow the magnitude of changes in (X_n). Our model will follow ARCH regime for the 'small' absolute values of white noise and GARCH regime for the others. The precise definition will be given in the next section. We will show that our model will response better to the sudden and unexpected 'jumps' in volatility sequence than ARCH, for instance. Because of its splitting reaction, we named the model *Split–ARCH*.

2. Definition and Main Properties

We introduce Split–ARCH subject to the experimental investigation some of which we shall display below. We shall specially point out one 'small sample' experiment – soybean meal price data from Product Exchange Novi Sad and a 'large sample' one – oil price data according to the Wall Street Journal source. Our model describes nonlinear behavior of volatility caused by the great fluctuation of price. The fluctuation of price implies market reaction that produces great oscillation in the volatility sequence. Split–ARCH will follow such oscillation.

The general definition of Split–ARCH will follow equation (1) and the following one:

$$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{n-i}^2 + \sum_{j=1}^{q} f_j(\sigma_{n-j}^2) I\left(\varepsilon_{n-j}^2 > c\right), \quad n \in D\,. \tag{4}$$

The order (p, q) of this model is analogues to the standard GARCH model.

The coefficients of the model satisfy the conditions $\alpha_0 > 0$ and $\alpha_i \geq 0$, while $f_j = f_j(u),\ u \geq 0$ is a nonnegative $\mathcal{F}_{n-j}-$ measurable function of the volatility sequence which will specify the reaction on the extremely large values in (ε_n). Obviously, it will be difficult to discuss the model and its properties, specially its application in the general case of f_j. So, further on, we shall investigate just the class of linear functions:

$$f_j(u) = \beta_0^{(j)} + \beta_1^{(j)} u, \quad j = 1, ..., q\,, \tag{5}$$

where $\beta_0^{(j)}, \beta_1^{(j)} \geq 0$.

The constant $c > 0$ will be chosen as a proper *critical value for the reaction*, i.e. it will be the level which will determine which value of the noise will be statistically significant to let the inclusion of the previous value of the volatility in the autoregression sum of (4).

As it is well known, according to the χ_1^2 distribution for the elements of (ε_n^2), it will be easily verified that the level c and the significant level m_c will be connected in the following way

$$m_c = P(\varepsilon_n^2 > c) = \frac{1}{\sqrt{2\pi}} \int_c^{\infty} x^{-1/2} e^{-x/2} dx\,. \tag{6}$$

So, if we set the significant level $\alpha : 0 < \alpha < 1$, the level for the reaction will be determined as

$$m_c = \alpha$$

and vice versa.

We shall set now some properties which will be used in estimating parameters of Split–ARCH. In order to prove the stationarity of the model, we shall follow the methodology used for standard GARCH model (see, for instance, [8]). The stochastic difference equation of order one:

$$Y_{n+1} = A_n Y_n + B_n \tag{7}$$

will represent Split–ARCH iff

$$Y_n = \left(\sigma_n^2, \ldots, \sigma_{n-q+1}^2, X_{n-1}^2, \ldots, X_{n-p+1}^2\right)',$$

$$A_n = \begin{pmatrix} \alpha_1\, \varepsilon_n^2 + \beta_1^{(1)} \psi_n(c) & \beta_1^{(2)} \psi_{n-1}(c) & \ldots & \beta_1^{(q)} \psi_{n-q+1}(c) & \alpha_2 & \ldots & \alpha_{p-1} & \alpha_p \\ 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \varepsilon_n^2 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

$$B_n = \left(\alpha_0 + \sum_{j=1}^{q} \beta_0^{(j)} \psi_{n-k+1}(c),\, 0, \ldots, 0\right)', \qquad \psi_n(c) = I\left(\varepsilon_n^2 > c\right),$$

where $n \geq \max\{p, q\}$.

Next, we shall set the conditions for the wide sense stationarity of the model.

Theorem 2.1. *Let the model, Split–ARCH, be defined by the equations (1), (4) and (5). Then, the following conditions are equivalent:*

- *(i) The polynomial*

$$P(\lambda) = \lambda^M - \sum_{j=1}^{M} \gamma_j \, \lambda^{M-j},$$

where

$$M = \max\{p, q\}, \quad \gamma_j = \begin{cases} \alpha_j + m_c \beta_1^{(j)}, & 1 \le j \le \min\{p, q\} \\ \alpha_j, & q < p \wedge q < j \le p \\ m_c \beta_1^{(j)}, & p < q \wedge p < j \le q \end{cases},$$

has the roots $\lambda_1, ..., \lambda_M$ which satisfy the condition

$$|\lambda_j| < 1, \quad \forall j = 1, ..., M. \tag{8}$$

- *(ii) The time series (X_n^2) is wide sense stationary with mean value:*

$$E\left(X_n^2\right) = \left(\alpha_0 + m_c \sum_{j=1}^{q} \beta_0^{(j)}\right) \left(1 - \sum_{i=1}^{p} \alpha_i - m_c \sum_{j=1}^{q} \beta_1^{(j)}\right)^{-1} \tag{9}$$

and correlation function $\rho(h) = Corr\left(X_n^2, X_{n+h}^2\right), \ h \ge 0$ which satisfies the equation:

$$\rho(h) = \sum_{j=1}^{M} \gamma_j \, \rho(h - j), \quad h \ge M \tag{10}$$

with the initial conditions:

$$\rho(0) = 1, \quad \rho(h) - \sum_{j=1}^{M} \gamma_j \, \rho(h - j) = 0, \quad 0 < h < M.$$

- *(iii)* $$\sum_{j=1}^{M} \gamma_j = \sum_{i=1}^{p} \alpha_i + m_c \sum_{j=1}^{q} \beta_1^{(j)} < 1.$$

Proof. See the Appendix.□

Further investigation and the application will concern only the simplest case, $p = q = 1$. The best explanation for this choice is the goodness of fit of the model to the real data.

Let $f_1(u) = \beta_0 + \beta_1 u$, i.e. let $f_1(\sigma_n^2)$ be the linear function of volatility, which will define the volatility sequence as the following threshold model

$$\sigma_n^2 = \alpha_0 + \alpha_1 X_{n-1}^2 + \left(\beta_0 + \beta_1 \sigma_{n-1}^2\right) \psi_{n-1}(c), \tag{11}$$

where $\psi_{n-1}(c)$ is the threshold function:

$$\psi_{n-1}(c) = \begin{cases} 0, & \varepsilon_{n-1}^2 \le c \ , \\ 1, & \varepsilon_{n-1}^2 > c \ . \end{cases}$$

It means that

$$\sigma_n^2 = \begin{cases} \alpha_0 + \alpha_1 X_{n-1}^2, & \varepsilon_{n-1}^2 \le c \ , \\ \\ \alpha_0 + \beta_0 + \alpha_1 X_{n-1}^2 + \beta_1 \sigma_{n-1}^2, & \varepsilon_{n-1}^2 > c \ . \end{cases} \tag{12}$$

Let us remark that the equation (12) enables us to apply the standard ARCH or GARCH procedure to solve the volatility sequence subject to the value of the noise (ε_n^2). This fact will make us easier the investigation of main stochastic properties of the sequence (X_n^2). According to the Theorem 2.1, the sequence (X_n^2) is stationary iff $\alpha_1 + m_c\beta_1 < 1$, and

$$E(X_n^2) = \frac{\alpha_0 + \beta_0 m_c}{1 - \alpha_1 - \beta_1 m_c}. \tag{13}$$

Also, using the fact that

$$E(X_n^4) = 3E(\sigma_n^4) = \frac{3\left(\alpha_0^2 + \beta_0^2 m_c^2\right)(1 + \alpha_1 + \beta_1 m_c)}{(1 - \alpha_1 - \beta_1 m_c)\left(1 - 3\alpha_1^2 - \beta_1^2 m_c^2 - 2\alpha_1\beta_1 m_c\right)},$$

we can compute the stationarity value of Kurtosis:

$$K = \frac{E(X_n^4)}{\left(E(X_n^2)\right)^2} = \frac{3\left(1 - (\alpha_1 + \beta_1 m_c)^2\right)}{1 - 3\alpha_1^2 - \beta_1^2 m_c^2 - 2\alpha_1\beta_1 m_c} \ge 3$$

which indicates peaky density functions for the elements of the sequence (X_n). This is the same with ARCH. Also, $K = 3$ iff $\alpha_1 = 0$ and that is just the case when the Split–ARCH reduces to the white noise.

The correlation function $\rho(h)$ can be determined almost in the same way as it was done with ARCH/GARCH models. It is easy to verify that

$$\begin{cases} \rho(1) = \alpha_1 \left(1 - \alpha_1\beta_1 m_c - \beta_1^2 m_c^2\right)\left(1 - 2\alpha_1\beta_1 m_c - \beta_1^2 m_c^2\right)^{-1} \\ \\ \rho(h) = (\alpha_1 + \beta_1 m_c)^{h-1}\rho(1), \qquad h > 1 \end{cases} \tag{14}$$

and it is obvious that the correlation function decreases with geometric rate to 0, like it was with ARCH/GARCH type models.

Finally, we shall set the theorem which is the implication of some well known results (see, for instance, [8]) concerning necessary and sufficient conditions for strong stationarity of the GARCH type models. In the case of Split–ARCH$(1,1)$ the following proposition is valid:

Theorem 2.2. *Let the model Split–ARCH be defined by the equations (1) and (11). The stochastic difference equation (7) has unique, strong stationary and ergodic solution of the form*

$$Y_n = B_n + \sum_{k=1}^{\infty} A_{n-1} \dots A_{n-k} B_{n-k-1}, \quad n \in D \tag{15}$$

iff $E\left(\ln(\alpha_1 \varepsilon_n^2 + \beta_1 \psi_n(c))\right) < 0$.

Proof. See the Appendix.□

3. Estimation of Parameters. Application of the Model

We shall generate now Split–ARCH$(1,1)$ subject to the data set.

The sequence (X_n) may represents the log–returns of any financial index (price) (P_n) and is defined as

$$X_n = \ln\left(\frac{P_n}{P_{n-1}}\right), \quad n > 0\,.$$

According to the previous results, we can use the part of only one realization of the log–return process:

$$X_t = x_t\,,\ t = 1, \dots, N \quad (x_0 = 0)\,. \tag{16}$$

Suppose also that the unknown parameter $\theta = (\alpha_0, \alpha_1, \beta_0, \beta_1)' \in \mathbb{R}^4$ belongs to the set

$$\Theta = \{\theta\,|\alpha_1 + \beta_1 m_c < 1\}$$

which is the available set of parameters subject to the stationarity condition (Theorem2.1) of Split–ARCH.

As the estimation procedure, we shall use the conditional least squares method and minimize the sum

$$S_N = \sum_{t=1}^{N} \left[X_t^2 - E\left(X_t^2\,|\mathcal{F}_{t-1}\right)\right]^2\,. \tag{17}$$

First of all, we shall stratify the sample (16) according to the fact that our model performs in two regimes:

$$A_{N,c} = \left\{X_t \,|\, \varepsilon_{t-1}^2 \le c\right\}, \quad B_{N,c} = \left\{X_t \,|\, \varepsilon_{t-1}^2 > c\right\} .$$

Further on, we shall use the notation

$$\boldsymbol{a} = (\alpha_0, \alpha_1)', \quad \Theta_1 = \{\boldsymbol{a} \,|0 < \alpha_1 < 1\} \tag{18}$$

$$\boldsymbol{b} = (\alpha_0 + \beta_0, \alpha_1 + \beta_1)', \quad \Theta_2 = \{\boldsymbol{b} \,|0 < \alpha_1 + \beta_1 < 1\} . \tag{19}$$

If we proceed now the above mentioned method of conditional least squares regressing the elements of the sequence (X_t^2) on the volatility values $\sigma_t^2 = E\left(X_t^2 \,|\mathcal{F}_{t-1}\right)$, the result will be as follows.

Split–ARCH will obey the ARCH structure on the data set $A_{N,c}$. So,

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$$

and the sum of squares (17) will become

$$S_N'(\alpha_0, \alpha_1) = \sum_{X_t \in A_{N,C}} \left(X_t^2 - \alpha_0 - \alpha_1 X_{t-1}^2\right)^2 . \tag{20}$$

From here we have the estimates of parameters

$$\widehat{\boldsymbol{a}}_N = \begin{pmatrix} \widehat{\alpha}_0 \\ \widehat{\alpha}_1 \end{pmatrix} = \begin{pmatrix} N_1 & \sum X_{t-1}^2 \\ \sum X_{t-1}^2 & \sum X_{t-1}^4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum X_t^2 \\ \sum X_t^2 X_{t-1}^2 \end{pmatrix} \tag{21}$$

and $N_1 = |A_{N,c}| = \sum_{t=1}^{N} I(\varepsilon_{t-1}^2 \le c)$, where all the summations are subject on t such that $X_t \in A_{N,c}$.

From the other side, the elements of set $B_{N,c}$ satisfy the relation

$$\sigma_n^2 = \alpha_0 + \beta_0 + \alpha_1 X_{n-1}^2 + \beta_1 \sigma_{n-1}^2$$

meaning that the model is of GARCH type. So, a common way of estimating parameters is some iterative method, like Newton–Raphson's procedure, for instance. Meanwhile, instead of that, we can use the maximum likelihood estimator for the elements of the volatility sequence:

$$\widehat{\sigma}_t^2 = X_t^2, \quad 1 \le t \le N$$

and, after that, determine the regression coefficients $\boldsymbol{b}$ applying the least squares optimization procedure on the specified sum

$$(22)\qquad S''_N(\alpha_0+\beta_0,\alpha_1+\beta_1)=\sum_{X_t\in B_{N,C}}\left[X_t^2-(\alpha_0+\beta_0)-(\alpha_1+\beta_1)X_{t-1}^2\right]^2 .$$

This implies

$$(23)\qquad \widehat{\boldsymbol{b}}_N=\begin{pmatrix} N_2 & \sum X_{t-1}^2 \\ \sum X_{t-1}^2 & \sum X_{t-1}^4 \end{pmatrix}^{-1}\cdot\begin{pmatrix} \sum X_t^2 \\ \sum X_t^2X_{t-1}^2 \end{pmatrix}$$

and $N_2=|B_{N,c}|=N-N_1$, where the summations are subject on t such that $X_t\in B_{N,c}$. Finally, (21) and (23) imply

$$\begin{pmatrix} \widehat{\beta}_0 \\ \widehat{\beta}_1 \end{pmatrix}=\widehat{\boldsymbol{b}}_N-\widehat{\boldsymbol{a}}_N .$$

This two–step procedure asserts the asymptotic properties of the estimates which can be formulated as the following proposition.

Theorem 3.1. *Let for some $N_0>0$ and all $N\ge N_0$ the following conditions be satisfied*

$$\widehat{\mathbf{a}}_N\in\Theta_1,\quad \widehat{\mathbf{b}}_N\in\Theta_2,\quad \widehat{\theta}_N=\left(\widehat{\mathbf{a}}_N,\ \widehat{\mathbf{b}}_N-\widehat{\mathbf{a}}_N\right)'\in\Theta .$$

Then, $\widehat{\boldsymbol{a}}_N$ and $\widehat{\boldsymbol{b}}_N$ are strong consistent and asymptotically normal estimates of the parameters $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Proof. See the Appendix.□

This estimating procedure can be easily applied to the real data set of some financial time series. One estimating result of this estimating procedure is given in the following table:

Model	ARCH	Split–ARCH (stratums)	
		I	II
Sample (N)	2500	1853	647
Parameters	0.000548	0.000619	0.000843
	0.306	0.0108	0.319
Correlation	98.48%	98.81%	

The sample size is 2 500 elements of the one realization of log–returns of the market price of oil according to WTI Posted, data base *Scotia Group* and *Wall Street Journal*.

In order to compare the results, first of all, we set the estimating procedure for the standard ARCH model on the set of data. The result was as follow:

$$\begin{cases} X_n = \sigma_n \varepsilon_n, \\ \sigma_n^2 = 5.482 \cdot 10^{-4} + 0.306 \cdot X_{n-1}^2 . \end{cases}$$

The registered correlation between the empirical data series and the one generated by the model was 98.48%. After that, the real data set was processed in two–step Split–ARCH modelling scheme. The stratification was done using the estimates

$$\widehat{\varepsilon}_n = X_n / \hat{\sigma}_n, \quad n = 1, \ldots, N ,$$

where $\hat{\sigma}_n$ was the empirical standard deviation of the sample. The illustration is given in Figure 2 (up and left). As a critical value for the reaction, we used the mean value of the χ_1^2 distribution and

$$c = E(\varepsilon_n^2) = 1 .$$

That was the starting point for the Split–ARCH estimation (and prediction) of the real data set:

$$\begin{cases} X_n = \sigma_n \varepsilon_n, \\ \sigma_n^2 = 6.187 \cdot 10^{-4} + 0.0108 \cdot X_{n-1}^2 + \left(2.243 \cdot 10^{-4} + 0.3082 \cdot \sigma_{n-1}^2\right) \psi_{n-1}(1) . \end{cases}$$

The correlation coefficient to the real data is somewhat greater (98.81%) for the last model than the ARCH one. Also, one can see in Figure 2 that the fluctuation of Split–ARCH values is more likely the real data values than when comparing ARCH approximation to the same data set.

We got the similar results for the soybean meal real data set of sample size 190.

The empirical variance for this data set is illustrated in Figure 3 and the estimated ARCH and Split–ARCH models as well as the log–return volatility of the real data, in Figure 4. The ARCH approximation was

$$\begin{cases} X_n = \sigma_n \varepsilon_n, \\ \sigma_n^2 = 1.2835 \cdot 10^{-3} + 0.6278 \cdot X_{n-1}^2 \end{cases}$$

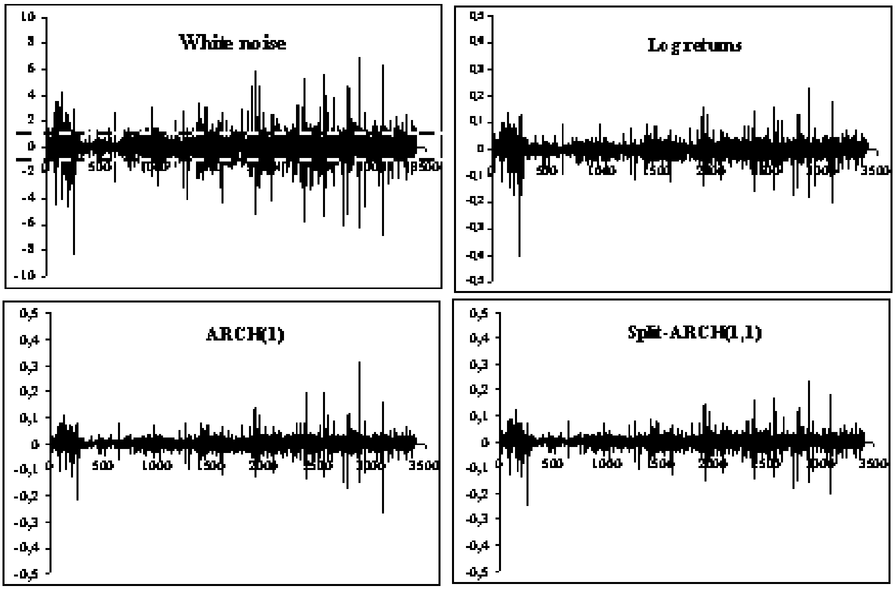

Figure 2: Comparative illustrations for the original oil data series, ARCH and Split-ARCH model.

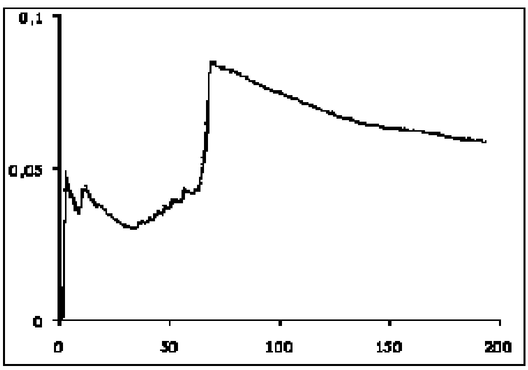

Figure 3: Empirical volatility of soybean meal price. (Source: Product Exchange Novi Sad)

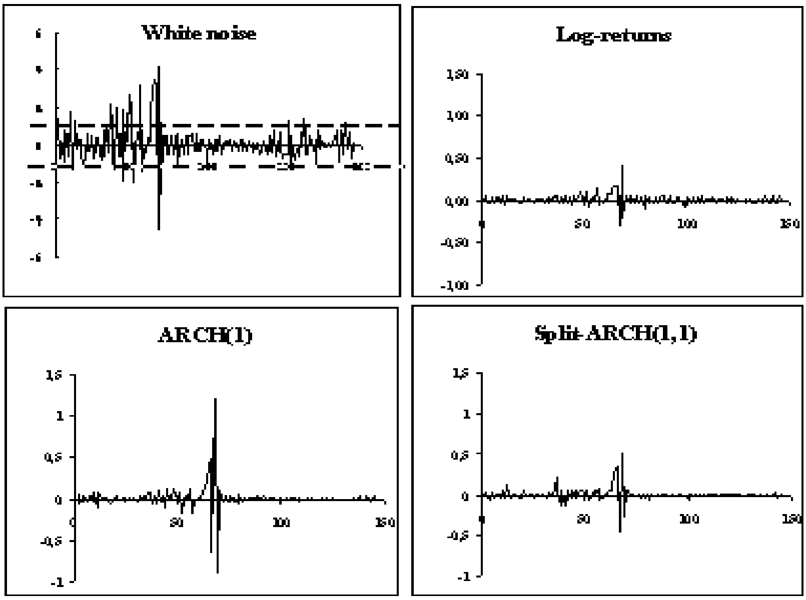

Figure 4: Comparative illustrations for the original soybean meal data series, ARCH and Split-ARCH model.

and the Split–ARCH one was

$$\begin{cases} X_n = \sigma_n \varepsilon_n, \\ \sigma_n^2 = 7.453 \cdot 10^{-4} + 0.4572 \cdot X_{n-1}^2 + \left(6.1687 \cdot 10^{-3} + 0.1215 \cdot \sigma_{n-1}^2\right) \psi_{n-1}(1) . \end{cases}$$

4. Concluding Remarks

The popularity of nonlinear quasi–Gaussian models can be explained, as we emphasized before, mostly by the fact that they gave the explanation to many different features of financial indexes (as, for instance, clustering and peaky behavior of the empirical density function). Meanwhile, there are many phenomena which can not be explained in this way.

Many of the contemporary researchers criticize Gaussianity assumption for the data set. So, the practitioners often use models with some non–Gaussian distribution (Student distribution, for instance) which they find more convenient in displaying fluctuation in the data set.

Such type of generalization is also convenient for the Split–ARCH model. Even more than that, the Gaussianity assumption has been made here just to simplify the estimating procedure of the residuals $(\hat{\varepsilon}_n)$, preciously, to determine the threshold constant c for the purpose of stratifying the realization of the process. The provident investigation should include testing concerning white noise distribution. But, the constructing procedure described in this paper will stay the same even in the case of non–Gaussianity assumption.

5. Appendix

Proof of Theorem 2.1.

- $(i) \Rightarrow (ii)$: According to (7), it is valid for any $n, k > 0$:

$$E(Y_{n+k}) = (I + A + A + \cdots + A^{k-1})\, B + A^k\, E(Y_n) \tag{24}$$

where

$$A = E(A_n) = \begin{pmatrix} \alpha_1 + \beta_1^{(1)} m_c & \beta_1^{(2)} m_c & \ldots & \beta_1^{(q)} m_c & \alpha_2 & \ldots & \alpha_{p-1} & \alpha_p \\ 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

$$B = E(B_n) = \left(\alpha_0 + m_c \sum_{j=1}^{q} \beta_0^{(j)},\ 0, \ldots, 0 \right)'.$$

After some computation it will be seen that

$$det\,(A-\lambda I)=(-1)^{p+q-1}\left(\lambda^{p+q-1}-\sum_{i=1}^{p}\alpha_i\lambda^{p+q-i-1}-\sum_{j=1}^{q}\beta_1^{(j)}m_c\lambda^{p+q-j-1}\right),$$

and

$$det\,(A-\lambda I)=(-1)^{p+q-1}\lambda^m P(\lambda),$$

where $m=\min\{p-1,q-1\}$. It means that the matrix A has m trivial eigenvalues ($\lambda_1^{(t)}=\cdots=\lambda_m^{(t)}=0$), while the rest of them, $(\lambda_1,\ldots,\lambda_M)$, are the roots of the characteristics polynomial $P(\lambda)$. The convergence then follows according to the assumption (8):

$$\sum_{j=0}^{k-1}A^j\to(I-A)^{-1}\,,\quad A^k\to\mathbb{O}\,,\quad k\to\infty\,.$$

The equality (24) then becomes:

$$E(Y_n)=(I-A)^{-1}\,B=\left(\alpha_0+m_c\sum_{j=1}^{q}\beta_0^{(j)}\right)\left(1-\sum_{i=1}^{p}\alpha_i-m_c\sum_{j=1}^{q}\beta_1^{(j)}\right)^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

i.e. (9) is valid.

The relation (10) can be proved in the similar way. The correlation function $\rho(h)$ will be calculated from the relation

$$\rho(h)=\frac{R(h)-E(X_n^2)^2}{R(0)-E(X_n^2)^2}\quad,\quad h\geq 0\,,$$

where

$$R(h)=E(X_n^2X_{n+h}^2)=E(X_n^2)+\sum_{j=1}^{M}\gamma_j\,R(h-j),\quad R(0)=E(X_n^4).$$

- *(ii)* ⇒ *(iii)*: As $\alpha_0>0$ and $\beta_0^{(j)}\geq 0\,, j=1,\ldots,q$, according to

$$E\left(X_n^2\right)=\left(\alpha_0+m_c\sum_{j=1}^{q}\beta_0^{(j)}\right)\left(1-\sum_{i=1}^{p}\alpha_i-m_c\sum_{j=1}^{q}\beta_1^{(j)}\right)^{-1}>0\,,$$

it will be

$$1-\sum_{i=1}^{p}\alpha_i-m_c\sum_{j=1}^{q}\beta_1^{(j)}>0$$

and that is obviously (iii).

- $(iii) \Rightarrow (i)$: Let

$$\mathcal{S}_r(A)=\max_j\{\lambda_j\},$$

the spectral radius of matrix A defined in (i). Then

$$\mathcal{S}_r(A)\leq ||A||$$

where we may set

$$||A||=\max\left\{\sum_{j=1}^{M}\gamma_j,1\right\}=1.$$

If we suppose that $\mathcal{S}r(A)=1$, then for some $\varphi\in[0,2\pi)$ there exists an eigenvalue $\lambda'=e^{i\varphi}$ which satisfies

$$P(\lambda')=e^{iM\varphi}-\sum_{j=1}^{M}\gamma_j\,e^{i(M-j)\varphi}=0\,.$$

After that, according to

$$\left|\,e^{iM\varphi}\right|\leq\sum_{j=1}^{M}\gamma_j\,\left|\,e^{i(M-j)\varphi}\right|\,,$$

it will be $\sum_{j=1}^{M}\gamma_j\geq 1$, which contradicts (iii). So,

$$\mathcal{S}_r(A)<1$$

and according to the above, it is equivalent to (i).□

Proof of Theorem 2.2 As $A_n=\alpha_1\varepsilon_n^2+\beta_1\psi_n(c)$, $B_n=\alpha_0\varepsilon_n^2+\beta_0\psi_n(c)$ and $Y_n=\sigma_n^2$, the proposition follows directly from the theorem 2.4 in [2] and theorem 1 in [7].□

Proof of Theorem 3.1 If we introduce the sequence

$$\upsilon_t = X_t^2 - \sigma_t^2, \qquad t = 1, \dots N$$

we shall have

$$E(\upsilon_t \mid \mathcal{F}_{t-1}) = E\left(X_t^2 \mid \mathcal{F}_{t-1}\right) - \sigma_t^2 = 0$$

i.e. (υ_t) , as a martingale difference, is the sequence of uncorrelated random variables. Then

$$X_t^2 = \sigma_t^2 + \upsilon_t = \alpha_0 + \beta_0\psi_{t-1}(c) + (\alpha_1 + \beta_1\psi_{t-1}(c))\, X_{t-1}^2 + \upsilon_t - \beta_1\psi_{t-1}(c)\upsilon_{t-1}$$

is the ARMA linear sequence with random coefficients and non–Gaussian "noise" (υ_t). This representation might be used for computing the spectral density of the X_t^2:

$$f(\omega) = \frac{Var(\upsilon_t)}{2\pi} \cdot \frac{1 - 2\beta_1 m_c \cos\omega + \beta_1^2 m_c^2}{1 - 2(\alpha_1 + \beta_1 m_c)\cos\omega + (\alpha_1 + \beta_1 m_c)^2}\,,$$

from here, we have:

$$f(0) = \frac{Var(\upsilon_t)}{2\pi}\left(\frac{1 - \beta_1 m_c}{1 - \alpha_1 - \beta_1 m_c}\right)^2.$$

Because of $\alpha_1 + \beta_1 m_c < 1$ for all $\theta \in \Theta$, the function $f(\omega)$ is continuous in $\omega = 0$. Then the sequences (X_t^2) and (υ_t) are ergodic and stationary .

From the other side, using the representation (21) we have

$$\widehat{\boldsymbol{a}}_N - \boldsymbol{a} = \begin{pmatrix} 1 & \frac{1}{N_1}\sum X_{t-1}^2 \\ \frac{1}{N_1}\sum X_{t-1}^2 & \frac{1}{N_1}\sum X_{t-1}^4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{1}{N_1}\sum \upsilon_t \\ \frac{1}{N_1}\sum \upsilon_t X_{t-1}^2 \end{pmatrix}, \qquad X_t \in A_{N,\,c}.$$

As

$$N_1 \overset{a.s.}{\to} \infty, \quad \text{when} \quad N \to \infty\,,$$

we may apply the ergodic theorem on the random sums of sequences (υ_t) and (X_t^2) (see, for instance [4]). Then, we shall have, when $N \to \infty$,

$$\begin{pmatrix} \frac{1}{N_1}\sum \upsilon_t \\ \frac{1}{N_1}\sum \upsilon_t X_{t-1}^2 \end{pmatrix} \overset{a.s.}{\to} 0, \quad \text{and} \quad \begin{pmatrix} 1 & \frac{1}{N_1}\sum X_{t-1}^2 \\ \frac{1}{N_1}\sum X_{t-1}^2 & \frac{1}{N_1}\sum X_{t-1}^4 \end{pmatrix}^{-1} \overset{a.s.}{\to} \Gamma^{-1}$$

where the random sums include only $X_t \in A_{N,\,c}$ and the second–moment matrix:

$$\Gamma = E\left(\mathbf{X_t}\mathbf{X_t'}\right), \qquad \mathbf{X_t} = \left(1, X_{t-1}^2\right)'$$

does not depend on t, for all $\boldsymbol{a}$ from the set of stationarity Θ_1.

These two convergences yield

$$\widehat{\boldsymbol{a}}_N - \boldsymbol{a} \overset{a.s.}{\to} 0, \quad N \to \infty$$

i.e. the estimator $(\widehat{\boldsymbol{a}}_N)$ is strictly consistent.

Now, we shell show the asymptotic normality of the sequence $(\widehat{\boldsymbol{a}}_N)$. We can write:

$$\sqrt{N_1}\,(\widehat{\boldsymbol{a}}_N - \boldsymbol{a}) = \mathbb{U}_{N_1}^{-1}\mathbb{V}_{N_1}$$

where:

$$\mathbb{U}_{N_1} = \frac{1}{N_1}\cdot\begin{pmatrix} N_1 & \sum X_{t-1}^2 \\ \sum X_{t-1}^2 & \sum X_{t-1}^4 \end{pmatrix}, \quad \mathbb{V}_{N_1} = \frac{1}{\sqrt{N_1}}\cdot\begin{pmatrix} \sum \upsilon_t \\ \sum \upsilon_t X_{t-1}^2 \end{pmatrix}.$$

For each $\mathbf{v} = (v_0, v_1)' \in \mathbb{R}^2$, the random sequence:

$$\sqrt{N}\mathbf{v}'\mathbb{V}_N = \sum_{t=1}^{N} \upsilon_t(v_0 + v_1 X_{t-1}^2)$$

is the martingale and, according to the Billingsley's central limit theorem for martingales, we have:

$$\mathbf{v}'\mathbb{V}_N \overset{d}{\to} \mathcal{N}(0, \mathbf{v}'\Lambda\mathbf{v})$$

where

$$\Lambda = E(\mathbf{Y_t}\mathbf{Y_t'}), \quad \mathbf{Y_t} = \upsilon_t\left(1, X_{t-1}^2\right)'$$

and Λ does not depend on t. From this, using the Cramer–Wold device, we have:

$$\mathbb{V}_N \overset{d}{\to} \mathcal{N}(0, \Lambda).$$

On the other hand, we have,

$$\frac{N_1}{N} = \frac{1}{N}\sum_{t=1}^{N} I\left(\varepsilon_{t-1}^2 \le c\right) \overset{a.s.}{\to} F(c), \quad N \to \infty$$

where $F(c) = P\left(\varepsilon_t^2 \le c\right) < \infty$. Now, we can apply the central limit theorem for random sums. We have

$$\mathbb{V}_{N_1} \xrightarrow{d} \mathcal{N}(0, \Lambda)$$

and, finally, because of $A_{N_1}^{-1} \xrightarrow{a.s.} \Gamma^{-1}$, we get,

$$\sqrt{N_1}\,(\widehat{\boldsymbol{a}}_N - \boldsymbol{a}) \xrightarrow{d} \mathcal{N}(0, \Gamma^{-1}\Lambda\ \Gamma^{-1}).$$

In a similar way, it can be proved the strong consistency and asymptotic normality of the sequence $\widehat{\boldsymbol{b}}_N$. □

6. Acknowledgment

The authors are grateful to Product Exchange from Novi Sad for consigning the data set of soybean meal.

REFERENCES

[1] Bollerslev, T. Generalized Autoregressive Conditional Heteroscedasticity *Journal of Financial Economics* (1986), **31**, 307–327

[2] Bougerol, P., N. Picard Strict Stationarity of Generalized Autoregressive Processes *The Annals of Probability* (1992), **20(4)** 1714–1730

[3] Brandt, A. The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients *Applied Probability* (1986), **18**, 211–220

[4] Embrechts, P. et al. Modeling Extremal Events *Application of Mathematics* Springer (2001)

[5] Engle, R. F. Autoregressive Conditional Heteroscedasticity With Estimates of the Variance of United Kingdom Inflation *Econometrica* (1982), **50(4)**, 987–1007

[6] Fornari, F., A. Mele Sign- And Volatility Switching- ARCH Models: Theory and Applications to International Stock Markets *Journal of Appl.*

Econometrics (1997), **12** 49–65

[7] Francq, C., M. Roussignol, J. M. Zakoian Conditional Heteroskedasticity Driven by Hidden Markov Chains *Journal of Time Series Analysis* (2001), **22(2)** 197–220

[8] Mikosch, T. Modelling Dependence and Tails of Financial Time Series *Laboratory of Actuarial Mathematics* University of Copenhagen (2001)

[9] Zakoian, J. M. Threshold Heteroskedastic Models *Journal of Economic Dynamics and Control* (1994), bf 18, 931–955.

Biljana Č. Popović
Department of Statistics,
Faculty of Sciences and Mathematics,
18000 Niš, Višegradska 33,
Serbia and Montenegro
e-mail: `biljanap@junis.ni.ac.yu`

Vladica S. Stojanović
Faculty of Economics,
Zubin Potok,
Serbia and Montenegro
e-mail: `vlada70@verat.net`