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EXTREMES OF BIVARIATE GEOMETRIC VARIABLES WITH APPLICATION TO BISEXUAL BRANCHING PROCESSES

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We obtain a limit theorem for the row maximum of a triangular array of bivariate geometric random vectors. An application of this limit theorem is provided for maximum family size within a generation of a bisexual branching process with varying geometric offspring laws.

1. Introduction

It is well known ([1], [18]) that the most commonly used discrete probability distributions (Poisson, uniform, geometric, negative-binomial, binomial) are not attracted to any max-stable law when the parameters are fixed. Considering triangular arrays of random variables where the parameters vary with the number of the raw allows one to obtain non-degenerate limiting distributions under appropriate normalizations. Anderson et al. [2] showed how this could be done for the Poisson distribution. Nadarajah and Mitov [18] performed the same for the other four discrete distributions.

Similar problem arises for bivariate discrete distributions. It is known (see, for example, Theorem 5.2.3 in [4]) that if there exist normalizing sequences which yield a non-degenerate limit for the maximum of some iid bivariate random vectors then the same sequences give non-degenerate limits for the marginals of the

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maximum. Coles and Pauli [3] considered the problem of finding non-degenerate limit distributions for the maximum of bivariate Poisson random vectors by using the results of [2].

The aim of this note is to find a non-degenerate limit distribution for the maximum of bivariate geometric random vectors. We also provide an application of this theorem for the maximum family size within a generation for bisexual branching processes in varying environment. This kind of branching processes have been studied by the very prolific work of Professor M. Molina and his colleagues [16], [5]–[11], [13]–[15]. In the present note we consider the particular case when the mating function is $L(x, y) = \min(x, y)$ and the offspring of mating has a bivariate geometric distribution. Applying the result for the maximum of bivariate geometric variables we obtain a limit theorem for the offspring of the most prolific mating living in the nth generation. The result relates to those in [17] for Galton-Watson branching processes.

The paper is organized as follows. In Section 2 we give the construction of a bivariate geometric distribution following Marshall and Olkin [12]. In Section 3 two limit theorems for the maxima of bivariate geometric random vectors are proved. These results are used in Section 4 to prove a limit theorem for bisexual branching processes in varying geometric environment.

2. Marshall and Olkin's Bivariate Geometric

The bivariate geometric distribution can be constructed in different ways. The following construction is given by Marshall and Olkin [12].

Consider a vector (U, V) having Bernoulli marginals. This vector has only four possible values $(1, 1)$, $(1, 0)$, $(0, 1)$ and $(0, 0)$ with probabilities p_{11}, p_{10}, p_{01} and p_{00}, respectively. The marginal probabilities are

$$\begin{aligned}
\Pr(U = 1) &= p_{1+} = p_{11} + p_{10}, \\
\Pr(U = 0) &= p_{0+} = p_{01} + p_{00}, \\
\Pr(V = 1) &= p_{+1} = p_{11} + p_{01}, \\
\Pr(V = 0) &= p_{+0} = p_{10} + p_{00}.
\end{aligned}$$

For a sequence (U_1, V_1), (U_2, V_2), … (U_n, V_n), … of iid bivariate Bernoulli random vectors let f and m denote the number of 0's before the first 1 in the sequences $U_1, U_2, \ldots, U_n, \ldots$ and $V_1, V_2, \ldots, V_n, \ldots$, respectively. Clearly f and m each has a geometric distribution, and in general they will not be independent. Marshall and Olkin's bivariate distribution is given by

$$\Pr(f = l, m = k) = \begin{cases} p_{00}^{l} p_{10} p_{+0}^{k-l-1} p_{+1}, & 0 \leq l < k, \\ p_{00}^{l} p_{11}, & l = k, \\ p_{00}^{k} p_{01} p_{0+}^{l-k-1} p_{1+}, & 0 \leq k < l \end{cases}$$

and

$$\bar{F}(l,k) = \Pr(f \geq l, m \geq k) = \begin{cases} p_{00}^{l} p_{+0}^{k-l}, & 0 \leq l < k, \\ p_{00}^{l}, & 0 \leq l = k, \\ p_{00}^{l} p_{0+}^{l-k}, & 0 \leq k < l. \end{cases} \tag{1}$$

The marginal pmfs and the marginal survival functions of f and m are

$$\Pr(f = l) = p_{1+} p_{0+}^{l}, \quad \Pr(m = k) = p_{+1} p_{+0}^{k},$$

and

$$\bar{F}_f(l) = \Pr(f \geq l) = p_{0+}^{l}, \qquad \bar{F}_m(k) = \Pr(m \geq k) = p_{+0}^{k}, \tag{2}$$

respectively, for $l \geq 0$ and $k \geq 0$.

Using (1) and (2), the joint cdf can be written as

$$F(x,y) = 1 - \bar{F}_f(x) - \bar{F}_m(y) + \bar{F}(x,y)$$

$$= \begin{cases} 1 - p_{0+}^{[x]+1} - p_{+0}^{[y]+1} + p_{00}^{[x]+1} p_{+0}^{[y]-[x]}, & 0 \leq x < y, \\ 1 - p_{0+}^{[x]+1} - p_{+0}^{[y]+1} + p_{00}^{[x]+1}, & 0 \leq x = y, \\ 1 - p_{0+}^{[x]+1} - p_{+0}^{[y]+1} + p_{00}^{[y]+1} p_{0+}^{[x]-[y]}, & 0 \leq y < x. \end{cases} \tag{3}$$

3. Bivariate Geometric Maxima

Let (f_1, m_1), (f_2, m_2), ... (f_k, m_k), ... be iid copies of the vector (f, m) defined in the previous section. Suppose that $\{\nu_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $\nu_n \to \infty$ as $n \to \infty$. Define $(M_{\nu_n}^{f}, M_{\nu_n}^{m})$ by

$$M_{\nu_n}^{f} = \max\{f_1, f_2, \ldots, f_{\nu_n}\}$$

and

$$M^m_{\nu_n} = \max\{m_1, m_2, \ldots, m_{\nu_n}\}$$

for $n = 1, 2, \ldots$.

As mentioned in Section 1, there are no sequences $(a^f(n), b^f(n)))$ and $(a^m(n), b^m(n))$, $n = 1, 2, \ldots$ such that

$$\lim_{n\to\infty} \Pr\left(M^f_{\nu_n} \le a^f(n)x + b^f(n), M^m_{\nu_n} \le a^m(n)y + b^m(n)\right) = H(x, y)$$

for a non-degenerate limit H. Assuming the existence of such sequences (Theorem 5.2.3, [4]) would imply the same for the maxima of the marginals, i.e.

$$\lim_{n\to\infty} \Pr\left(M^f_{\nu_n} \le a^f(n)x + b^f(n)\right) = H(x, \infty) \tag{4}$$

and

$$\lim_{n\to\infty} \Pr\left(M^m_{\nu_n} \le a^m(n)y + b^m(n)\right) = H(\infty, y), \tag{5}$$

which cannot be true (see [1]). On the other hand, if we consider a rectangular array of iid bivariate geometric vectors where the parameters vary together with n we can prove the existence of such sequences, corresponding with the results in [18] for the univariate geometric distribution. More precisely, we consider a rectangular array $\{(f_i(n), m_i(n)), i = 1, 2, \ldots, \nu_n\}$, $n = 1, 2, \ldots$ of independent random vectors which are identically distributed in each row, i.e.

$$\Pr(f_i(n) = l, m_i(n) = k) = \begin{cases} p_{00}(n)^l p_{10}(n) p_{+0}(n)^{k-l-1} p_{+1}(n), & 0 \le l < k, \\ p_{00}(n)^l p_{11}(n), & l = k, \\ p_{00}(n)^k p_{01}(n) p_{0+}(n)^{l-k-1} p_{1+}(n), & 0 \le k < l, \end{cases} \tag{6}$$

where $p_{00}(n)$, $p_{10}(n)$, $p_{01}(n)$, and $p_{11}(n)$ satisfy the conditions

$$p_{1+}(n) = p_{11}(n) + p_{10}(n) \to 0 \tag{7}$$

and

$$p_{+1}(n) = p_{11}(n) + p_{01}(n) \to 0 \tag{8}$$

as $n \to \infty$. It is clear that $p_{11}(n) \to 0$, $p_{10}(n) \to 0$, $p_{01}(n) \to 0$, and $p_{00}(n) \to 1$ as $n \to \infty$. We assume further that

$$(9)\quad p_{11}(n) = o\left(\frac{1}{\log \nu_n}\right), \; \frac{p_{10}(n)}{p_{11}(n)} = o\left(\frac{1}{\log \nu_n}\right), \text{ and } \frac{p_{01}(n)}{p_{11}(n)} = o\left(\frac{1}{\log \nu_n}\right)$$

as $n \to \infty$. Now we are ready to prove the following theorem.

Theorem 1. *Assume (6), (7), (8) and (9). Then*

$$(10)\qquad \lim_{n\to\infty} \Pr\left(M^f_{\nu_n} \le \frac{x + \log \nu_n}{p_{11}(n)}, M^m_{\nu_n} \le \frac{y + \log \nu_n}{p_{11}(n)}\right) = H(x,y),$$

where

$$H(x,y) = \exp\left[-\exp(-x) - \exp(-y) + \exp\{-\max(x,y)\}\right].$$

Proof. Suppose that $x < y$. Obviously,

$$\frac{x + \log \nu_n}{p_{11}(n)} < \frac{y + \log \nu_n}{p_{11}(n)}$$

for every n. Using (3) and (6), one obtains

$$\begin{aligned} &\Pr\left(M^f_{\nu_n} \le \frac{x + \log \nu_n}{p_{11}(n)}, M^m_{\nu_n} \le \frac{y + \log \nu_n}{p_{11}(n)}\right) \\ = &\left(1 - p_{0+}(n)^{\left[\frac{x + \log \nu_n}{p_{11}(n)}\right]+1} - p_{+0}(n)^{\left[\frac{y + \log \nu_n}{p_{11}(n)}\right]+1}\right. \\ &\left. + p_{00}(n)^{\left[\frac{x + \log \nu_n}{p_{11}(n)}\right]+1} p_{+0}(n)^{\left[\frac{y + \log \nu_n}{p_{11}(n)}\right] - \left[\frac{x + \log \nu_n}{p_{11}(n)}\right]}\right)^{\nu_n}. \end{aligned}$$

Taking logarithms and expanding in Taylor's series, the above can be reduced to

$$\begin{aligned} &-\log \Pr\left(M^f_{\nu_n} \le \frac{x + \log \nu_n}{p_{11}(n)}, M^m_{\nu_n} \le \frac{y + \log \nu_n}{p_{11}(n)}\right) \\ = \;&\nu_n \left(p_{0+}(n)^{\left[\frac{x + \log \nu_n}{p_{11}(n)}\right]+1} + p_{+0}(n)^{\left[\frac{y + \log \nu_n}{p_{11}(n)}\right]+1}\right. \\ (11)\qquad &\left. - p_{00}(n)^{\left[\frac{x + \log \nu_n}{p_{11}(n)}\right]+1} p_{+0}(n)^{\left[\frac{y + \log \nu_n}{p_{11}(n)}\right] - \left[\frac{x + \log \nu_n}{p_{11}(n)}\right]}\right)(1 + o(1)) \end{aligned}$$

as $n \to \infty$. If $\{x\}$ denotes the fractional part of the real number x (i.e. $[x] = x - \{x\}$) then it is not difficult to see that

$$\begin{aligned}
& \lim_{n\to\infty} \nu_n \left(p_{0+}(n)^{\left[\frac{x+\log\nu_n}{p_{11}(n)}\right]+1} + p_{+0}(n)^{\left[\frac{y+\log\nu_n}{p_{11}(n)}\right]+1} \right) \\
= & \lim_{n\to\infty} \nu_n \left(1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log\nu_n}\right)\right)\right)^{\frac{x+\log\nu_n}{p_{11}(n)} - \left\{\frac{x+\log\nu_n}{p_{11}(n)}\right\}+1} \\
& + \lim_{n\to\infty} \nu_n \left(1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log\nu_n}\right)\right)\right)^{\frac{y+\log\nu_n}{p_{11}(n)} - \left\{\frac{y+\log\nu_n}{p_{11}(n)}\right\}+1} \\
= & \lim_{n\to\infty} \nu_n \left(1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log\nu_n}\right)\right)\right)^{\frac{x+\log\nu_n}{p_{11}(n)}} \\
& \times \lim_{n\to\infty} \left(1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log\nu_n}\right)\right)\right)^{-\left\{\frac{x+\log\nu_n}{p_{11}(n)}\right\}+1} \\
& + \lim_{n\to\infty} \nu_n \left(1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log\nu_n}\right)\right)\right)^{\frac{y+\log\nu_n}{p_{11}(n)}} \\
& \times \lim_{n\to\infty} \left(1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log\nu_n}\right)\right)\right)^{-\left\{\frac{y+\log\nu_n}{p_{11}(n)}\right\}+1} \\
= & \exp(-x) + \exp(-y). \qquad (12)
\end{aligned}$$

Using (9) and $p_{00}(n) = 1 - p_{11}(n) - p_{10}(n) - p_{01}(n)$, one can see

$$p_{00}(n) = 1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log\nu_n}\right)\right)$$

as $n \to \infty$, which can be used in the same way as above to prove that

$$\nu_n p_{00}(n)^{\left[\frac{x+\log\nu_n}{p_{11}(n)}\right]} \to \exp(-x) \qquad (13)$$

as $n \to \infty$. Similarly, using (9) and $p_{+0}(n) = 1 - p_{11}(n) - p_{01}(n)$, one can see

$$p_{+0}(n) = 1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log\nu_n}\right)\right)$$

as $n \to \infty$, from which it follows that

$$
\begin{aligned}
& p_{+0}(n)^{\left[\frac{y+\log \nu_n}{p_{11}(n)}\right]-\left[\frac{x+\log \nu_n}{p_{11}(n)}\right]} \\
= \quad & \left(1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log \nu_n}\right)\right)\right)^{\frac{y-x}{p_{11}(n)}} \\
& \times \left(1 - p_{11}(n)\left(1 + o\left(\frac{1}{\log \nu_n}\right)\right)\right)^{\left\{\frac{y+\log \nu_n}{p_{11}(n)}\right\}-\left\{\frac{x+\log \nu_n}{p_{11}(n)}\right\}} \\
(14) \quad \to \quad & \exp(x-y)
\end{aligned}
$$

as $n \to \infty$. Combining (13) and (14), one obtains

$$
(15) \lim_{n\to\infty} \nu_n \left\{ p_{00}(n)^{\left[\frac{x+\log \nu_n}{p_{11}(n)}\right]+1} p_{+0}(n)^{\left[\frac{y+\log \nu_n}{p_{11}(n)}\right]-\left[\frac{x+\log \nu_n}{p_{11}(n)}\right]} \right\} = \exp(-y)
$$

for $x < y$. Finally, from (11), (12) and (15), one obtains

$$
\begin{aligned}
& \lim_{n\to\infty} \log \Pr\left(M^f_{\nu_n} \le \frac{x+\log \nu_n}{p_{11}(n)}, M^m_{\nu_n} \le \frac{y+\log \nu_n}{p_{11}(n)}\right) \\
(16) \qquad & \qquad = -\exp(-x) - \exp(-y) + \exp(-y).
\end{aligned}
$$

Similarly, for $x > y$, one can show that

$$
\begin{aligned}
& \lim_{n\to\infty} \log \Pr\left(M^f_{\nu_n} \le \frac{x+\log \nu_n}{p_{11}(n)}, M^m_{\nu_n} \le \frac{y+\log \nu_n}{p_{11}(n)}\right) \\
(17) \qquad & \qquad = -\exp(-x) - \exp(-y) + \exp(-x).
\end{aligned}
$$

Combining (16) and (17) completes the proof of (10). □

Let $\{N_n\}_{n=1}^{\infty}$ be a sequence of integer valued non-negative random variables independent of the vectors $(f_1(n), m_1(n)), (f_2(n), m_2(n)), \ldots$ in the nth row of the rectangular array. The following theorem gives the distribution of the maximum in the case of a random indexing sequence.

Theorem 2. *If the conditions of Theorem 1 are satisfied and* $\frac{N_n}{\nu_n}$ *converges in probability, as* $n \to \infty$, *to a proper random variable* W, *non-degenerate at zero, i.e.* $\Pr(W < \infty) = 1$ *and* $\Pr(W > 0) > 0$, *then*

$$
\lim_{n\to\infty} \Pr\left(M^f_{N_n} \le \frac{x+\log \nu_n}{p_{11}(n)}, M^m_{N_n} \le \frac{y+\log \nu_n}{p_{11}(n)}\right) = \int_0^\infty \{H(x,y)\}^z \, d\Pr(W \le z).
$$

4. Maximum Family Size of A Bisexual Branching Process

In this section we use the rectangular array $(f_i(n), m_i(n))$, $i = 1, 2, \ldots$, $n = 1, 2, \ldots$ of independent random vectors with bivariate geometric distributions defined in Section 3 to define a bisexual branching process as follows. Assuming the conditions (7), (8), and

$$\sum_{n=1}^{\infty} p_{11}(n) < \infty \tag{18}$$

and the mating function $L(x, y) = \min(x, y)$, define

$$\begin{aligned} Z_0 &= N > 0, \\ (F_{n+1}, M_{n+1}) &= \sum_{i=1}^{Z_n} (f_i(n+1), m_i(n+1)), \\ Z_{n+1} &= \min(F_{n+1},\ M_{n+1}) \end{aligned} \tag{19}$$

for $n = 0, 1, 2, \ldots$. Define the sequence $\{X_{n1}\}_{n=1}^{\infty}$ by $X_{n1} := \min\{f_1(n), m_1(n)\}$ and denote by

$$r_{nj} = j^{-1} E[Z_{n+1} | Z_n = j]$$

for $n = 1, 2, \ldots$ and $j = 1, 2, \ldots$, the mean growth rate per mating unit. Denote also

$$m_0 = 1, \quad m_n = \prod_{i=0}^{n-1} r_{i1}, \quad r_n = \sup_{j>0} r_{nj} \tag{20}$$

for $n = 1, 2, \ldots$.

From (6) and (1), we obtain

$$r_{n1} = EX_{n1} = E[\min\{f_1(n), m_1(n)\}] = \sum_{k=1}^{\infty} \Pr(\min\{f_1(n), m_1(n)\} \geq k)$$

$$= \sum_{k=1}^{\infty} \Pr(f_1(n) \geq k, m_1(n) \geq k) = \sum_{k=1}^{\infty} p_{00}(n)^k = \frac{p_{00}(n)}{1 - p_{00}(n)}$$

$$= \frac{p_{00}(n)}{p_{11}(n) + p_{10}(n) + p_{01}(n)} \tag{21}$$

for $n = 1, 2, \ldots$. It is also not difficult to check that $r_{n1} = \inf_{j>0} r_{nj}$. From (7), (8), (21) and (20) it follows

$$r_{n1} \to \infty \text{ and } m_n \to \infty \tag{22}$$

as $n \to \infty$. Denote further by $\nu_n = [m_n]$ and assume that for $\nu_n = [m_n]$, (9) holds.

Remark 1. *An example of probabilities $p_{11}(n)$, $p_{10}(n)$, and $p_{01}(n)$ such that all the above conditions are satisfied is*

$$p_{11}(n) = (n+1)^{-a}$$

and

$$p_{10}(n) = p_{01}(n) = (n+1)^{-a-b}$$

for $a > 1$ and $b > 1$. In this case,

$$m_n = \prod_{k=0}^{n-1} \left(\frac{(k+1)^a}{1 + 2(k+1)^{-b}} \right)$$

for $n \geq 1$, which yields

$$\begin{aligned} \log m_n &= \sum_{k=0}^{n-1} \log \left(\frac{(k+1)^a}{1 + 2(k+1)^{-b}} \right) \\ &= \sim an \log n \end{aligned}$$

as $n \to \infty$. Now it is easy to check that all of the conditions above are satisfied.

Theorem 3 provides the main result of this section.

Theorem 3. *If the conditions (6), (7), (8), (9) and (18) hold then*

$$\lim_{n\to\infty} \Pr \left(\frac{\max_{i=1}^{Z_n}(f_i(n)) + \log[m_n]}{p_{11}(n)} \leq x, \frac{\max_{i=1}^{Z_n}(m_i(n)) + \log[m_n]}{p_{11}(n)} \leq y \right)$$
$$= \int_0^\infty \{H(x,y)\}^z \, d\Pr(\bar{W} \leq z),$$

where $H(x, y)$ is given by Theorem 1.

P r o o f. For the proof we need to verify the following hypotheses required by Theorems 3.1 and 3.2 in [[16]]:

(i) The series $\sum_{k=0}^{\infty}(1 - r_k^{-1} r_{k1})$ converges.

(ii) There exists a random variable X such that $E[X \log^+ X] < \infty$ and $\Pr(r_{n1}^{-1} X_{n1} \leq u) \geq \Pr(X \leq u)$ for all $u \geq 0$ and $n = 1, 2, \ldots$

(iii) There exist constants $A > 0$ and $c > 1$ such that $m_{n+j}/m_j \geq Ac^n$ for $j = 1, 2, \ldots$ and $n = 1, 2, \ldots$.

If the above hypotheses are satisfied then Z_n/m_n converges almost surely as $n \to \infty$ to a finite and nonnegative random variable $\bar{W}$ non-degenerate at zero (i.e. $E[\bar{W}|Z_0 = N] < \infty$ and $\Pr(\bar{W} > 0\} > 0$).

(i)$'$ For $j = 1, 2, \ldots,$

$$\begin{aligned} jr_{nj} &= E\left[\min\left\{f_1(n) + f_2(n) + \ldots + f_j(n), m_1(n) + m_2(n) + \ldots + m_j(n)\right\}\right] \\ &\leq E\left[\frac{\{f_1(n) + f_2(n) + \ldots + f_j(n)\} + \{m_1(n) + m_2(n) + \ldots + m_j(n)\}}{2}\right] \\ &= \frac{1}{2}\sum_{l=1}^{j}\left(E\left[f_l(n)\right] + E\left[m_l(n)\right]\right) \\ &= \frac{j}{2}\left(E\left[f_1(n)\right] + E\left[m_1(n)\right]\right) \\ &= \frac{j}{2}\left(\frac{p_{0+}(n)}{p_{1+}(n)} + \frac{p_{+0}(n)}{p_{+1}(n)}\right). \end{aligned}$$

Therefore,

$$(23) \quad r_{n1} \leq r_{nj} \leq r_n^* = \frac{1}{2}\left(\frac{p_{00}(n) + p_{01}(n)}{p_{10}(n) + p_{11}(n)} + \frac{p_{00}(n) + p_{10}(n)}{p_{01}(n) + p_{11}(n)}\right),$$

which implies $r_{n1} \leq r_n \leq r_n^*$ for every $n = 1, 2, \ldots$. Using (18), (21), and (23), it is not difficult to see that

$$\begin{aligned} 1 - \frac{r_{n1}}{r_n^*} &= 1 - \frac{2p_{00}(n)}{p_{11}(n) + p_{10}(n) + p_{01}(n)}\left(\frac{p_{00}(n) + p_{01}(n)}{p_{10}(n) + p_{11}(n)} + \frac{p_{00}(n) + p_{10}(n)}{p_{01}(n) + p_{11}(n)}\right)^{-1} \\ &= 1 - \frac{2[1 - p_{11}(n)(1 + o(1))]}{p_{11}(n)(1 + o(1))}\left(\frac{2[p_{11}(n)(1 + o(1))]}{[p_{11}(n)(1 + o(1))]^2}\right)^{-1} \\ &= 1 - (1 - p_{11}(n)(1 + o(1))) = p_{11}(n)(1 + o(1)), \end{aligned}$$

which yields

$$\sum_{n=1}^{\infty}\left(1-\frac{r_{n1}}{r_n^*}\right) \quad < \quad \infty.$$

Because $r_n \leq r_n^*$, it follows that

$$\sum_{n=1}^{\infty}\left(1-\frac{r_{n1}}{r_n}\right) \quad < \quad \infty.$$

Hence, (i) is satisfied and we have that

$$\frac{Z_n}{m_n} \quad \to \quad \bar{W}$$

almost surely, where $\Pr(\bar{W} \in [0,\infty)) = 1$. Since $m_n \to \infty$ as $n \to \infty$, we also have

$$\frac{Z_n}{[m_n]} \quad \to \quad \bar{W}$$

almost surely.

(ii)′ Consider the sequence $r_{n1}^{-1}X_{n1}$. By repeating the arguments of Theorem 1, one obtains

$$\begin{aligned}\Pr\left(r_{n1}^{-1}X_{n1} \geq u\right) &= \Pr\left(X_{n1} \geq ur_{n1}\right)\\ &= \Pr\left(\min\left\{f_{n1}, m_{n1}\right\} \geq ur_{n1}\right)\\ &= p_{00}(n)^{[ur_{n1}]}\\ &= \left\{1-p_{11}(n)(1+o(1))\right\}^{[u/(p_{11}(n)(1+o(1)))]}\\ &\to \exp(-u) \qquad (24)\end{aligned}$$

as $n \to \infty$, where the convergence is uniform on $[0,\infty)$. From (24), it follows that there exists n_0 such that

$$\Pr\left(r_{n1}^{-1}X_{n1} \geq u\right) \quad \leq \quad (1+\varepsilon)\exp(-u) \qquad (25)$$

for all $n > n_0$ and $u \in [0,\infty)$, where $\varepsilon = \exp(1-p_{00}(1)) - 1 > 0$. Furthermore,

$$\begin{aligned}\Pr\left(r_{n1}^{-1}X_{n1} \geq u\right) &= p_{00}(n_0)^{\left[\dfrac{u}{1-p_{00}(n_0)}\right]}\\ &\leq p_{00}(n_0)^{\dfrac{u}{1-p_{00}(1)}-1}\\ &= \frac{\exp(-\alpha u)}{p_{00}(n_0)} \qquad (26)\end{aligned}$$

for all $n = 1, 2, \ldots, n_0$ and $u \in [0, \infty)$, where $\alpha = -\log p_{00}(n_0)/(1 - p_{00}(1)) > 0$. From (25) and (26), it is clear that

$$\Pr\left(r_{n1}^{-1} X_{n1} \geq u\right) \leq \max\left\{\bar{G}_1(u), \bar{G}_2(u)\right\}$$

for all $n = 1, 2, \ldots$ and $u \geq 0$, where $\bar{G}_1$ and $\bar{G}_2$ are the survivor functions on $[0, \infty)$ defined by

$$\bar{G}_1(u) = \begin{cases} 1, & u \in [0, 1 - p_{00}(1)), \\ (1+\varepsilon)\exp(-u), & u \in [1 - p_{00}(1), \infty), \end{cases}$$

and

$$\bar{G}_2(u) = \begin{cases} 1, & u \in [0, 1 - p_{00}(1)), \\ \dfrac{\exp(-\alpha u)}{p_{00}(n_0)}, & u \in [1 - p_{00}(1), \infty), \end{cases}$$

respectively. But $\max\{\bar{G}_1(u), \bar{G}_2(u)\}$ is a survivor function with exponential tail, so, it must have finite moments of all orders. Hence, (ii) is satisfied.

(iii)$'$ Using the relations $r_{n1} \to \infty$ and $m_n \to \infty$ as $n \to \infty$, it is easy to verify that there exist constants $A > 0$ and $c > 1$ as described in the hypothesis (iii).

The rest of the proof is an immediate application of Theorem 2. □

5. Concluding Remarks

We have obtained a limiting distribution for the maximum family size within a generation of a bisexual branching process with varying geometric offspring laws. This distribution depends on the random variable $\bar{W}$ introduced in [16]. Unfortunately, there is no explicit formulas for the distribution function of $\bar{W}$, even in particular cases. Our attempt to find the distribution function in the case considered above has been unsuccessful too.

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