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HYPOELLIPTICITY OF ANISOTROPIC PARTIAL DIFFERENTIAL EQUATIONS

Giuseppe De Donno

ABSTRACT. We propose an approach based on methods from microlocal analysis, for characterizing the hypoellipticity in C^∞ and Gevrey G^λ classes of semilinear anisotropic partial differential operators with multiple characteristics, in dimension $n \geq 3$. Conditions are imposed on the lower order terms of the linear part of the operator; we also consider C^∞ nonlinear perturbations, see Theorem 1.1 and Theorem 1.4 below.

1. Introduction

We consider a class of semilinear anisotropic equations with multiple characteristics in n variables $z = (x, y) = (x_1, \dots, x_{n'}, y_1, \dots, y_{n''})$, $n \geq 3$ (for related results in the case $n = 2$, see De Donno-Oliaro [3]), belonging to the set Ω , neighborhood of a point $z_0 \in \mathbb{R}^n$, in the case when no assumptions of Levi-type are imposed on the lower order terms; then, as well known, the main properties of the operators depend heavily on the lower order terms of their symbol. We consider operators of the form:

$$(1.1) \quad P(x, y, D_x, D_y)u + G(x, y; \partial_x^\gamma \partial_y^j u) \Big|_{\left| \frac{\gamma}{\rho'} \right| + |j| < k^*} = 0,$$

where the linear part is given by:

$$(1.2) \quad P(x, y, D_x, D_y) = \sum_{|\alpha|=m} a_\alpha(z) D_y^\alpha - \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_\beta(z) D_x^\beta + \sum_{k^* \leq \left| \frac{\gamma}{\rho'} \right| + |j| < m} c_{\gamma j}(z) D_x^\gamma D_y^j$$

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with $m \in \mathbb{Z}_+$, $m \geq 4$, and the anisotropic weight $\rho = (\rho', 1) = (\rho_1, \dots, \rho_{n'}, \underbrace{1, \dots, 1}_{n''})$, $0 < \rho_i \leq 1$, $i = 1, \dots, n'$; $\alpha = (\alpha_1, \dots, \alpha_{n''})$, $j = (j_1, \dots, j_{n''}) \in \mathbb{Z}_+^{n''}$, $\beta = (\beta_1, \dots, \beta_{n'})$, $\gamma = (\gamma_1, \dots, \gamma_{n'}) \in \mathbb{Z}_+^{n'}$, $0 < k^* < m$, $\left| \frac{\beta}{\rho'} \right| := \sum_{i=1}^{n'} \beta_i \frac{1}{\rho_i}$; we shall also say that $\left| \frac{\gamma}{\rho'} \right| + |j|$ is the anisotropic order of $D_x^\gamma D_y^j$, so the nonlinearity involves derivatives of anisotropic order less than k^* . We give for (1.1) and (1.2) results of hypoellipticity and for (1.2) of Gevrey hypoellipticity too; the arguments in our proofs are based mainly on microlocal tools, allowing relevant simplifications in the study: pseudo-differential operators, wave front sets and $S_{\rho, \delta}^m$ techniques. We consider C^∞ nonlinearity G , C^∞ coefficients in (1.2) and in the following we also suppose that the principal symbol of (1.2) is real and elliptic (with respect to the η variables), i.e.:

$$(1.3) \quad c_1 |\eta|^m \leq \left| \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha \right| \leq C_1 |\eta|^m,$$

for positive constants c_1 , C_1 , and

$$(1.4) \quad \Re \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_\beta(z) \xi^\beta \neq 0 \text{ for } \xi \neq 0, z \in \Omega,$$

$$(1.5) \quad G(z; t) = \sum_{r \in \mathbb{Z}_+^M} C_r(z) t^r, \quad C_r \in C^\infty(\Omega), \quad t \in \mathbb{Z}^M,$$

where, for every compact $K \subset \Omega$, $\sup_{z \in K} |D^\alpha C_r(z)| \leq C_{\alpha, K} \lambda_r$ and moreover $\tilde{F}(t) = \sum_r \lambda_r t^r$ is entire analytic.

We recall that the nonzero hypothesis on $\Re \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_\beta(z) \xi^\beta$ is a nondegeneracy condition with invariant meaning, usually required in the study of hypoellipticity (and local solvability) of the linear operator (1.2) in C^∞ and G^λ Gevrey spaces, see for example Liess-Rodino [10], De Donno-Rodino [4], concerning Gevrey hypoellipticity for 2 variables PDE's with higher multiplicity. As standard, the Gevrey anisotropic space $G^\lambda(\Omega)$, $\lambda = (\lambda_1, \dots, \lambda_n)$, is defined by the estimates:

$$(1.6) \quad \sup_K |\partial_z^\alpha f(z)| \leq C_K^{|\alpha|+1} (\alpha_1!)^{\lambda_1} \cdots (\alpha_n!)^{\lambda_n}, \text{ for every } K \subset\subset \Omega,$$

where $\lambda_i \geq 1$ for $i = 1, \dots, n$. Let us also observe that if $\Im \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_\beta(z) \xi^\beta \neq 0$ then

the operator is quasi-elliptic; the results of hypoellipticity (and local solvability) are well known in this case.

Moreover we define the anisotropic characteristic manifold

$$(1.7) \quad \Sigma := \{(z, \zeta) \in \Omega \times (\mathbb{R}^n \setminus 0) : \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha - \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta = 0\}.$$

We may regard the next results as an extension of De Donno-Oliaro [3] in which hypoellipticity (and local solvability) are proved in the case of 2-variables equations. Let us state the main results.

Theorem 1.1. *Let us fix k^* in (1.1) $m - \frac{1}{2} < k^* < m$ in such a way that there exists at least one n -uple $(\gamma^*, j^*) \in \mathbb{Z}_+^{n'} \times \mathbb{Z}^{n''}$ such that $\left|\frac{\gamma^*}{\rho'}\right| + |j^*| = k^*$. We suppose $a_\alpha(z), b_\beta(z), c_{\gamma j}(z) \in C^\infty(\Omega)$, and assume that for $(z, \zeta) \in \Sigma$ the following conditions hold:*

- i) $\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \neq 0$ for $\xi \neq 0, \eta \neq 0$;
- ii) $\Im \sum_{k^* < \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j \cdot \Im \sum_{\left|\frac{\gamma^*}{\rho'}\right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \geq 0$;
- iii) $\Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta \cdot \Im \sum_{\left|\frac{\gamma^*}{\rho'}\right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \leq 0$.

Assume moreover that (1.3) and (1.4) hold. Then (1.2) is C^∞ -hypoelliptic.

Remark 1.2. *The operator $P(x, y, D_x, D_y)$ is hypoelliptic with loss of regularity of $\iota = m - k^*$ derivatives.*

Remark 1.3. *Taking analytic coefficients in Theorem 1.1 we obtain G^λ -hypoellipticity of the operator P in (1.2) for $\lambda_i \geq \frac{1}{k^* - (m-1)}$, cf. De Donno-Rodino [4] regarding the isotropic case.*

In the following it will be convenient to use the Sobolev anisotropic space H_ρ^s , where as before $\rho = (\rho', 1)$, defined by

$$\|f\|_{H_\rho^s} := \left(\int (1 + \sum_{i=1}^{n'} |\xi_i|^{2\rho_i} + |\eta|^2)^s |(\mathcal{F}_{z \rightarrow \zeta} f)(\zeta)|^2 d\zeta \right)^{\frac{1}{2}} < \infty$$

$\mathcal{F}_{z \rightarrow \zeta} f$ being the Fourier transform of $f(z)$. For $s > \frac{1 + \sum_{i=1}^{n'} \frac{1}{\rho_i}}{2}$, the space H_ρ^s is an algebra, cf. the inhomogeneous Schauder estimates in Garello [6].

Theorem 1.4. *Under the above assumptions on $P(x, y, D_x, D_y)$ and G , let u be a solution of (1.1) which belongs to $H_{\rho, \text{loc}}^s(\Omega)$, for $s \geq s_0$, where s_0 is a sufficiently large fixed real number. Then $u \in C^\infty(\Omega)$.*

As examples of operators satisfying Theorem 1.1 we consider in \mathbb{R}^4 :

(1.8)

$$\left(D_{y_1}^2 + D_{y_2}^2\right)^{bp} - (1 - i|z|^2) \left(D_{x_1}^{2a} + D_{x_2}^{2b}\right)^{p-1} + i \left(D_{x_1}^{2a} + D_{x_2}^{2b}\right)^{p-2} \left(D_{y_1}^2 + D_{y_2}^2\right)^b,$$

where $p, a, b \in \mathbb{N}$, $p \geq 4b + 2$, $1 \leq a \leq b$; we have $\rho_1 = \frac{a(p-1)}{bp}$, $\rho_2 = \frac{p-1}{p}$, $k^* = 2bp - \frac{2b}{p-1}$, $(\gamma^*, j^*) = (2a(p-2-k), 2bk, 2(b-h), 2h)$ for $k = 0, \dots, p-2$, $h = 0, \dots, b$. In \mathbb{R}^3 , of even order operator we take

$$(1.9) \quad D_{y_1}^{2bp} - (1 - i|z|^{2l})(D_{x_1}^{2a} + D_{x_2}^{2b})^{p-1} + i(D_{x_1}^{2a} + D_{x_2}^{2b})^{p-2} D_{y_1}^{2b},$$

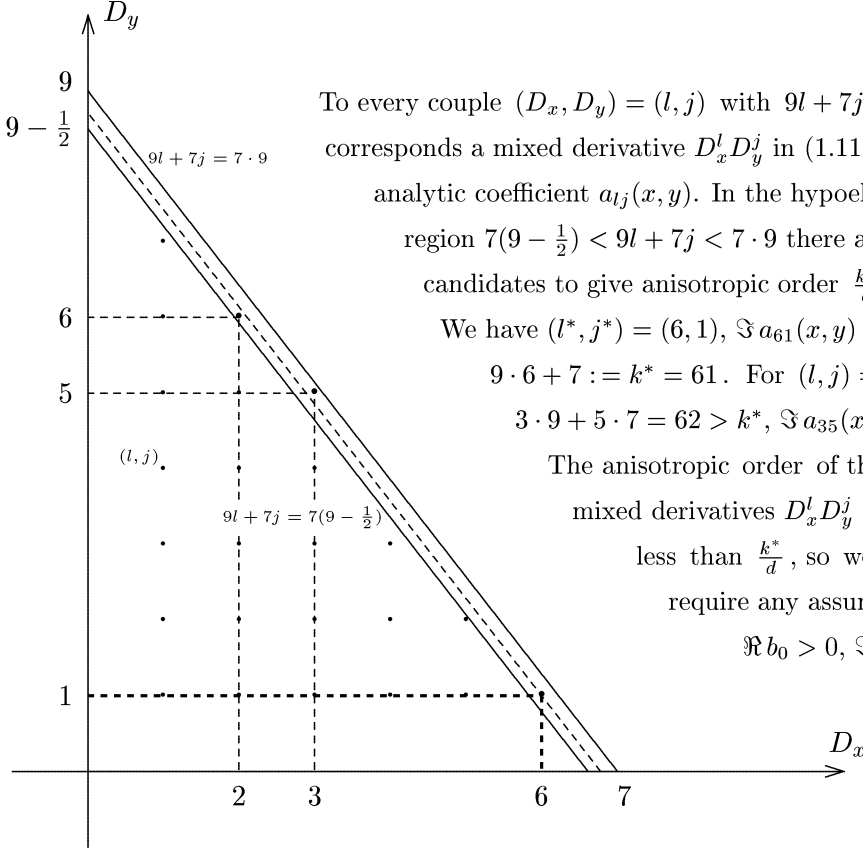
where $p, a, b, \rho_1, \rho_2, k^*$ are the same of the previous example, but $(\gamma^*, j^*) = (2a(p-2-k), 2bk, 2b)$ for $k = 0, \dots, p-2$. About of odd order operator we consider

$$(1.10) \quad D_{y_1}^{2bp+1} - (1 - i|z|^{2l})(D_{x_1}^{2a} + D_{x_2}^{2b})^p + i(D_{x_1}^{2a} + D_{x_2}^{2b})^{p-1} D_{y_1}^{2b},$$

where $p, a, b \in \mathbb{N}$, $p \geq 3$, $1 \leq a \leq b$; we have $\rho_1 = \frac{2ap}{2bp+1}$, $\rho_2 = \frac{2bp}{2bp+1}$, $k^* = 2bp + 1 - \frac{1}{p}$, $(\gamma^*, j^*) = (2a(p-1-k), 2bk, 2b)$ for $k = 0, \dots, p-1$. We may add in (1.8)-(1.10) arbitrary nonlinear C^∞ perturbation of lower anisotropic order satisfying the hypotheses (1.5), and we obtain that (1.8)-(1.10) are C^∞ hypoelliptic. In the following picture, which resembles the Newton polygon pictures, we show in the case of two variables (x, y) the geometrical meaning of the hypothesis *ii*) in Theorem 1.1. We consider the operator of order $m = 9$ with $d = 7$:

$$(1.11) \quad D_y^9 - (1 - iy^{2k})D_x^7 + y^h D_x^3 D_y^5 + iD_x^6 D_y + \sum_{\frac{9}{7}l+j < \frac{61}{7}} a_{lj}(x, y) D_x^l D_y^j,$$

that is C^∞ and Gevrey hypoelliptic.



To every couple $(D_x, D_y) = (l, j)$ with $9l + 7j < 7 \cdot 9$, corresponds a mixed derivative $D_x^l D_y^j$ in (1.11) having analytic coefficient $a_{lj}(x, y)$. In the hypoellipticity region $7(9 - \frac{1}{2}) < 9l + 7j < 7 \cdot 9$ there are three candidates to give anisotropic order $\frac{k^*}{d} = \frac{k^*}{7}$.

We have $(l^*, j^*) = (6, 1)$, $\Im a_{61}(x, y) > 0$ and $9 \cdot 6 + 7 = k^* = 61$. For $(l, j) = (3, 5)$, $3 \cdot 9 + 5 \cdot 7 = 62 > k^*$, $\Im a_{35}(x, y) \equiv 0$.

The anisotropic order of the other mixed derivatives $D_x^l D_y^j$ is $\frac{9l+7j}{d}$, less than $\frac{k^*}{d}$, so we do not require any assumptions.

$\Re b_0 > 0$, $\Im b_0 \leq 0$.

In the next section 2. we prove Theorem 1.1 using $S_{\rho, \delta}^m$ estimates; Theorem 1.4 is proved in Section 3..

2. Hypoellipticity for a class of differential polynomials.

In this Section we begin to prove $S_{\rho, \delta}^m$ estimates for a pseudo-differential model in n variables, $n = n' + n''$, $n \geq 3$ (for related results in the case $n = 2$ see De Donno-Oliaro [3]). We recall that an operator P is said to be hypoelliptic at (a neighborhood $\Omega \subseteq \mathbb{R}^n$ of) a point z_0 when $\text{sing supp } Pu = \text{sing supp } u$ for all $u \in \mathcal{E}'(\Omega)$. We take $m \in \mathbb{Z}_+$, $m \geq 4$ and the anisotropic weight $\rho = (\rho', 1) = (\rho_1, \dots, \rho_{n'}, \underbrace{1, \dots, 1}_{n''})$, $0 < \rho_i \leq 1$, $i = 1, \dots, n'$.

Let the function in $\Omega \times \mathbb{R}^n$

$$(2.1) \quad p(z, \zeta) = \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha - \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_\beta(z) \xi^\beta + \\ + \sum_{k^* \leq \left| \frac{\gamma}{\rho'} \right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j + \sigma(z, \zeta),$$

be the symbol of the pseudo-differential operator

$$P(z, D) = \sum_{|\alpha|=m} a_\alpha(z) D_y^\alpha - \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_\beta(z) D_x^\beta + \\ + \sum_{k^* \leq \left| \frac{\gamma}{\rho'} \right| + |j| < m} c_{\gamma j}(z) D_x^\gamma D_y^j + \Xi(z, D),$$

where $z = (x, y) \in \mathbb{R}^{n'} \times \mathbb{R}^{n''}$, $\zeta = (\xi, \eta) = (\xi_1, \dots, \xi_{n'}, \eta_1, \dots, \eta_{n''}) \in \mathbb{R}^{n'} \times \mathbb{R}^{n''}$ is the dual variable of z ; $\left| \frac{\beta}{\rho'} \right| := \sum_{i=1}^{n'} \beta_i \frac{1}{\rho_i}$; $a_\alpha : \Omega \rightarrow \mathbb{R}$, $b_\beta, c_{\gamma j} : \Omega \rightarrow \mathbb{C}$, are in $C^\infty(\Omega)$, $\beta = (\beta_1, \dots, \beta_{n'})$, $\gamma = (\gamma_1, \dots, \gamma_{n'}) \in \mathbb{Z}_+^{n'}$, $\alpha = (\alpha_1, \dots, \alpha_{n''})$, $j = (j_1, \dots, j_{n''}) \in \mathbb{Z}_+^{n''}$. We define the following sets for $k \in \mathbb{Q}_+$, $0 < k < m$:

$$I_k = \left\{ (\gamma, j) \in \mathbb{Z}_+^{n'} \times \mathbb{Z}_+^{n''}, : \left| \frac{\gamma}{\rho'} \right| + |j| = k \right\}$$

and let k^* be such that $(m - \frac{1}{2}) < k^* < m$. We use the notation k^- for all $k < k^*$ and k^+ for all $k > k^*$. The symbol $\sigma(z, \zeta)$ in (2.1) belonging in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is such that

$$|D_z^p D_\zeta^q \sigma(z, \zeta)| \leq C_{p,q} \langle \zeta \rangle_\rho^{\bar{k} - \left| \frac{q}{\rho} \right|}$$

where we understand $p = (p', p'')$, $q = (q', q'') \in \mathbb{Z}_+^{n'} \times \mathbb{Z}_+^{n''}$, $\left| \frac{q}{\rho} \right| := \sum_{i=1}^{n'} q'_i \frac{1}{\rho_i} + \sum_{i=1}^{n''} q''_i$; with $\bar{k} < k^*$; $\langle \zeta \rangle_\rho = \langle \xi \rangle_{\rho'} + |\eta| = \sum_{i=1}^{n'} (1 + |\xi_i|^{\rho_i}) + |\eta|$ is the anisotropic norm. Let Λ be a neighborhood of the anisotropic characteristic manifold Σ , (see (1.7)), and let Γ the set $\Omega \times \Lambda$. Then we state the following:

Theorem 2.1. *Assume I_{k^*} is not empty, and moreover for $(z, \zeta) \in \Gamma$:*

$$\begin{aligned}
 & i) \Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \neq 0, \quad \xi \neq 0, \eta \neq 0 \\
 & ii) \Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j| = k^+} a_{\gamma j}(z) \xi^\gamma \eta^j \geq 0 \\
 (2.2) \quad & \text{for every } k_+ \\
 & iii) \Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \Im \sum_{\left| \frac{\beta}{\rho'} \right| = m} b_\beta(z) \xi^\beta \leq 0, \\
 & iv) \Re \sum_{\left| \frac{\beta}{\rho'} \right| = m} b_\beta(z) \xi^\beta \neq 0, \quad \xi \neq 0
 \end{aligned}$$

Then for all $p, q \in \mathbb{Z}_+^n$, for all $K \subset\subset \Omega$ there exist positive constants $L_{p,q}$ and B such that:

$$(2.3) \quad \frac{|D_z^p D_\zeta^q p(z, \zeta)| \langle \zeta \rangle_\rho^{\mu \left| \frac{q}{\rho} \right| - \delta \left| \frac{p}{\rho} \right|}}{|p(z, \zeta)|} \leq L_{p,q}, \quad z \in K, |\zeta| > B, \zeta \in \mathbb{R}^n,$$

with $\mu = k^* - (m - 1)$, $\delta = m - k^*$. Observe that $\delta < \mu$ since we have assumed $k^* > (m - \frac{1}{2})$

Remark 2.2. *Hypothesis ii) implies that $\Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*}$ and $\Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j| = k^+} c_{\gamma j}(z) \xi^\gamma \eta^j$ are both positive or both negative (we observe that the sum $\Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j| = k^+} c_{\gamma j}(z) \xi^\gamma \eta^j$ may vanish, too).*

Remark 2.3. *We may obtain the estimates (2.3) for $p(z, \zeta)$ also in the case when the set I_{k^*} is empty, by requiring $|\Im \sigma(z, \zeta)| \geq \langle \zeta \rangle_\rho^{\bar{k}}$, $\bar{k} > m - \frac{1}{2}$ and by replacing $\Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*}$ with $\Im \sigma(z, \zeta)$ in the hypotheses ii), iii). We will give more details at the end of this section.*

Remark 2.4. *By formula (2.3) and by well known result, see for example Rodino-Mascarello ([12], Theorem 3.3.6), we have that the operator $P(z, D)$, associated to the symbol $p(z, \zeta)$ in (2.1), is C^∞ -hypoelliptic and for analytic coefficients Gevrey-hypoelliptic in $G^\lambda(\Omega)$, $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_i \geq \frac{1}{k^* - m + 1}$. The estimates (2.3) guarantee also the existence of a parametrix of $P(z, D)$.*

Remark 2.5. *We confine ourselves for simplicity to prove the estimates (2.3) for $|p| + |q| = |p'| + |p''| + |q'| + |q''| = 1$. The case $|p| + |q| > 1$ does not involve actual complications; cf. Wakabayashi([16], Theorem 2.6), Kajitani-Wakabayashi([11], Theorem 1.9) for the analytic frame.*

Proof of theorem 2.1. We estimate first the numerator of (2.3) and then we give some lemmas to estimate the denominator.

If $|p'| = 1$, we get

$$|D_{x_l} p(z, \zeta)| \langle \zeta \rangle_\rho^{-\delta \frac{1}{\rho_l}} \leq L_1 \left((|\eta|^m + \langle \xi \rangle_{\rho'}^m + \langle \xi \rangle_{\rho'}^{\bar{m}} |\eta|^{|j|}) \langle \zeta \rangle_\rho^{-\delta \frac{1}{\rho_l}} + \langle \zeta \rangle_\rho^{\bar{k} - \delta \frac{1}{\rho_l}} \right)$$

where $l = 1, \dots, n'$, $\bar{m} < m - |j|$ and $\bar{k} < k^*$; moreover, for $|p''| = 1$

$$|D_{y_h} p(z, \zeta)| \langle \zeta \rangle_\rho^{-\delta} \leq L_2 \left((|\eta|^m + \langle \xi \rangle_{\rho'}^m + \langle \xi \rangle_{\rho'}^{\bar{m}} |\eta|^{|j|}) \langle \zeta \rangle_\rho^{-\delta} + \langle \zeta \rangle_\rho^{\bar{k} - \delta} \right)$$

with $h = 1, \dots, n''$, for suitable constants L_1, L_2 .

If $|q'| = 1$,

$$|D_{\xi_l} p(z, \zeta)| \langle \zeta \rangle_\rho^{\mu \frac{1}{\rho_l}} \leq L_3 \left(\left(\langle \xi \rangle_{\rho'}^{m - \frac{1}{\rho_l}} + \langle \xi \rangle_{\rho'}^{\bar{m} - \frac{1}{\rho_l}} |\eta|^{|j|} \right) \langle \zeta \rangle_\rho^{\mu \frac{1}{\rho_l}} + \langle \zeta \rangle_\rho^{\bar{k} - \frac{1}{\rho_l} (1 - \mu)} \right),$$

$l = 1, \dots, n'$; and for $|q''| = 1$

$$|D_{\eta_h} p(z, \zeta)| \langle \zeta \rangle_\rho^\mu \leq L_4 \left((|\eta|^{m-1} + \langle \xi \rangle_{\rho'}^{\bar{m}} |\eta|^{|j|-1}) \langle \zeta \rangle_\rho^\mu + \langle \zeta \rangle_\rho^{\bar{k} - 1 + \mu} \right)$$

where $h = 1, \dots, n''$, with suitable constants L_3, L_4 .

Therefore, we note that $\bar{k} - (1 - \mu) \geq \bar{k} - \frac{1}{\rho_l} (1 - \mu)$, $l = 1, \dots, n'$ and $\bar{k} - (1 - \mu) = \bar{k} - \delta > \bar{k} - \frac{1}{\rho_l} \delta$ since $\rho + \delta = 1$. To prove (2.3), it will be then sufficient to show the boundedness in \mathbb{R}^n , for $|\zeta| > B$, of the functions

$$\begin{aligned} Q_1(\zeta) &= \frac{(|\eta|^m + \langle \xi \rangle_{\rho'}^m + \langle \xi \rangle_{\rho'}^{\bar{m}} |\eta|^{|j|}) \langle \zeta \rangle_\rho^{-\delta}}{|p(z, \zeta)|}, \\ Q_2(\zeta) &= \frac{(\langle \xi \rangle_{\rho'}^{\bar{m}} |\eta|^{|j|-1} + |\eta|^{m-1}) \langle \zeta \rangle_\rho^\mu}{|p(z, \zeta)|}, \\ Q_3(\zeta) &= \frac{(\langle \xi \rangle_{\rho'}^{\bar{m} - \frac{1}{\rho_l}} |\eta|^{|j|} + \langle \xi \rangle_{\rho'}^{m - \frac{1}{\rho_l}}) \langle \zeta \rangle_\rho^{\mu \frac{1}{\rho_l}}}{|p(z, \zeta)|}, \quad l = 1, \dots, n' \end{aligned}$$

$$Q_4(\zeta) = \frac{\langle \zeta \rangle_{\rho}^{\bar{k}-1+\mu}}{|p(z, \zeta)|}.$$

First we introduce three regions:

$$(2.4) \quad \begin{aligned} R_1 : & \quad c \langle \xi \rangle_{\rho'} \leq |\eta| \leq C \langle \xi \rangle_{\rho'} \\ R_2 : & \quad |\eta| \geq C \langle \xi \rangle_{\rho'} \\ R_3 : & \quad |\eta| \leq c \langle \xi \rangle_{\rho'} \end{aligned} \quad ,$$

for suitable constants c, C to be determined precisely later on, satisfying $0 < c \ll \min_{z \in K} G(z)$, $G(z) = \min\{|b_{\beta}(z)|\}_{|\frac{\beta}{\rho'}|=m}$, and $C \gg \max_{z \in K} F(z)$, $F(z) = \max\{|b_{\beta}(z)|\}_{|\frac{\beta}{\rho'}|=m}$. We understand the neighborhood $\Lambda \subset R_1$.

The following inequalities then hold:

$$(2.5) \quad \langle \zeta \rangle_{\rho}^{-\delta} \leq \begin{cases} C^{\delta} |\eta|^{-\delta} & , \quad \xi \in R_1 & (I) \\ |\eta|^{-\delta} & , \quad \xi \in R_2 & (II) \\ \langle \xi \rangle_{\rho'}^{-\delta} & , \quad \xi \in R_3 ; & (III) \end{cases}$$

and,

$$\langle \zeta \rangle_{\rho}^{\mu} \leq \begin{cases} C_1 |\eta|^{\mu} & , \quad \xi \in R_1 \\ C_2 |\eta|^{\mu} & , \quad \xi \in R_2 \\ C_3 \langle \xi \rangle_{\rho'}^{\mu} & , \quad \xi \in R_3 ; \end{cases}$$

note that (II) and (III) in (2.5) hold for all $\zeta \in \mathbb{R}^n$, but for our aim we may limit ourselves to consider them respectively in R_2 and in R_3 . By abuse of notation, in the following we shall also denote by R_1, R_2, R_3 the sets $\Omega \times R_1, \Omega \times R_2, \Omega \times R_3$; recall that $\Gamma = \Omega \times \Lambda$.

Lemma 2.6. *Let $p(z, \zeta)$ be the function (2.1) and assume that i), ii), iii) in (2.2) hold. Then there are positive constants $K_1 < 1, B$, such that:*

$$(2.6) \quad |p(z, \zeta)| \geq K_1 \left| \mathfrak{S} \sum_{\substack{|\frac{\gamma^*}{\rho'}| + |j^*| = k^*}} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \right| \quad (z, \zeta) \in \Gamma \cap R_1, \quad |\zeta| > B.$$

Proof. We have that

$$\begin{aligned}
 (2.7) \quad & |p(z, \zeta)|^2 = \\
 & \left(\sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha - \Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta + \Re \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j + \Re \sigma(z, \zeta) \right)^2 + \\
 & + \left(\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} + \right. \\
 & \left. + \Im \sum_{k^* < \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j - \Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta + \Im \sigma(z, \zeta) \right)^2.
 \end{aligned}$$

By removing the terms rising from the real part of $p(z, \zeta)$, we can write

$$|p(z, \zeta)|^2 \geq \left(\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \right)^2 + \sum_{i=1}^4 W_i(z, \zeta)$$

where

$$(2.8) \quad W_1(z, \zeta) = \left(\Im \sum_{k^* < \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j - \Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta + \Im \sigma(z, \zeta) \right)^2,$$

$$(2.9) \quad W_2(z, \zeta) = 2\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \Im \sum_{k^* < \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j,$$

$$(2.10) \quad W_3(z, \zeta) = -2\Im \sum_{\left|\frac{\gamma^*}{\rho'}\right| + |j^*| = k^*} c_{\gamma^* j^*}(z) \xi^{\gamma^*} \eta^{j^*} \Im \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta,$$

$$(2.11) \quad W_4(z, \zeta) = 2\mathfrak{S}\sigma(z, \zeta)\mathfrak{S} \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^*j^*}(z)\xi^{\gamma^*}\eta^{j^*}.$$

The function (2.8) is non-negative, (2.9) and (2.10) are also non-negative by hypotheses (ii), (iii) for all $(z, \zeta) \in R_1$.

Concerning (2.11), it holds

$$\begin{aligned} \left(\mathfrak{S} \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^*j^*}(z)\xi^{\gamma^*}\eta^{j^*} \right)^2 + W_4(z, \zeta) &\geq \\ &\geq (1 - \epsilon) \left(\mathfrak{S} \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^*j^*}(z)\xi^{\gamma^*}\eta^{j^*} \right)^2. \end{aligned}$$

In fact, for $|\zeta|$ sufficiently large

$$\begin{aligned} \frac{|W_4(z, \zeta)|}{\left| \mathfrak{S} \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^*j^*}(z)\xi^{\gamma^*}\eta^{j^*} \right|^2} &\leq \text{const} \frac{\langle \xi \rangle_{\rho'}^{k^* - |j^*|} |\eta|^{|j^*|} \langle \xi \rangle_{\rho'}^{\bar{k}}}{|\eta|^{2|j^*|} \langle \xi \rangle_{\rho'}^{2k^* - 2|j^*|}} \leq \\ &\leq \text{const} \frac{|\eta|^{k^* - \bar{k}}}{\eta^{2k^*}} < \epsilon, \quad |\zeta| > B; \end{aligned}$$

since $\bar{k} < k^*$.

Then

$$|p(z, \zeta)| \geq K_1 \left| \mathfrak{S} \sum_{\left|\frac{\gamma^*}{\rho'}\right|+|j^*|=k^*} c_{\gamma^*j^*}(z)\xi^{\gamma^*}\eta^{j^*} \right|, \quad (z, \zeta) \in R_1, \quad |\zeta| > B,$$

for a suitable positive constant K_1 . \square

Lemma 2.7. *Let $p(z, \zeta)$ be the function (2.1). Then there are positive constants $K_2 < 1$, B , such that:*

$$(2.12) \quad |p(z, \zeta)| \geq K_2 |\eta|^m, \quad (z, \zeta) \in R_2, \quad |\zeta| > B.$$

Proof. We write $|p(z, \zeta)|^2$ as in (2.7); by removing the terms arising from the imaginary part of $p(z, \zeta)$, we get

$$(2.13) \quad |p(z, \zeta)|^2 \geq \left(\sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha - \Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta \right)^2 + W_1(z, \zeta) + W_2(z, \zeta) + W_3(z, \zeta)$$

where

$$(2.14) \quad W_1(z, \zeta) = \left(\Re \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j + \Re \sigma(z, \zeta) \right)^2,$$

$$(2.15) \quad W_2 = 2\Re \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j \sum_{|\alpha|=m} a_\alpha \eta^\alpha - 2\Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta \Re \sum_{k^* \leq \left|\frac{\gamma}{\rho'}\right| + |j| < m} a_{\gamma j}(z) \xi^\gamma \eta^j$$

$$(2.16) \quad W_3 = 2 \sum_{|\alpha|=m} a_\alpha \eta^\alpha \Re \sigma(z, \zeta) - 2\Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta \Re \sigma(z, \zeta).$$

Observe first that for $\lambda > 0$ sufficiently small

$$\left(\sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha - \Re \sum_{\left|\frac{\beta}{\rho'}\right|=m} b_\beta(z) \xi^\beta \right)^2 > \lambda \eta^{2m};$$

(2.14) is non-negative. We denote (2.15) by $\Upsilon_1(z, \zeta) - \Upsilon_2(z, \zeta)$ and (2.16) by $\Upsilon_3(z, \zeta) - \Upsilon_4(z, \zeta)$. Then

$$|p(z, \zeta)|^2 \geq \lambda \left| \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha \right|^2 + \Upsilon_1(z, \zeta) - \Upsilon_2(z, \zeta) + \Upsilon_3(z, \zeta) - \Upsilon_4(z, \zeta).$$

Arguing on $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ in the same way as we have done in Lemma 2.6, it is possible to show that for all $\epsilon > 0$

$$\frac{\lambda}{2} \left| \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha \right|^2 + \Upsilon_1(z, \zeta) - \Upsilon_2(z, \zeta) \geq \frac{(\lambda - \epsilon)}{2} \left| \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha \right|^2, \quad (z, \zeta) \in R_2,$$

and

$$\frac{\lambda}{2} \left| \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha \right|^2 + \Upsilon_3(z, \zeta) - \Upsilon_4(z, \zeta) \geq \frac{(\lambda - \epsilon)}{2} \left| \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha \right|^2, \quad (z, \zeta) \in R_2.$$

Thus

$$|p(z, \zeta)| \geq K_2 \eta^m, \quad (z, \zeta) \in R_2, \quad |\zeta| > B.$$

□

Lemma 2.8. *Let $p(z, \zeta)$ be the function (2.1), such that iv) in (2.2) holds. Then there are positive constants $K_3 < 1, B$, such that:*

$$(2.17) \quad |p(z, \zeta)| \geq K_3 \langle \xi \rangle_{\rho'}^m, \quad (z, \zeta) \in R_3, \quad |\zeta| > B.$$

Proof. We apply again (2.13), (2.14), (2.15), (2.16) to $|p(z, \zeta)|^2$. Observe that in R_3 , arguing as above, since $c \ll \min_{z \in K} G(z)$, we obtain for a suitable constant $\mu > 0$

$$\left(\sum_{|\alpha|=m} a_\alpha \eta^\alpha - \Re \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_\beta(z) \xi^\beta \right)^2 > \mu \langle \xi \rangle_{\rho'}^{2m}.$$

About the terms in (2.14), (2.15) and (2.16), the remarks we have done in Lemma 2.7 hold by replacing $\lambda \left| \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha \right|^2$ with $\mu \langle \xi \rangle_{\rho'}^{2m}$. Then we have

$$|p(z, \zeta)| \geq K_3 \langle \xi \rangle_{\rho'}^m, \quad (z, \zeta) \in R_3, \quad |\zeta| > B.$$

□

Remark 2.9. *In the previous lemmas we have estimated the symbol function $|p(z, \zeta)|$ in (2.1), separately in the three regions (2.4). It is possible to obtain the following global result on $|p(z, \zeta)|$: there exists $m' \in \mathbb{R}, d > 0, B > 0$, such that*

$$(2.18) \quad |p(z, \zeta)| \geq d \langle \zeta \rangle_{\rho}^{m'}, \quad \text{in } \Gamma \text{ for } |\zeta| > B.$$

In fact, by remembering that under assumptions *i*), *ii*), *iii*), *iv*) in theorem 2.1 the estimates (2.6), (2.12), (2.17) hold, we obtain that $|p(z, \zeta)| \geq \text{const} |\eta|^{k^*}$ in R_1 and R_2 . Since in these regions $|\eta|^{k^*} = \frac{1}{2}|\eta|^{k^*} + \frac{1}{2}|\eta|^{k^*} \geq c\langle \xi \rangle_{\rho'}^{k^*} + \frac{1}{2}|\eta|^{k^*} \sim \text{const} (\langle \xi \rangle_{\rho'} + |\eta|)^{k^*} = \text{const} \langle \zeta \rangle_{\rho}^{k^*}$, we have that

$$|p(z, \zeta)| \geq d \langle \zeta \rangle_{\rho}^{k^*}.$$

In the same way we get

$$|p(z, \zeta)| \geq d \langle \zeta \rangle_{\rho}^m$$

in R_3 . Because $k^* < m$, we have $m' = k^*$ in (2.18).

We first consider $Q_1(\zeta)$ separately in the regions R_1 , R_2 , R_3 , to prove boundedness.

In R_1 by (2.5),(2.6) we get easily:

$$Q_1(\zeta) \leq \text{const} \frac{|\eta|^{\bar{m}+j-\delta} + |\eta|^{m-\delta}}{|\eta|^{k^*}} \leq L, |\zeta| > B$$

since $\delta \geq m - k^*$. We recall that $\bar{m} + j < m$. In the region R_2 we have that $|p(z, \zeta)| \geq |\eta|^m > |\eta|^{k^*}$. In R_3 , by using (2.5),(2.17), we have for a constant $\epsilon > 0$ which we may take as small as we want by fixing B sufficiently large:

$$Q_1(\zeta) \leq \text{const} \frac{\langle \xi \rangle_{\rho'}^{\bar{m}+j-\delta} + \langle \xi \rangle_{\rho'}^{m-\delta}}{\langle \xi \rangle_{\rho'}^m} < \epsilon, |\zeta| > B$$

We have therefore proved that $Q_1(\zeta)$ is bounded. Let us estimate $Q_2(\zeta)$ and $Q_3(\zeta)$. In the region R_2 we argue as before; in the regions R_1 , R_3 we obtain respectively

$$Q_2(\zeta) \leq \text{const} \frac{|\eta|^{\bar{m}-\frac{1}{\rho_l}+j+\mu\frac{1}{\rho_l}} + |\eta|^{m-\frac{1}{\rho_l}+\mu\frac{1}{\rho_l}}}{|\eta|^{k^*}} < \epsilon,$$

in R_1 for $|\zeta| > B$, since $\mu \leq k^* - (m - 1)$, $\bar{m} < m - j$,

$$Q_2(\zeta) \leq \text{const} \frac{\langle \xi \rangle_{\rho'}^{\bar{m}-\frac{1}{\rho_l}+j+\mu\frac{1}{\rho_l}} + \langle \xi \rangle_{\rho'}^{m-\frac{1}{\rho_l}+\mu\frac{1}{\rho_l}}}{\langle \xi \rangle_{\rho'}^{k^*}} < \epsilon,$$

in R_3 for $|\zeta| > B$.

For Q_3 we obtain that

$$(2.19) \quad Q_3(\zeta) \leq \text{const} \frac{|\eta|^{\bar{m}+j-1+\mu} + |\eta|^{m-1+\mu}}{|\eta|^{k^*}} \leq L, |\zeta| > B$$

since $\mu \leq k^* - m + 1$ in R_1 and

$$(2.20) \quad Q_3(\zeta) \leq \text{const} \frac{\langle \xi \rangle_{\rho'}^{\bar{m}+j-1+\mu} + \langle \xi \rangle_{\rho'}^{m-1+\mu}}{\langle \xi \rangle_{\rho'}^m} < \epsilon,$$

in R_3 for $|\zeta| > B$.

For Q_4 , we get $\frac{|\eta|^{\bar{k}-(1-\mu)}}{|\eta|^{k^*}} < \epsilon$ in R_1 since $\bar{k} < k^*$, $\mu < 1$. In R_3 : $\frac{\langle \xi \rangle_{\rho'}^{\bar{k}-(1-\mu)}}{\langle \xi \rangle_{\rho'}^{k^*}} < \epsilon$, $|\zeta| > B$

Now Lemma 2.6, Lemma 2.7, Lemma 2.8 and the estimate (2.18) complete the proof.

We shall use also the following variant of Theorem 1.1, where the role of I_{k^*} -terms is played by the pseudo-differential term $\sigma(z, \zeta)$ (see Remark 2.3). Namely, we fix now t with $0 < t < \frac{1}{2}$ and assume that $|\Im \sigma(z, \zeta)| \geq \langle \zeta \rangle_{\rho}^{\bar{k}}$ where $m - \frac{1}{2} < \bar{k} < m - t$, considering a symbol of the form:

$$(2.21) \quad p(z, \zeta) = \sum_{|\alpha|=m} a_{\alpha}(z) \eta^{\alpha} - \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_{\beta}(z) \xi^{\beta} + \sum_{m-t \leq \left| \frac{\gamma}{\rho'} \right| + |j| < m} c_{\gamma j}(z) \xi^{\gamma} \eta^j + \sigma(z, \zeta),$$

Theorem 2.10. *Let $p(z, \zeta)$ be the function (2.21) such that for $(z, \zeta) \in \Gamma$*

$$(2.22) \quad \begin{aligned} & i) \quad |\Im \sigma(z, \zeta)| \geq \langle \zeta \rangle_{\rho}^{\bar{k}}, \\ & ii) \quad \Im \sigma(z, \zeta) \Im \sum_{\left| \frac{\gamma}{\rho'} \right| + |j|=k_+} c_{\gamma j}(z) \xi^{\gamma} \eta^j \geq 0, \text{ for every } k_+ \geq m - t, \\ & iii) \quad \Im \sigma(z, \zeta) \Im \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_{\beta}(z) \xi^{\beta} \leq 0, \\ & iv) \quad \Re \sum_{\left| \frac{\beta}{\rho'} \right|=m} b_{\beta}(z) \xi^{\beta} \neq 0, \quad \xi \neq 0 \end{aligned}$$

Then for all $p, q \in \mathbb{Z}_+^n$, for all $K \subset\subset \Omega$ there exist such positive constants $L_{p,q}$ and B that:

$$(2.23) \quad \frac{|D_z^p D_\zeta^q p(z, \zeta)| \langle \zeta \rangle_\rho^{\mu \left| \frac{q}{\rho} \right| - \delta \left| \frac{p}{\rho} \right|}}{|p(z, \zeta)|} \leq L_{p,q}, z \in K, |\zeta| > B, \zeta \in \mathbb{R}^n,$$

with $\mu = \bar{k} - (m - 1)$, $\delta = m - \bar{k}$. Observe that $\delta < \mu$ since we have assumed $\bar{k} > (m - \frac{1}{2})$.

Proof. We have \bar{k} in the role of k^* in the proof of Theorem 2.1, by observing that in R_1

$$(2.24) \quad \begin{aligned} & |p(z, \zeta)|^2 \geq \\ & \left(\Im \sum_{m-t < \left| \frac{\gamma}{\rho'} \right| + |j| < m} a_{\gamma j}(z) \xi^\gamma \eta^j - \Im \sum_{\left| \frac{\beta}{\rho'} \right| = m} b_\beta(z) \xi^\beta + \Im \sigma(z, \zeta) \right)^2 \\ & \geq (\Im \sigma(z, \zeta))^2 \geq \langle \zeta \rangle_\rho^{2\bar{k}} \geq |\eta|^{2\bar{k}} \end{aligned}$$

since *ii*), *iii*) and *i*) hold; then arguing as in the proof of theorem 2.1 we obtain our result. Of course, the power m' in Remark 2.9 is given by \bar{k} :

$$(2.25) \quad |p(z, \zeta)| \geq d \langle \xi \rangle_\rho^{\bar{k}}, \quad |\zeta| > B.$$

□

3. The semilinear version

Let us consider now the semilinear equation

$$(3.1) \quad P(x, y, D_x, D_y)u + G(x, y; \partial_x^\gamma \partial_y^j u) \Big|_{\left| \frac{\gamma}{\rho'} \right| + |j| < k^*} = 0,$$

where $P(x, y, D_x, D_y)$ is the model operator considered in (1.2) having the symbol $p(z, \zeta)$ in $\Omega \times \mathbb{R}^n$:

$$(3.2) \quad p(z, \zeta) = \sum_{|\alpha|=m} a_\alpha(z) \eta^\alpha - \sum_{\left| \frac{\beta}{\rho'} \right| = m} b_\beta(z) \xi^\beta + \sum_{k^* \leq \left| \frac{\gamma}{\rho'} \right| + |j| < m} c_{\gamma j}(z) \xi^\gamma \eta^j$$

and such that the hypotheses of the Theorem 1.1 hold. Moreover G is of the type:

$$G(z; t) = \sum_{r \in \mathbb{Z}_+^M} C_r(z) t^r, \quad C_r \in C^\infty(\Omega), \quad t \in \mathbb{Z}^M,$$

where, for every compact $K \subset \Omega$, $\sup_{z \in K} |D^\alpha C_r(z)| \leq C_{\alpha,K} \lambda_r$ and moreover $\tilde{F}(t) = \sum_r \lambda_r t^r$ is entire analytic.

Theorem 3.1. *Under the above assumptions on $P(x, y, D_x, D_y)$ and G , let u be a solution of (3.1) which belongs to $H_{\rho,loc}^s(\Omega)$, for $s \geq s_0$, where s_0 is a sufficiently large fixed real number. Then $u \in C^\infty(\Omega)$.*

Proof. Observe that $D_x^\gamma D_y^j u \in H_{\rho,loc}^{s - (\frac{\gamma}{\rho'} + |j|)}(\Omega)$. Note that $\left\lfloor \frac{\gamma}{\rho'} \right\rfloor + |j| < k^*$ actually implies $\left\lfloor \frac{\gamma}{\rho'} \right\rfloor + |j| \leq k^* - \varepsilon$, with $\varepsilon > 0$. Then

$$P(x, y, D_x, D_y)u = -G(x, y; \partial_x^\gamma \partial_y^j u) \Big|_{\left\lfloor \frac{\gamma}{\rho'} \right\rfloor + |j| < k^*} \in H_{\rho,loc}^{s - k^* + \varepsilon}(\Omega)$$

using here the assumption $s \geq s_0$, see Garello ([6], remark 2.4). By remark 1.2 we have that $u \in H_{\rho,loc}^{s+\gamma}(\Omega)$. Using again Garello ([6], remark 2.4) we get that $P(x, y, D_x, D_y)u \in H_{\rho,loc}^{s-k^*+2\varepsilon}(\Omega)$ and in its own turn $u \in H_{\rho,loc}^{s+2\gamma}(\Omega)$. Repeating now the preceding argument we obtain $u \in \cap_{t \in \mathbb{R}^+} H_{\rho,loc}^t(\Omega)$, that is $u \in C^\infty(\Omega)$. \square

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