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STUDY ON ROBUSTNESS OF STREHLER-MILDVAN MODEL

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The probability that a device will work properly after a certain period of time can be studied using the Strehler-Mildvan model. Let us suppose that the functionality of a device depends on an unknown parameter X, which decreases progressively in time. The device stops working if X goes below a certain given value.

We can use the method of maximum likelihood to estimate the parameters of the model and to estimate the probability of proper work at a future moment using the survival function.

This model can be modified in order its robust performance to be improved. We will consider the breakdown properties of the model using the WLTE(k) Estimators and the theory of d-fullness of the set of subcompact functions.

1. The Strehler-Mildvan model

Here we will consider a robust modification of the Strehler-Mildvan model. Let us suppose that there is a population of individuals and our aim is to study the mortality function in the population. According to this model there is an unknown parameter X, which is critical for the vitality of an individual. Let us suppose (following the Strehler-Mildvan model) that this parameter decreases

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progressively in time $X = X_0(1 - ct)$, where X_0 is the value in the moment of birth.

Let T be a random value presenting the moment of death. As we know, the hazard function

$$\mu(x) = lim_{\Delta \to 0} \frac{P(x < T < x + \Delta \mid T > x)}{\Delta} = \frac{f(x)}{1 - F(x)}$$

represents the probability of death of the individual at a moment x. Here $f(x)$ and $F(x)$ are the probability density function and the cumulative function of T. Using the same notation we can represent the survival function

$$S(x) = P(X > x) = 1 - F(x) = \int_x^\infty f(s)\, ds = e^{-\int_0^x \mu(s) ds} = e^{-H(x)},$$

where

$$H(x) = \int_0^x \mu(s)\, ds = \int_0^x \frac{f(s)}{1 - F(s)} ds = -log(1 - F(x))$$

is the cumulative hazard function.

Let us suppose that in the moment of death T the individual examines a stress with intensity Y. This stress is exponentially distributed

$$P(Y > y) = e^{-\lambda y},$$

and T is a process with intensity K. The individual dies if the stress at the moment T is greater than the value of the parameter X, representing vitality. Obviously we have

$$\mu(x)\, \Delta x = K \Delta x P(Y > X(x)).$$

Therefore

$$\mu(x) = K e^{-\lambda X_0 (1 - cx)}. \tag{1}$$

The main problem in this theory is that the lifespan is limited. If we suppose that the hazard function follows the Gompertz law, $\mu(x) = ae^{bx}$, we have $a = Ke^{\lambda X_0}$ $b = \lambda X_0 c$ and

$$\log a = \log K - \frac{b}{c}. \tag{2}$$

Therefore we have a relation between parameters of the model

$$\log \mu(x) = \log a + bx = \log K - \frac{b}{c} + bx = \log K + b\left(x - \frac{b}{c}\right).$$

So, the maximum lifespan for this model is equal to $\frac{1}{c}$.

To resolve this problem we can use a modification of Strehler-Mildvan model. If we consider $1 - cx$ as approximation of e^{-cx} we will have that the critical parameter decreases as

$$X(x) = X_0 e^{-cx}.$$

Therefore the hazard function becomes

$$\mu(x) = Ke^{-\lambda X_0 e^{-cx}}.$$

The only difference between these two models is that in the second one we have larger values of the hazard function in the beginning of the lifespan. There is no difference in the computational methods used for the models.

2. Statistical model

A robust extension of the maximum likelihood estimators (MLE) that possesses a high breakdown point was introduced by Vandev and Neykov (1993). This modification considers the likelihood of individual observations as residuals and applies the basic idea of the LTS estimators of Rousseeuw (1984) using appropriate weights.

Generally speaking, Vandev and Neykov (1998) defined the $WLTE(k)$ estimators, $\widehat{\theta}$, for the unknown parameter $\theta \in \Theta^p$ as

$$\widehat{\theta} = \underset{\theta \in \Theta}{argmin} \sum_{i=1}^{k} w_i f_{\nu(i)}(\theta),$$

where $f_{\nu(1)}(\theta) \leq f_{\nu(2)}(\theta) \leq \cdots \leq f_{\nu(n)}(\theta)$ are the ordered values of $f_i = -\log \varphi(x_i, \theta)$ at θ, $\varphi(x_i, \theta)$ is a probability density, θ is an unknown parameter and $\nu = (\nu(1), \cdots, \nu(n))$ is the corresponding permutation of the indices, which may depend on θ. The weights $w_i \geq 0$, $i = 1, \cdots, k$, are such that an index $k = \max\{i : w_i > 0\}$ exists.

Vandev and Neykov (1998) proved that the finite sample breakdown point of the $WLTE(k)$ estimators is not less than $(n-k)/n$ if $n \geq 3d$, $(n+d)/2 \leq k \leq n-d$, when Θ^p is a topological space and the set $F = \{f_i(\theta), i = 1, \cdots, n\}$ is d-full.A finite set F of n functions is called d-full, according to Vandev (1993), if for each subset of cardinality d of F, the supremum of this subset is a subcompact function. A real valued function $g(\theta)$ is called subcompact, if its Lesbegue sets $L_g(C) = \{\theta : g(\theta) \leq C\}$ are compact for any constant C (see Vandev and Neykov, 1993).

For the sake of completeness, we draw the attention to the fact that the finite sample breakdown point of an estimator T, at the finite sample $X = \{x_i; i =$

$1, \cdots, n\}$, is defined as the largest fraction m/n for which the $\sup_{\tilde{X}} \left\| T(X) - T\left(\tilde{X}\right) \right\|$ is finite, where $\tilde{X}$ is a sample obtained from X by replacing any m of the points in X by arbitrary values (see Hampel et al. 1986, Rousseeuw and Leroy, 1987).

Thus, if one wants to study the breakdown point of the $WLTE(k)$ estimators for a particular distribution, one has to find out the index d of fullness of the corresponding set of log-density functions.

According to Atanasov and Neykov (2001) if D is an open subset of R^n, θ_0 belongs to the boundary of D and $g(\theta)$ is a real valued continuous function defined on D, then we have the following theorem.

Theorem 1. *The function $g(\theta)$ is subcompact if and only if for any sequence $\theta_i \to \theta_0$ $g(\theta_i) \to \infty$ when $i \to \infty$.*

Remark 1. If D is a compact set, then any continuous function defined on D is subcompact.

We will apply this concept for studying the robust properties of estimates of the parameters of the Strhler-Mildvan model.

3. Robust properties and estimation of the parameters

We will consider the case of clasical censoring of the data. The couples (T_i, Δ_i), $i = 1, \cdots, n$ represent the moment of the observation T_i and Δ_i is an indicator if the individual is alive or not ($\Delta_i = 1$ if, in the moment T_i the individual is alive and zero otherwise). Let us suppose that there is another set of random values $Z_1, \cdots, Z_n$, representing the moments Z, in which the individual dies. These values are unobservable. The observations we have are $\min(T_i, Z_i), i = 1, \cdots, n$, because if $T_i \leq Z_i$ we will observe that the individual is alive (we note this with $\Delta = 1$).

The density function for this data can be written as follows:

$$L(T, \Delta) = \left(f(T)(1 - G(T))\right)^{1-\Delta} \left(g(T)(1 - F(T))\right)^{\Delta}.$$

Here $f(T)$ and $F(T)$ are the density function and the cumulative density function of the moments of observations as defined above. The functions $g(T)$ and $G(T)$ are the density function and the cumulative density function for the unknown variable Z.

Using this notation the likelihood function for this model is

$$L\left(\{T_i, \Delta_i\}_{i=1}^{n}\right) = \prod_{i=1}^{n} \left(f(T_i)(1 - G(T_i))\right)^{1-\Delta_i} \left(g(T_i)(1 - F(T_i))\right)^{\Delta_i}.$$

As the unknown variable Z does not give any additional information, we will restrict the likelihood function over the observable variable T. Therefore the likelihood function becomes

$$L\left(\{T_i, \Delta_i\}_{i=1}^{n}\right) = \prod_{i=1}^{n} f\left(T_i\right)^{1-\Delta_i}\left(1 - F\left(T_i\right)\right)^{\Delta_i}.$$

The $WLT(k)$ estimator for theunknown paremeter $\theta = (K, \lambda, X_0, c)$ of the model can be calculated as follows:

$$\widehat{\theta} = \underset{\theta \in \Theta}{argmin} \sum_{i=1}^{k} w_i(-(1 - \Delta_{\nu(i)}) \log f(T_{\nu(i)}) - \Delta_{\nu(i)} \log(1 - F(T_{\nu(i)}))),$$

where $\Theta = \{K > 0, \lambda > 0, X_0 \in R, c > 0\}$, k is the trimming factor and ν is a proper permutation of indexes.

In order to study the breakdown point properties of the Strehler-Mildvan model we have to find out the index of fullnes of the set of log-likelihood curves for the model

$$g(T_i, \Delta_i, \theta) = (\Delta_i - 1) \log f(T_i) - \Delta_i \log(1 - F(T_i)).$$

This function can be rewriten as

$$g(T_i, \Delta_i, \theta) = (\Delta_i - 1) \log \mu(T_i, \theta) S(T_i, \theta) - \Delta_i \log S(T_i, \theta) =$$

$$= (\Delta_i - 1) \log \mu(T_i, \theta) - \log S(T_i, \theta) =$$

$$= (\Delta_i - 1) \log K e^{-\lambda X_0 (1 - c T_i)} + H(T_i, \theta).$$

This function (as well as any supemum of such functions) does not satisfy the conditions of the Theorem 1., so the set of functions $F = \{g(T_i, \Delta_i, \theta), i = 1, \cdots, n\}$ is not full for any index of fullnes d.

Therefore, according to the Remark 1., in order to obtain a robust estimator of the parameters of the model, we have to restrict θ to a compact set $\Theta^c = \{0 \geq K \geq K_u, 0 \geq \lambda \geq \lambda_u, X_0 \in C, 0 \geq c \geq c_u\}$, where K_u, λ_u, c_u are real values and C is a compact subset of R. In this case it is obvious that the index of fullnes of the set $F = \{g(T_i, \Delta_i, \theta), i = 1, \cdots, n\}$ is equal to 4. Therefore, according to Vandev and Neykov (1998), the breakdown point of these estimators is not less than $(n - k)/n$ if $n \geq 12$ and $(n + 4)/2 \leq k \leq n - 4$.

4. Discussion

These ideas can be applied for studying the surety period of devices which allow a single test of functionality (Atanasov, 2003). To estimate the survival function we use a maximum likelihood estimation of unknown parameters using the trimmed likelihood function for censored data.

This technique can be applied also in the classical studying of demographic models. In such studies, using the weights of the estimator, we can study more than one subpopulations (Atanasov, 2002).

It is also possible that, a continuous heterogeneity in the population is applied.

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