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Pliska Stud. Math. Bulgar. **16** (2004), 89-99

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ON A STOCHASTIC PARTIAL DIFFERENTIAL EQUATION WITH A NOISY TERM

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We review results obtained in [13] and [14] on a one-dimensional Burgers-type stochastic differential equation involving fractional power of the Laplacian in its linear part, perturbed by a white noise term, with Dirichlet boundary conditions. We discuss existence of weak solutions and regularity of solutions.

1. Introduction

We consider the stochastic partial differential equation

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= \frac{\partial^2}{\partial x^2}u(t,x) + f(t,x,u(t,x)) + \frac{\partial}{\partial x}g(t,x,u(t,x)) \\ &+ \sigma(t,x,u(t,x))\frac{\partial^2}{\partial t\partial x}W(t,x)\end{aligned}$$

with Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0,\ t \geq 0$$

and initial condition

$$u(0,x) = u_0(x),\ x \in [0,1],$$

2000 *Mathematics Subject Classification:* 60H15, 60H40

Key words: Burgers equation, white noise, weak and strong solutions, Hilbert space regularity.

where $\frac{\partial^2}{\partial t\partial x}W(t,x)$ is a space-time white noise (see [21] for the definition and properties of white noise), $u_0 \in L^2([0,1])$ and $f \equiv f(t,x,y)$, $g \equiv g(t,x,y)$, $\sigma \equiv \sigma(t,x,y)$ are Borel-measurable functions on $\mathbb{R}_+ \times [0,1] \times \mathbb{R}$. A solution of the above equation is an $L^2([0,1])$-valued continuous process, adapted to the filtration generated by the white noise, which solves the equation in a weak sense (see below).

When $f = \sigma = 0$ and $g(t,x,y) = y^2/2$ the above equation is called Burgers equation. It has been proposed as a model for turbulent fluid motion (see [4, 5, 11]). When $g = 0$ the equation is a stochastic reaction-diffusion equation which has been studied intensively (see e.g. [8, 21, 3, 15] and the references therein).

When $f = 0$, $g(t,x,y) = y^2/2$ and $\sigma \neq 0$, we obtain the Burgers equation perturbed by a space-time white noise. It has been studied by several authors under Lipschitz conditions on σ (see e.g. [1, 7, 6, 10] and the references therein).

Burgers equation involving fractional powers $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, $\alpha \in (0,2]$, of the Laplacian in its linear part has also been studied in connection with models of several hydrodynamical phenomena (see e.g. [20], [9], [6] and the references therein for applications).

In [6] Biller, Funaki and Woyczynski studied existence, uniqueness, regularity and asymptotic behavior of solutions of the multidimensional fractal Burgers-type equation

$$\frac{\partial}{\partial t}u(t,x) = \nu\Delta_\alpha u(t,x) - a\nabla u^r(t,x), \tag{1}$$

where $x \in \mathbb{R}^d$, $d \geq 1$, $\alpha \in (0,2]$, $r \geq 1$, and $a \in \mathbb{R}^d$ is a fixed vector. For $\alpha > 3/2$ and $d = 1$ they proved existence of a unique regular weak solution of (1) with initial conditions in $H^1(\mathbb{R})$.

In [13] it is proved existence of a weak solution of the one-dimensional stochastic Burgers equation perturbed by a white noise term with a non-Lipschitz coefficient

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \Delta u(t,x) + \lambda\nabla u^2(t,x) + \gamma\sqrt{u(t,x)(1-u(t,x))}\,\frac{\partial^2}{\partial t\partial x}W(t,x), \\ u(t,0) &= u(t,1) = 0, \\ u(0,x) &= f(x),\ x \in [0,1], \end{aligned} \tag{2}$$

where $f : [0,1] \to [0,1]$ is continuous and $\frac{\partial^2}{\partial t\partial x}W(t,x)$ is the space-time white noise. Equation (2) is interpreted in the weak sense, which means that for each $\varphi \in C^2([0,1])$,

$$\int_{[0,1]} u(t,x)\varphi(x)\,dx = \int_{[0,1]} u(0,x)\varphi(x)\,dx + \int_{[0,1]} u(t,x)\varphi''(x)\,dx - \tag{3}$$

$$-\lambda\int_0^t\int_{[0,1]} u^2(s,x)\varphi'(x)\,dx\,ds+\gamma\int_0^t\int_{[0,1]}\sqrt{u(s,x)(1-u(s,x))}\varphi(x)\,W(ds,dx).$$

The method of proof in [13] consists in approximating (2) by finite systems of stochastic differential equations possessing a unique strong solution. Using bounds for the fundamental solution of the discrete Laplacian, it is shown tightness of the approximating systems, and that each weak limit is a weak solution of (2).

In this paper we concentrate mainly on the one-dimensional fractal Burgers equation

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= \Delta_\alpha u(t,x)+\lambda\nabla u^2(t,x)+\gamma\sqrt{u(t,x)(1-u(t,x))}\,\frac{\partial^2}{\partial t\partial x}W(t,x),\\ u(t,0) &= u(t,1)=0,\ x\in[0,1],\end{aligned}\tag{4}$$

where the random positive initial condition $u(0,x)$ is bounded by 1.

Due to the presence of non-Lipschitz coefficients, existence and uniqueness of a weak solution of (4) cannot be achieved by classical results. Following the method of proof of [13], in this paper we consider a discrete version of (4). Using the method of [13], we prove existence of a strong solution of the corresponding finite system of SDEs. We discuss also existence and regularity properties of a weak solutuons proved in [14].

2. Notations and basic results

We recall some notations from [2]. Let $\mathbf{S}=[0,1)$ and let $\mathbf{T}$ denote the quotient space obtained from $[0,1]$ by identifying 0 and 1. We put $\varphi_0(x)=1$ for $x\in[0,1]$, and

$$\varphi_n(x)=\sqrt{2}\cos(\pi nx),\quad \psi_n(x)=\sqrt{2}\sin(\pi nx),\quad x\in[0,1],\quad n=2,4,\dots.$$

This system of functions, which we also denote by e_m, $m=0,1,2,...$, is the usual orthonormal basis in $L^2([0,1])$. Moreover, for all n, $\Delta e_n=-\pi^2n^2e_n$. For any $\beta\in\mathbb{R}$ we define H_β as the Hilbert space obtained from $L^2(S)$ by completion with respect to the norm

$$|f|_\beta=\left(\sum\langle f,e_m\rangle^2(1+\pi^2m^2)^\beta\right)^{1/2},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $L^2(S)$.

For any integer $N\geq 1$, let $H(N)$ denote the set of functions $f:[0,1]\to\mathbb{R}$ which are constant on $[\frac{k}{N},\frac{k+1}{N})$ for $k=0,1,2,...,N-1$. Clearly we have $H(N)\subset L^2([0,1])$.

Without loss of generality we assume that $\lambda = \gamma = 1$.

Let us fix a positive integer N, and consider the discretized version of (4), namely the system

$$\begin{aligned} \frac{\partial}{\partial t} X^N(t,r) = & \quad \Delta_{N,\alpha} X^N(t,r) + \nabla_N X^N(t,r)^2 + \\ & + \sqrt{N X^N(t,r)(1 - X^N(t,r))}\, dB_N(t,r), \\ X^N(0,r) = & \quad X(0,r), \;\; r = 0, \tfrac{1}{N}, ..., \tfrac{N-1}{N}, \;\; t \geq 0, \end{aligned} \tag{5}$$

where $\Delta_{N,\alpha}$ is the fractional power of the discrete Laplacian, and $\{B_N(t,r)\}_r$ is a sequence of independent Brownian motions. Our results are the following theorems.

Theorem 1. *(a) For any positive initial random condition $X^N(0)$ bounded by 1, there exists a unique strong solution $X^N(t)$ of (5) in the space* $C([0,\infty), L^2([0,1])$.

(b) The distributions of $\{X^N\}$ are relatively compact on $C((0,\infty) : H_\beta)$ if $\beta \leq 0, \alpha > \beta + 3/2$, and on $C([0,\infty) : H_\beta)$ for $\alpha > \beta + 3/2, \beta < -1/2$.

(c) For any $\alpha > 3/2$, equation (4) has a weak solution in the space $C((0,\infty)$, $L^2([0,1]))$.

Remark 1. Theorem 1 is consistent with results obtained in [6] for the case $\gamma = 0$.

Theorem 2. *The weak solution $X(t)$ in Theorem 1 has a modification which is Holder continuous in time: it satisfies*

$$P\left(\sup_{0 < s_0 \leq s < t \leq T} \frac{|X(t) - X(s)|_\beta}{|t-s|^\delta} < \infty \right) = 1$$

for each $0 < \delta < [(2\alpha - 2\beta - 3)/(2\alpha)] \wedge 1/2$, $3/2 < \alpha \leq 2$, and $\beta < (2\alpha - 3)/2$.

Remark 2. In particular, when $\alpha = 2$ and $0 \leq \beta < 1/2$, we can take $0 < \delta < \frac{1-2\beta}{4}$, and obtain

$$P(X \in C((0,\infty) : H_\beta)) = 1.$$

Thus $X(t)$ is smoother than an $L^2([0,1])$ function for $t > 0$. This is due to the regularization property of the Laplacian.

Remark 3. We are going to prove only part (a) of Theorem 1. The remaining results, including Theorem 2, are proved in [14].

3. Proof of Theorem 1(a)

Let us write $x_r^N(t) = X^N(t,r)$. The system (5) then can be written in the more compact form

$$dx_i^N(t) = \left(\sum_{j=1}^{N} a_{ij}^N x_j^N(t) + b_{ij}^N x_j^N(t)^2 \right) dt + \sqrt{N x_i^N(t) \left(1 - x_i^N(t)\right)}\, dB_i(t) \tag{6}$$

where

$$b_{ij}^N = \begin{cases} N & \text{if } \quad j = i+1, \\ -N & \text{if } \quad j = i, \\ 0 & \qquad \text{otherwise,} \end{cases}$$

and a_{ij}^N are the coefficients of $\Delta_{N,\alpha}$.

For example, in the case $\alpha = 2$, (5) takes the form:

$$\begin{aligned} \frac{\partial}{\partial t} X^N\left(t, \frac{k}{N}\right) &= \Delta_N X^N\left(t, \frac{k}{N}\right) + \nabla_N \left(X^N\left(t, \frac{k}{N}\right)^2 \right) \\ &+ \sqrt{N X^N\left(t, \frac{k}{N}\right) \left(1 - X^N\left(t, \frac{k}{N}\right)\right)}\, dB_k(t), \end{aligned} \tag{7}$$

$1 \leq k \leq N$, $t \geq 0$.

Here $\{B_k(t)\}_{1 \leq k \leq N}$ is an infinite system of independent one-dimensional Brownian motions and ∇_N and Δ_N are, respectively, the discrete approximations of the first and second derivative with respect to the variable x:

$$\Delta_N X^N\left(t, \frac{k}{N}\right) = \frac{X^N\left(t, \frac{k+1}{N}\right) - 2X^N\left(t, \frac{k}{N}\right) + X^N\left(t, \frac{k-1}{N}\right)}{\frac{1}{N^2}},$$

$$\nabla_N\, h\left(s, \frac{k}{N}\right) = \frac{h\left(s, \frac{k+1}{N}\right) - h\left(s, \frac{k}{N}\right)}{\frac{1}{N}}, \quad 1 \leq k \leq N.$$

Substituting the above expressions in equation (7), we obtain the finite-dimensional system of stochastic differential equations

$$\begin{aligned} dx_i^N(t) &= N^2\left[x_{i+1}^N(t) - 2x_i^N(t) + x_{i-1}^N(t)\right] + N x_{i+1}^N(t)^2 - N x_i^N(t)^2 \\ &+ \sqrt{N x_i^N(t)\left(1 - x_i^N(t)\right)}\, dB_i(t),\ i = 1, ..., N, \end{aligned}$$

which can be written as (6) with with

$$a_{ij}^N = \begin{cases} N^2 & \text{if} \quad j = i+1, i-1, \\ -2N^2 & \text{if} \quad j = i, \\ 0 & \quad\ \ \text{otherwise} \end{cases}$$

and

$$b_{ij}^N = \begin{cases} N & \text{if} \quad j = i+1, \\ -N & \text{if} \quad j = i, \\ 0 & \quad\ \ \text{otherwise.} \end{cases}$$

In the general case $0 < \alpha \leq 2$ we also will have $\sum_i a_{ij}^N = 0$ because the operator $\Delta_{N,\alpha}$ is symmetric.

Note that for system (6) we cannot apply standard results on existence and uniqueness of solution because Lipschitz assumptions on the drift and diffusion coefficients fail. We prove the following result.

Lemma 1. *For any initial random condition* $X^N(0) = (x_1^N, ..., x_N^N) \in [0,1]^N$, *the system*

$$\text{(8)} \qquad \begin{aligned} dx_i^N(t) &= \left(\sum_j a_{ij}^N x_j^N(t) + \sum_j b_{ij}^N x_j^N(t)^2 \right) dt + \\ &\qquad + \sqrt{N x_i^N(t)(1 - x_i^N(t))}\, dB_i(t) \\ x_i^N(0) &= x_i,\ i = 1, ..., N, \end{aligned}$$

admits a unique strong solution $X^N(t) = (x_1^N(t), \ldots, x_N^N(t)) \in C([0,\infty), [0,1]^N)$.

Proof. Let us consider the re-scaled system

$$\text{(9)} \qquad \begin{aligned} dx_i^N(t) &= \left(\sum_j a_{ij}^N x_j^N(t) + \sum_j b_{ij}^N x_j^N(t)^2 \right) dt + \sqrt{g(x_i^N(t))}\, dB_i(t) \\ x_i^N(0) &= x_i,\ i = 1, ..., N, \end{aligned}$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = Nx(1-x)$ for $0 \leq x \leq 1$, and $g(x) = 0$ otherwise. Since the coefficients of (9) are continuous, by Skorohod's existence theorem ([19, 12]) we conclude that on some probability space there exists a weak solution $X^N(t)$ of (9). We will prove that each weak solution $X^N(t) = (x_1^N(t), \ldots, x_N^N(t))$ of this system is bounded: $x_i^N(t) \in [0,1]$ for all $i = 1, \ldots, N$ and $t \geq 0$, thus showing that $X^N(t)$ also solves (8).

First we show that $x_i^N(t) \geq 0$ for each $i = 1 \ldots, N$. Since the coefficients of the system are non-Lipschitz, the solution may explode in finite time. Let $\tau_1 \leq \infty$ denote the explosion time of the solution. If some of the solution coordinates are negative, then there exists a random time $0 < \tau_2 \leq \infty$ such that for $0 < t \leq \tau_2$ all such coordinates are between -1 and 0. This is so because there is only finite number of coordinates, and they are continuous.

In order to obtain pathwise uniqueness of weak solutions we shall use the local time techniques of Le Gall combined by the classical method of Ikeda and Watanabe (see e.g. [18], Chapter V, §43) We state the following result of Le Gall ([16]).

Lemma 2. *Let $Z \equiv \{Z(t),\ t \geq 0\}$ be a real-valued semimartingale. Suppose that there exists a function $\rho : [0, \infty) \to [0, \infty)$ such that $\int_0^\varepsilon \frac{du}{\rho(u)} = +\infty$ for all $\varepsilon > 0$, and $\int_0^t \frac{1_{\{Z_s>0\}}}{\rho(Z_s)}\, d\langle Z\rangle_s < \infty$ for all $t > 0$ a.s. Then the local time at zero of Z, $L_t^0(Z)$, is identically zero for all t a.s.*

Applying Lemma 2 to $x_i^N(t)$ with $\rho(u) = u$, and using the Tanaka's formula (see [17]), after summation we obtain for $x_i^N(t)_- := \max[0, -x_i^N(t)]$,

$$
\begin{aligned}
\sum_{i=1}^N x_i^N(t)_- &= -\int_0^t \sum_{i=1}^N 1_{x_i^N(s)<0} \sum_{j=1}^N (a_{ij}^N x_j(s) + b_{ij}^N x_j(s)^2)\, ds \\
&\leq \int_0^t \sum_{i,j=1}^N 1_{x_i^N(s)<0} a_{ij}^N x_j(s)_-\, ds + N \int_0^t \sum_{i=1}^N 1_{x_i^N(s)<0} x_i(s)^2\, ds \\
&\leq \int_0^t \sum_{i,j=1}^N a_{ij}^N x_j(s)_-\, ds + N \int_0^t \sum_{i=1}^N x_i^N(s)_-\, ds \\
&= N \int_0^t \sum_{i=1}^N x_i^N(s)_-\, ds,
\end{aligned}
$$

where we used that $\sum_i a_{ij}^N = 0$ to obtain the last equality. Then by Gronwall's lemma we obtain that $\sum_{i=1}^N x_i^N(t)_- = 0$, and hence that the solution is non-negative for each $t \geq 0$. By a similar argument applied to $(1 - x_i^N(t))_-$, it follows that $x_i^N(t) \leq 1$ for each $1 \leq i \leq N$.

Let $X^{1,N} = (x_1^{1,N}, \ldots, x_N^{1,N})$ and $X^{2,N} = (x_1^{2,N}, \ldots, x_N^{2,N})$ be two weak solutions of (8) with the same initial conditions and the same Brownian motions.

Then,

$$
\begin{aligned}
&x_i^{1,N}(t) - x_i^{2,N}(t) = \\
&\quad = \int_0^t \left[\sum_j a_{ij}^N \left(x_j^{1,N}(s) - x_j^{2,N}(s) \right) + b_{ij}^N \left(x_j^{1,N}(s)^2 - x_j^{2,N}(s)^2 \right) \right] ds + \\
&\qquad + \int_0^t \left[\sqrt{N x_i^{1,N}(s)(1 - x_i^{1,N}(s))} - \sqrt{N x_i^{2,N}(s)(1 - x_i^{2,N})(s)} \right] dB_i(s), \\
&i = 1, \dots, N.
\end{aligned}
$$

Since

$$
\langle X \rangle_t = \int_0^t \left[\sqrt{N x_i^{1,N}(s)(1 - x_i^{1,N}(s))} - \sqrt{N x_i^{2,N}(s)(1 - x_i^{2,N})(s)} \right]^2 ds
$$

and

$$
\begin{aligned}
&\int_0^t \frac{\left[\sqrt{N x_i^{1,N}(s)(1 - x_i^{1,N}(s))} - \sqrt{N x_i^{2,N}(s)(1 - x_i^{2,N})(s)} \right]^2}{x_i^{1,N}(s) - x_i^{2,N}(s)} \times \\
&\qquad \times 1_{x_i^{1,N}(s) - x_i^{2,N}(s) > 0} \, ds \le \int_0^t 2N 1_{x_i^{1,N}(s) - x_i^{2,N}(s) > 0} \, ds \ < \ 2Nt
\end{aligned}
$$

(where we used that $(\sqrt{x(1-x)} - \sqrt{y(1-y)})/(x-y) < 2$ for $x, y \in [0,1]$, $x > y$, which follows from L'Hospital rule), we can apply Lemma 3.2 to $Z(t) = x_i^{1,N}(t) - x_i^{2,N}(t)$ with $\rho(x) = x$. Therefore, $L_t^0 \left(x_i^{1,N}(s) - x_i^{2,N}(s) \right) = 0$ for all $i \in \{1, \dots, N\}$.

Applying Tanaka's formula again,

$$
\begin{aligned}
\left| x_i^{1,N}(t) - x_i^{2,N}(t) \right| &= \int_0^t \operatorname{sgn} \left(x_i^{1,N}(s) - x_i^{2,N}(s) \right) \times \\
&\quad \times \left[\sum_j a_{ij}^N \left(x_j^{1,N}(s) - x_j^{2,N}(s) \right) + b_{ij}^N \left(x_j^{1,N}(s)^2 - x_j^{2,N}(s)^2 \right) \right] ds + \\
&\quad + \int_0^t \operatorname{sgn} \left(x_i^{1,N}(s) - x_i^{2,N}(s) \right) \left[\sqrt{N x_i^{1,N}(s) \left(1 - x_i^{1,N}(s) \right)} - \right. \\
&\quad \left. - \sqrt{N x_i^{2,N}(s) \left(1 - x_i^{2,N}(s) \right)} \right] \cdot dB_i(s), \quad i = 1, \dots, N.
\end{aligned}
$$

Since a_{ij}^N and b_{ij}^N are bounded, it follows that

$$\begin{aligned}
&\mathbb{E}\sum_{i=1}^{N}\left|x_i^{1,N}(t)-x_i^{2,N}(t)\right| \\
&\leq \int_0^t \mathbb{E}\sum_{i=1}^{N}\left|\sum_j a_{ij}^N\left(x_j^{1,N}(s)-x_j^{2,N}(s)\right)+b_{ij}^N\left(x_j^{1,N}(s)^2-x_j^{2,N}(s)^2\right)\right| ds \\
&\leq \int_0^t K(N)\,\mathbb{E}\sum_{i=1}^{N}\left|x_i^{1,N}(s)-x_i^{2,N}(s)\right| ds,
\end{aligned}$$

where $K(N)$ is a constant depending on N. From Gronwall's inequality we conclude that

$$\mathbb{E}\sum_{i=1}^{d}\left|x_i^{1,N}(t)-x_i^{2,N}(t)\right| = 0$$

for all $t \geq 0$, thus proving pathwise uniqueness.

By a classical theorem of Yamada and Watanabe [22], this is sufficient for existence of a unique strong solution of (8).

Acknowledgement The author wishes to thank the organizers of the Sozopol Probability School 2003 for their hospitality and for giving her the opportunity to participate.

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