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ПЛИСКА
БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

ON THE STRONG ASYMPTOTIC BEHAVIOR OF ASYMPTOTICALLY NONEXPANSIVE SEMI-GROUPS IN BANACH SPACES

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Dedicated to the late Y. A. Tagamlitzki

1. Introduction. Throughout this paper X denotes a uniformly convex Banach space and C is a nonempty closed subset of X. Let J be an unbounded subset of $[0, \infty)$ satisfying

(1.1) $$t+s\in J \text{ for every } t,\ s\in J$$

and

(1.2) $$t-s\in J \text{ for every } t,\ s\in J \text{ with } t>s.$$

A family $\{T(t):\ t\in J\}$ of $T(t):\ C\to C$ is called an asymptotically nonexpansive semi-group on C, if

1.3) $$T(t+s)=T(t)T(s) \text{ for every } t,\ s\in J$$

and there exists a function $a(\cdot):\ J\to[0, \infty)$ with $\lim_{t\to\infty}a(t)=1$ such that

(1.4) $$\| T(t)x-T(t)y\|\leqq a(t)\|x-y\| \text{ for every } x,\ y\in C \text{ and } t\in J.$$

In particular, if $a(t)\equiv 1$, then $\{T(t):\ t\in J\}$ is called a nonexpansive semi-group on C. The set of fixed points of $\{T(t):\ t\in J\}$ will be denoted by F i. e. $F=\{x\in C:\ T(t)x=x \text{ for every } t\in J\}$.

The purpose of this note is to investigate the strong convergence of $T(t)x$ $(x\in C)$ as $t\to\infty$. The main result is stated as follows.

Theorem 1. *Let $\{T(t):\ t\in J\}$ be an asymptotically nonexpansive semi-group on C and let $x\in C$. Then the conditions (i) and (ii) are equivalent:*

(*i*) *$T(t)x, t\in J$, converges strongly as $t\to\infty$.*

(*ii*) $\lim_{t\to\infty}\|T(t+h)x-T(t)x\|=0$ *for every $h\in J$ and there exists a $x_0\in F$ such that*

(1.5) $$\liminf_{t,s\to\infty}(\liminf_{h\to\infty}\|T(t+h)x+T(s+h)x-2x_0\|)$$
$$\leqq\liminf_{t,s\to\infty}\|T(t)\,x+T(s)x-2x_0\|.$$

As a direct consequence of Theorem 1 we obtain the following theorem which has been proved by Taniguti, Takahashi and Abe [3].

Theorem 2. *Let $\{T(t):\ t\in J\}$ be an asymptotically nonexpansive semi-group on C and let $x\in C$. Suppose that*

(a_1) *there exist $x_0\in F$, $\varphi\in C_{11}[0, \infty)$, $\psi\in C[0, \infty)$ and a nonnegative function b defined on J with $\lim_{h\to\infty}b(h)=1$ such that*

$$\varphi(\| T(h)u+T(h)v-2x_0\|)\leq\varphi(b(h)\|u+v-2x_0\|)$$
$$+[\psi(b(h)\|u-x_0\|)-\psi(\|T(h)u-x_0\|)+\psi(b(h)\|v-x_0\|)-\psi(\|T(h)v-x_0\|)]$$

for every $u, v \in \{T(t)x: t \in J\}$ *and* $h \in J$, *where* $C_{11}[0, \infty)$ *denotes the set of increasing continuous functions defined on* $[0, \infty)$, *and*

(a_2) $\lim_{t\to\infty} \| T(t+h)x - T(t)x \| = 0$ for every $h \in J$.

Then $T(t)x$, $t \in J$, *converges strongly as* $t \to \infty$ (*to an element of* F).

Remarks. 1) Suppose that $T: C \to C$ is nonexpansive, $0 \in C$, $T0 = 0$ and there is a $c \geq 0$ such that

$$(1.6) \qquad \| Tu + Tv \|^2 \leq \| u+v \|^2 + c(\| u \|^2 - \| Tu \|^2 + \| v \| - \| Tv \|^2)$$

for every $u, v \in C$. Then, the nonexpansive semi-group $\{T^n: n=1, 2\dots\}$ satisfies condition (a_1) in Theorem 2 with $x_0 = 0$, $\varphi(t) = t^2$, $\psi(t) = ct^2$ and $b(n) \equiv 1$. Condition (1.6) has been considered in [2], and note that (1.6) with $c=0$ is satisfied, if T is odd, i. e. $C = -C$ and $T(-u) = -Tu$ for $u \in C$.

2) See [3] for applications of Theorem 2.

2. Proofs of the Theorems. We start with the

Proof of Theorem 1. First, suppose that (*i*) holds. Let $\lim_{t\to\infty} T(t)x = u$. By $\| T(t+s)x - T(s)u \| \leq a(s) \| T(t)x - u \|$, we obtain that $T(s)u = u$ for every $s \in J$, i. e. $u \in F$ so that $F \neq \emptyset$. Clearly, $\lim_{t\to\infty} \| T(t+h)x - T(t)x \| = 0$ for every $h \in J$ and (1.5) is satisfied for every $x_0 \in F$.

Next, suppose that (*ii*) holds. We note that $\| T(t)x - x_0 \|$ is convergent as $t \to \infty$. In fact, $\| T(t+h)x - x_0 \| = \| T(h)T(t)x - T(h)x_0 \| \leq a(h) \| T(t)x - x_0 \|$. Letting $h \to \infty$, we have

$$\limsup_{t\to\infty} \| T(t)x - x_0 \| = \limsup_{h\to\infty} \| T(t+h)x - x_0 \| \leq \| T(t)x - x_0 \|$$

for every $t \in J$ and hence, $\limsup_{t\to\infty} \| T(t)x - x_0 \| \leq \liminf_{t\to\infty} \| T(t)x - x_0 \|$.

Set $d = \lim_{t\to\infty} \| T(t)x - x_0 \|$. Now

$$(2.1) \quad 2\| T(t+h)x - x_0 \| \leq \| T(t+h)x - T(s+h)x \| + \| T(t+h)x + T(s+h)x - 2x_0 \|$$

for every t, s and h in J. Since

$$\lim_{h\to\infty} \| T(t+h)x - T(s+h)x \| = 0 \text{ for every } t, s \in J$$

(by the assumption $\lim_{t\to\infty} \| T(t+h)x - T(t)x \| = 0$ for $h \in J$), by taking the lim inf as $h \to \infty$ in (2.1) we obtain

$$2d \leq \liminf_{h\to\infty} \| T(t+h)x + T(s+h)x - 2x_0 \| \text{ for every } t, s \in J.$$

Hence, by (1.5)

$$2d \leq \liminf_{t,s\to\infty} \| T(t)x + T(s)x - 2x_0 \| \leq \limsup_{t,s\to\infty} \| T(t)x + T(s)x - 2x_0 \|$$
$$\leq \lim_{t,s\to\infty} [\| T(t)x - x_0 \| + \| T(s)x - x_0 \|] = 2d.$$

Consequently,

$$\lim_{t,s\to\infty} \| (T(t)x - x_0) + (T(s)x - x_0) \| = 2d.$$

So, by uniform convexity of X and $\lim_{t\to\infty} \| T(t)x - x_0 \| = d$, we have

$$\lim_{t,s\to\infty} \| T(t)x - T(s)x \| = \lim_{t,s\to\infty} \| (T(t)x - x_0) - (T(s)x - x_0) \| = 0.$$

Hence, $T(t)x$ converges strongly as $t \to \infty$. This completes the proof.

Proof of Theorem 2. It is sufficient to show that (1.5) is satisfied. It follows from (a_1) that

$$\varphi(\| T(t+h)x + T(s+h)x - 2x_0 \|) \leq \varphi(b(h) \| T(t)x + T(s)x - 2x_0 \|)$$

$$+[\psi(b(h)\|T(t)x-x_0\|)-\psi(\|T(t+h)x-x_0\|)+\psi(b(h)\|T(s)x-x_0\|)$$
$$-\psi(\|T(s+h)x-x_0\|)]$$

for every t, s and h in J. As shown in the proof of Theorem 1, $\|T(t)x-x_0\|$ is convergent as $t\to\infty$. Set $\lim_{t\to\infty}\|T(t)x-x_0\|=d$. Letting $h\to\infty$ in the inequality above, we have

$$\text{(2.2)}\quad \varphi(\liminf_{h\to\infty}\|T(t+h)x+T(s+h)x-2x_0\|)\leqq\varphi(\|T(t)x+T(s)x-2x_0\|)$$
$$+[\psi(\|T(t)x-x_0\|)-\psi(d)+\psi(\|T(s)x-x_0\|)-\psi(d)]$$

for every t, $s\in J$. Now, by taking the lim inf as t, $s\to\infty$ in (2.2), we obtain

$$\varphi(\liminf_{t,s\to\infty}(\liminf_{h\to\infty}\|T(t+h)x+T(s+h)x-2x_0\|))$$
$$\leqq\liminf_{t,s\to\infty}\varphi(\|T(t)x+T(s)x-2x_0\|)=\varphi(\liminf_{t,s\to\infty}\|T(t)x+T(s)x-2x_0\|).$$

Therefore,

$$\liminf_{t,s\to\infty}(\liminf_{h\to\infty}\|T(t+h)x+T(s+h)x-2x_0\|)$$
$$\leqq\liminf_{t,s\to\infty}\|T(t)x+T(s)x-2x_0\|,$$

. e. (1.5) is satisfied. This completes the proof.

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