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## ON THE STRONG ASYMPTOTIC BEHAVIOR OF ASYMPTOTICALLY NONEXPANSIVE SEMI-GROUPS IN BANACH SPACES

## ISAO MIYADERA

Dedicated to the late Y. A. Tagamlitzki

1. Introduction. Throughout this paper X denotes a uniformly convex Banach space and C is a nonempty closed subset of X. Let  $\mathcal{F}$  be an unbounded subset of  $[0, \infty)$  satisfying

$$(1.1) t+s \in J for every t, s \in J$$

a nd

$$(1.2) t-s \in J for every t, s \in J with t>s.$$

A family  $\{T(t): t \in J\}$  of  $T(t): C \rightarrow C$  is called an asymptotically nonexpansive semi-group on C, if

1.3) 
$$T(t+s) = T(t)T(s) \text{ for every } t, s \in J$$

and there exists a function  $a(\cdot)$ :  $J \rightarrow [0, \infty)$  with  $\lim_{t \to \infty} a(t) = 1$  such that

(1.4) 
$$||T(t)x-T(t)y|| \le a(t) ||x-y||$$
 for every x, y \(\inC\) and t \(\inJ\).

In particular, if a(t) = 1, then  $\{T(t): t \in J\}$  is called a nonexpansive semigroup on C. The set of fixed points of  $\{T(t): t \in J\}$  will be denoted by F i. e.  $F = \{x \in C: T(t)x = x \text{ for every } t \in J\}$ .

The purpose of this note is to investigate the strong convergence of

T(t)x  $(x \in C)$  as  $t \to \infty$ . The main result is stated as follows.

Theorem 1. Let  $\{T(t): t \in J\}$  be an asymptotically nonexpansive semigroup on C and let  $x \in C$ . Then the conditions (i) and (ii) are equivalent:

(i)  $T(t)x, t \in J$ , converges strongly as  $t \to \infty$ . (ii)  $\lim_{t \to \infty} ||T(t+h)x - T(t)x|| = 0$  for every  $h \in J$  and there exists a  $x_0 \in F$  such that

(1.5) 
$$\liminf_{t,s\to\infty} ||T(t+h)x + T(s+h)x - 2x_0||)$$

$$\leq \liminf_{t,s\to\infty} ||T(t)x + T(s)x - 2x_0||.$$

As a direct consequence of Theorem 1 we obtain the following theorem which has been proved by Taniguti, Takahashi and Abe [3].

Theorem 2. Let  $\{T(t): t \in J\}$  be an asymptotically nonexpansive semi-

group on C and let x (C. Suppose that

(a<sub>1</sub>) there exist  $x_0 \in F$ ,  $\phi \in C_{11}[0, \infty)$ ,  $\psi \in C[0, \infty)$  and a nonnegative function b defined on J with  $\lim_{h\to\infty} b(h) = 1$  such that

$$\varphi(||T(h)u + T(h)v - 2x_0||) \le \varphi(b(h)||u + v - 2x_0||)$$

$$+\left[\psi(b(h)\parallel u-x_0\parallel)-\psi(\parallel T(h)u-x_0\parallel)+\psi(b(h)\parallel v-x_0\parallel)-\psi(\parallel T(h)v-x_0\parallel)\right]$$

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tor every  $u, v \in \{T(t)x : t \in J\}$  and  $h \in J$ , where  $C_{11}[0, \infty)$  denotes the set of increasing continuous functions defined on  $[0, \infty)$ , and

$$(a_2)$$
  $\lim_{t\to\infty} ||T(t+h)x-T(t)x||=0$  for every  $h \in J$ .

Then T(t)x,  $t \in J$ , converges strongly as  $t \to \infty$  (to an element of F). Remarks. 1) Suppose that  $T: C \to C$  is nonexpansive,  $0 \in C$ , T0 = 0 and there is a  $c \ge 0$  such that

$$(1.6) || Tu + Tv ||^{2} \le || u + v ||^{2} + c(|| u ||^{2} - || Tu ||^{2} + || v || - || Tv ||^{2})$$

for every  $u, v \in C$ . Then, the nonexpansive semi-group  $\{T^n : n=1, 2...\}$  satisfies condition  $(a_1)$  in Theorem 2 with  $x_0 = 0$ ,  $\varphi(t) = t^3$ ,  $\psi(t) = ct^2$  and  $\varphi(t) = 1$ . Condition (1.6) has been considered in [2], and note that (1.6) with  $\varphi(t) = 0$  is considered in [2]. satisfied, if T is odd, i. e. C = -C and T(-u) = -Tu for  $u \in C$ .

2) See [3] for applications of Theorem 2.

2. Proofs of the Theorems. We start with the

Proof of Theorem 1. First, suppose that (i) holds. Let  $\lim_{t\to\infty} T(t)x = u$ . By  $||T(t+s)x-T(s)u|| \le a(s) ||T(t)x-u||$ , we obtain that T(s)u=u for every  $s \in J$ , i. e.  $u \in F$  so that  $F \neq \emptyset$ . Clearly,  $\lim_{t \to \infty} ||T(t+h)x-T(t)x|| = 0$  for every

h ( J and (1.5) is satisfied for every  $x_0 \in F$ . Next, suppose that (ii) holds. We note that  $|T(t)x-x_0||$  is convergent as  $t \to \infty$ . In fact,  $||T(t+h)x-x_0|| = ||T(h)T(t)x-T(h)x_0|| \le a(h) ||T(t)x-x_0||$ . Letting  $h \to \infty$ , we have

$$\lim \sup_{t\to\infty} ||T(t)x-x_0|| = \lim \sup_{t\to\infty} ||T(t+t)x-x_0|| \le ||T(t)x-x_0||$$

for every  $t \in J$  and hence,  $\limsup_{t\to\infty} ||T(t)x-x_0|| \le \liminf_{t\to\infty} ||T(t)x-x_0||$ . Set  $d = \lim_{t \to \infty} || T(t)x - x_0 ||$ . Now

 $(2.1) 2 || T(t+h)x - x_0 || \le || T(t+h)x - T(s+h)x || + || T(t+h)x + T(s+h)x - 2x_0 ||$ for every t, s and h in J. Since

$$\lim_{h\to\infty} ||T(t+h)x-T(s+h)x||=0$$
 for every  $t, s\in J$ 

(by the assumption  $\lim_{t\to\infty} ||T(t+h)x-T(t)x||=0$  for  $h\in J$ ), by taking the  $\lim\inf_{t\to\infty} ||T(t+h)x-T(t)x||=0$ as  $h \rightarrow \infty$  in (2.1) we obtain

$$2d \le \lim \inf_{h \to \infty} ||T(t+h)x + T(s+h)x - 2x_0||$$
 for every  $t, s \in J$ .

Hence, by (1.5)

$$2d \leq \lim \inf_{t,s\to\infty} || T(t)x + T(s)x - 2x_0 || \leq \lim \sup_{t,s\to\infty} || T(t)x + T(s)x - 2x_0 || \leq \lim_{t,s\to\infty} [|| T(t)x - x_0 || + || T(s)x - x_0 ||] = 2d.$$

Consequently,

$$\lim_{t,s\to\infty} || (T(t)x-x_0)+(T(s)x-x_0)|| = 2d.$$

So, by uniform convexity of X and  $\lim_{t\to\infty} ||T(t)x-x_0||=d$ , we have

$$\lim_{t,s\to\infty} || T(t)x - T(s) x || = \lim_{t,s\to\infty} || (T(t)x - x_0) - (T(s)x - x_0) || = 0.$$

Hence, T(t)x converges strongly as  $t\rightarrow\infty$ . This completes the proof.

Proof of Theorem 2. It is sufficient to show that (1.5) is satisfied. It follows from  $(a_1)$  that

$$||\phi(||T(t+h)x+T(s+h)x-2x_0||) \le ||\phi(b(h)||T(t)x+T(s)x-2x_0||)$$

+ 
$$[\psi(b(h) || T(t)x - x_0 ||) - \psi(|| T(t+h)x - x_0 ||) + \psi(b(h) || T(s)x - x_0 ||) - \psi(|| T(s+h)x - x_0 ||)]$$

for every t, s and h in J. As shown in the proof of Theorem  $1, ||T(t)x-x_0||$  is convergent as  $t\to\infty$ . Set  $\lim_{t\to\infty} ||T(t)x-x_0||=d$ . Letting  $h\to\infty$  in the inequality above, we have

(2.2) 
$$\varphi(\liminf_{h\to\infty} || T(t+h)x + T(s+h)x - 2x_0 ||) \le \varphi(|| T(t)x + T(s)x - 2x_0 ||) + [\psi(|| T(t)x - x_0 ||) - \psi(d) + \psi(|| T(s)x - x_0 ||) - \psi(d)]$$

for every t,  $s \in J$ . Now, by taking the  $\lim \inf as t$ ,  $s \to \infty$  in (2.2), we obtain  $\varphi(\lim \inf_{t,s\to\infty} (\lim \inf_{h\to\infty} || T(t+h)x + T(s+h)x - 2x_0 ||))$ 

$$\leq \lim \inf_{t,s\to\infty} \varphi(\|T(t)x+T(s)x-2x_0\|) = \varphi(\lim \inf_{t,s\to\infty} \|T(t)x+T(s)x-2x_0\|).$$

Therefore,

$$\lim \inf_{t,s\to\infty} (\lim \inf_{h\to\infty} || T(t+h)x + T(s+h)x - 2x_0 || )$$

$$\leq \lim \inf_{t,s\to\infty} || T(t)x + T(s)x - 2x_0 ||,$$

. e. (1.5) is satisfied. This completes the proof.

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