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# ON GENERALIZED ORLICZ SEQUENCE SPACES OF FOURIER COEFFICIENTS FOR TRIGONOMETRIC GAP SERIES. I 

J. MUSIELAK

To the memory of
Y. A. Tagamlitzki


#### Abstract

We investigate the operator associating with a function $f \in L_{2 \pi}^{p}, 1<p \leq 2$, the sequence of Fourier cocfficients of $f$ with respect to a trigonometric gap system, as well as an operator from a modular space $X_{\rho_{s}}{ }_{(\varphi)}$ to the generalized Orlicz sequence space $l^{\varphi}$.


1. Let $\left(n_{k}\right)$ be an increasing sequence of positive integers. We take an increasing function $l(x), x \geqq 0$ such that $l(k)=n_{k}$ for $k=1,2, \ldots$, and we denote by $m(x)$ the inverse function of $l$. We write $A_{v}=\left\{k \in N: 2^{v-1} \pi \leq n_{k}<2^{v} \pi\right\}$, $v=1,2,3, \ldots$, and we put $k_{0}=[m(\pi)]+1$, where $[x]$ denotes the integer part of $x$. Then, $n_{k_{0}}$ is the least integer in $A_{1}$. Let $\left|A_{v}\right|$ be the number of elements of $A_{v}$; then, $\left|A_{v}\right|<\left[m\left(2^{v} \pi\right)-m\left(2^{v-1} \pi\right)\right]+1=N_{v}$ for $v \in N$.

Let

$$
\sum_{k=1}^{\infty}\left(a_{k}(f) \cos n_{k} x+b_{k}(f) \sin n_{k} x\right)
$$

be the Fourier series of a function $f \in L_{2 \pi}^{p}, 1<p \leq 2$, with respect to the trigonometric gap system $\cos n_{1} x, \sin n_{1} x, \cos n_{2} x, \sin n_{2} x, \ldots$ in $\langle 0,2 \pi\rangle$. With every $f \in L_{2 \pi}^{p}$ we associate the sequence $c(f)=a_{k_{0}}(f), b_{k_{0}}(f), a_{k_{0}+1}(f), b_{k_{0}+1}(f), \ldots$ with some fixed index $k_{0}$. We shall investigate the linear operator $c: f \rightarrow c(f)$ as an operator from some modular space $X_{\rho_{\varphi}}^{(s)}$ to a generalized Orlicz sequence space $l^{\varphi}$, generated by a sequence $\varphi=\left(\varphi_{n}\right)_{n=1}^{\infty}$ of $\varphi$-functions $\varphi_{n}$ (for the tet. minology, see [2]), i. e. the space of sequences $c=\left(c_{k}\right)_{k=k_{0}}^{\infty}$ such that $\rho(\lambda c)$ $=\Sigma_{n} \varphi_{n}\left(\lambda\left|c_{n}\right|\right)<\infty$ for a $\lambda>0$.

The following assumptions on the sequence $\varphi$ will be fundamental.
A.1. There exists a constant $C \geqq 1$ and a sequence of integers $(m(v))$ with $m(v) \in A_{v}$ such that $\varphi_{v}(u) \leqq C \varphi_{m(v)}(u)$ for $u \geq 0$ and $v \in A_{v}$;
A.2. The functions $\overline{\varphi_{n}}(u)=\varphi_{n}\left(u^{1 / q}\right), u \geq 0$, where $1 / p+1 / q=1$, are concave-

Let us remark that A. 1 is certainly satisfied, if $\left(\varphi_{n}(u)\right)_{n=1}^{\infty}$ is an increasing (decreasing) sequence for all $u \geq 0$. Moreover, it is easily observed that if $\varphi$ satisfies A.2, then

$$
\begin{equation*}
\varphi_{n}(2 u) \leq 2^{1 / q} \varphi_{n}(u) \text { for } u \geq 0, n \in N . \tag{*}
\end{equation*}
$$

In the following, we denote by $\omega_{p}$ the $p$-th modulus of continuity of $f$ in $L_{2 \pi}^{p}$, i. e.

$$
\omega_{p}(f, \delta)=\sup _{|h| \leq \delta}\left(\int_{0}^{2 \pi}|f(x+h)-f(x)|^{p} d x\right)^{1 / p}
$$

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2. We prove now the following:

Theorem 1. Let $\varphi=\left(\varphi_{n}\right)_{n=1}^{\infty}$, satisfy A.1 and A.2. Then, for every $f \in L_{2 \pi}^{p}, 1<p \leq 2$, there holds the inequality

$$
\rho(c(f)) \leq \sum_{k=1}^{\infty} \rho_{k}^{(\varphi)}(f)=\rho_{s}^{(\varphi)}(f),
$$

where

$$
\rho_{v}^{(\varphi)}(f)=2 C N_{v} \varphi_{m(v)}\left\{N^{-1 / q} \omega_{p}\left(\frac{1}{4} f, \frac{1}{2^{v}}\right)\right\}
$$

or $v \in N$, with $1 / p+1 / q=1$.
Proof. Applying the Hausdorff-Young inequality to the function $F_{h}(x)=f(x+h)-f(x-h)$ and taking into account the formulae

$$
a_{k}\left(F_{h}\right)=2 b_{k}(f) \sin n_{k} h, \quad b_{k}\left(F_{h}\right)=-2 a_{k}(f) \sin n_{k} h,
$$

we obtain the inequality

$$
\left\{\sum_{k=1}^{\infty}\left(\left|a_{k}(f)\right|^{q}+\left|b_{k}(f)\right|^{q}\right)\left|\sin n_{k} h\right|^{q\}^{1 / q}} \leq \frac{1}{2}\left\{\frac{1}{\pi} \int_{0}^{2 \pi}\left|F_{h}(x)\right|^{p} d x\right\}^{1 / p} .\right.
$$

Restricting the summation on the left-hand side to $k \in A_{v}$ and observing that $\left|\sin n_{k} 2^{-v-1}\right| \geqq 2^{-1 / 2}$ for $k \in A_{v}$, we obtain

$$
\begin{gather*}
\left\{\underset{k \in A_{v}}{\sum}\left(\left|a_{k}(f)\right|^{q}+\left|b_{k}(f)\right|^{q}\right)\right\}^{1 / q^{\prime}}  \tag{**}\\
\leq \frac{1}{\sqrt{2}}\left\{\frac{1}{\pi} \int_{0}^{2 \pi}\left|F_{2}^{-v-1}(x)\right|^{p} d x\right\}^{1 / p} \leq \frac{1}{\sqrt{2}} \frac{1}{\pi^{1 / p}} \omega_{p}\left(f, \frac{1}{2^{v}}\right) .
\end{gather*}
$$

Now, we have by Jensen's inequality for concave functions

$$
\begin{gathered}
\sum_{k \in A_{v}}^{\sum}\left(\varphi_{k}\left(\left|a_{k}(f)\right|\right)+\varphi_{k}\left(\left|b_{k}(f)\right|\right)\right) \\
\leq C_{k \in A_{v}}^{\Sigma}\left(\bar{\varphi}_{m(k)}\left(\left|a_{k}(f)\right|\right)+\bar{\varphi}_{m(k)}\left(\left|b_{k}(f)\right|\right)\right) \\
\leq 2 C\left|A_{v}\right| \bar{\varphi}_{m(v)}\left\{\frac{1}{2\left|A_{v}\right|} \sum_{k \in A_{v}}\left(\left|a_{k}(f)\right|^{\eta}+\left|b_{k}(f)\right|^{q}\right)\right\} \\
\leq 2 C\left|A_{v}\right| \bar{\varphi}_{m(v)}\left\{\frac{1}{2\left|A_{v}\right|} \frac{1}{\sqrt{2^{q}}} \frac{1}{\pi^{q / p}} \omega_{o}^{q}\left(f, \frac{1}{2^{v}}\right)\right\} \\
\leq 2 C\left|\dot{A}_{v}\right| \bar{\varphi}_{m(v)}\left\{\frac{1}{\left|A_{v}\right|} \omega_{p}^{q}\left(\frac{1}{4} f, \frac{1}{2^{v}}\right)\right\} .
\end{gathered}
$$

Since $\bar{\varphi}_{m(v)}$ are concave, then $\bar{\varphi}_{m(v)}(u) / u$ are nonincreasing. Hence,

$$
\underset{k \in A_{v}}{\Sigma}\left(\varphi_{k}\left(\left|a_{k}(f)\right|\right)+\varphi_{k}\left(\left|b_{k}(f)\right|\right)\right) \leq 2 C N_{v} \varphi_{m(v)}\left\{N^{-1 / q} \omega_{p}\left(\frac{1}{4} f, \frac{1}{2^{v}}\right)\right\}=\rho_{v}^{(\varphi)}(f) .
$$

This gives

$$
\rho(c(f))=\sum_{v=1}^{\infty} \underset{k \in A_{v}}{\sum}\left(\varphi_{k}\left(\left|a_{k}(f)\right|\right)+\varphi_{k}\left(\left|b_{k}(f)\right|\right)\right) \leq \sum_{v=1}^{\infty} \rho_{v}^{(\varphi)}(f)=\rho_{s}^{(\varphi)}(f) .
$$

Taking as a special case $\varphi_{n}(u)=n^{\beta}|u|^{\gamma}$ with any real $\beta$ and for $0<\gamma \leq q$, we obtain from Theorem 1 the following

Corollary 1. If $0<\gamma \leq q, \beta$ real and

$$
\sum_{v=1}^{\infty} m(v)^{\beta} N_{v}^{1-\gamma / q} \omega_{p}^{\gamma}\left(f, \frac{1}{2^{v}}\right)<\infty,
$$

then

$$
\sum_{n=1}^{\infty} n^{\beta}\left(\left|a_{n}(f)\right|^{\gamma}+\left|b_{n}(f)\right|^{\gamma}\right)<\infty .
$$

This Corollary generalizes a number of well-known results on Fourier series (see e. g. [4, Chapter VI, § 3]; also [1, p. 149, Theorem 3.1]).

Following [1], one may consider also special cases with $k^{r}=O\left(n_{k}\right)$ for an $r>0$ and $k \in N$, or $n_{k+1} / n_{k} \geq \alpha>1$ for $k \in N$.
3. We are going to apply Theorem 1 in order to investigate the continuity of the linear operator $c: f \rightarrow c(f)$. Obviously, $\rho_{s}^{(\varphi)}$ is a pseudomodular in the space $L_{2 \pi}^{p}$, thus generating the modular space

$$
X_{\rho_{s}^{(\varphi)}}=\left\{f \in L_{2 \pi}^{p}: \rho_{s}^{(\varphi)}(\lambda, f) \longrightarrow 0 \text { as } \lambda \rightarrow 0+\right\}
$$

(see [2, Def. 1.4]).
The following results is obtained applying Theorem 1, immediately:
Theorem 2. Under assumptions A.1 and A.2, $c: f \rightarrow c(f)$ is a linear operator, continuous from $X_{p_{s}}^{(\varphi)}$ to $l^{\varphi}$.

Let us remark that due to the inequalities (*), modular convergence and norm convergence are equivalent in both spaces $X_{\rho_{s}}^{(\rho)}$ and $l^{\varphi}$, so there is no need to distinguish between them.

Theorem 2 generalizes results of [3] concerning trigonometric Fourier series, if we put $n_{k}=k$.
4. Now, let $\varphi=\left(\varphi_{n}\right)_{n=1}^{\infty}$ and $\psi=\left(\psi_{n}\right)_{n=1}^{\infty}$ be two sequences of $\varphi$-functions satisfying A. 1 with the same $m(v)$. Let us consider the following assumption (see [2, 8.1]):
A.3. There exist positive numbers $\delta, K_{1}, K_{2}$ and a sequence $\left(\varepsilon_{k}\right)$ with $\varepsilon_{k} \geq 0, \quad \sum_{1}^{\infty} \varepsilon_{k}<\infty$ such that for every $u \geq 0$ and $k \in N$ the inequality $\varphi_{k}(u)<\delta$ implies

$$
\psi_{k}(u) \leq K_{1} \varphi_{k}\left(K_{9} u\right) .
$$

Let us note that A. 3 is the necessary and sufficient condition, in order that $l^{\varphi} \subset l^{\psi}$ continuously (see [2, Theorem 8.5]).

Theorem 3. If A.3 holds, then $X_{\rho s}(\varphi) \subset X_{\rho_{s}(\Psi)}$, and this imbedding is continuous both with respect to the modular convergencies, as well as to norm convergencies.

Proof. Let $f \in X_{\rho_{s}^{(p)}}$, then $\rho_{s}^{(\varphi)}(\lambda f) \rightarrow 0$ as $\lambda \rightarrow 0+$, whence $\rho_{s}^{(\varphi)}(\lambda f)<\delta$ for $0<\lambda<\lambda_{1}$ with some $\lambda_{1}>0$. Hence, $\rho_{v}^{(\varphi)}(\lambda f)<\delta$ for $0<\lambda<\lambda_{1}, v \in N$, and so

$$
\varphi_{m(v)}\left\{N_{v}^{-1 / q} \omega_{p}\left(\cdot \frac{1}{4} \lambda f, \frac{1}{2^{v}}\right)\right\}<\delta .
$$

By A.3,

$$
\psi_{m(v)}\left\{N_{v}^{-1 / q} \omega_{p}\left(\frac{1}{4} \lambda f, \frac{1}{2^{v}}\right)\right\} \leq K_{1} \varphi_{m(v)}\left\{K_{2} N_{v}^{-1 / q} \omega_{p}\left(\frac{1}{4} \lambda f, \frac{1}{2^{v}}\right)\right\}
$$

for $v \in N, 0<\lambda<\lambda_{1}$. Thus $\rho_{s}^{(\psi)}(\lambda f) \leq K_{1} \rho_{s}^{(\varphi)}\left(K_{2} \lambda f\right)$ for $0<\lambda<\lambda_{1}$, which shows that $f \in X_{\rho_{s}}^{(\varphi)}$. Now, let $f_{n} \in X_{\rho_{s}^{(\varphi)}}, f_{n} \rightarrow 0$ in $X_{\rho_{s}}^{(\varphi)}$ in the sense of modular convergence (resp. norm convergence). Fromi $f_{n} \rightarrow 0$ it follows that $\rho_{s}^{(\varphi)}\left(K_{2} \lambda f_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$ for some $\lambda>0$ (resp. for every $\lambda>0$ ). Taking such a $\lambda>0$ fixed, we choose an index $N$ such that $\rho_{s}^{(\rho)}\left(\lambda f_{n}\right)<\delta$ for $n \geq N$. Arguing as above, we obtain $\rho_{s}^{(\psi)}\left(\lambda f_{n}\right) \leq K_{1} \rho_{s}^{(\rho)}\left(K_{2} \lambda f_{n}\right)$ for $n \geq N$. Hence, $\rho_{s}^{(\psi)}\left(\lambda f_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$ for a $\lambda>0$ (resp. for all $\lambda>0$ ). This means that $f_{n} \rightarrow 0$ in $X_{p_{s}}(\psi)$ in the sense of modular convergence (resp. norm convergence).

Remark 1. From Theorems 2 and 3 and from [2, Theorem 8.5], we may put our results together in the form of the following diagram:

Remark 2. All the above results may be extended to the case of almost periodic functions, taking noninteger values of $n_{k}$ (see [1]).

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