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GENERALIZED DISCERNIBILITY FUNCTION BASED ATTRIBUTE REDUCTION IN INCOMPLETE DECISION SYSTEMS

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Abstract. A rough set approach for attribute reduction is an important research subject in data mining and machine learning. However, most attribute reduction methods are performed on a complete decision system table. In this paper, we propose methods for attribute reduction in static incomplete decision systems and dynamic incomplete decision systems with dynamically-increasing and decreasing conditional attributes. Our methods use generalized discernibility matrix and function in tolerance-based rough sets.

1. Introduction. Rough set theory was introduced by Zdzislaw Pawlak [9]. In practical problems, there are many cases where decision tables contain missing values for at least one conditional attribute in the value set of that attribute and these decision tables are called *incomplete decision tables.* To extract

ACM Computing Classification System (1998): I.5.2, I.2.6.
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decision rules directly from incomplete decision tables, Marzena Kryszkiewicz [4] has extended the equivalent relation in classical rough set theory to tolerance relation and proposed *tolerance rough set*. Using this tolerance rough set, many researchers have proposed different concepts of reduct based on different measures and proposed attribute reduction methods in incomplete decision tables: *reduct based on generalized decision* [4], *reduct based on positive region* [14], *reduct based on information quantity* [2], *reduct based on metric* [5, 6], *distribution reduct, assignment reduct* [10, 13], *reduct based on discernibility* matrix [7], *reduct based on* tolerance matrix [3].

Based on the idea of discernibility matrix and discernibility function in traditional rough set theory as proposed by Skowron [8], in this paper we introduce generalized discernibility matrix and function. Using generalized discernibility function, we propose attribute reduction methods in two cases: static incomplete decision tables and dynamic incomplete decision

The structure of this paper is as follows. Section 2 presents some basic concepts in tolerance rough set and some concepts of reduct in incomplete decision tables. Section 3 presents attribute reduction methods in incomplete decision tables based on generalized discernibility function. Section 4 presents attribute reduction methods in incomplete decision tables in additional cases and removes the attribute set. The conclusion and future research are presented in the last section.

2. Basic concepts. In this section, we present some basic concepts about tolerance rough set which have been proposed by Marzena Kryszkiewicz [4] and some concepts about reducts of incomplete decision tables.

An *information system* is a pair $IS = (U, A)$, where the set U denotes the *universe of objects* and A is the set of *attributes*, i.e., mappings of the form $a : U \to V_a$. V_a is called the *value set* of attribute a. If V_a contains a missing value for at least one attribute $a \in A$, then IS is called an incomplete information system, otherwise it is complete. Further on, we will denote the missing value by $*$. An incomplete decision table (IDS) is an incomplete information system $IDS = (U, A \cup \{d\})$ where d, $d \notin A$ and $* \notin V_d$, is a distinguished attribute called *decision attribute*, and the elements of A are called *conditional attributes*.

Let $IIS = (U, A)$ be an incomplete information system. For any attribute set $P \subseteq A$. We define a binary relation on U as follows:

$$\begin{aligned} &SIM\left(P\right) \\ &= \left\{(u,v) \in U \times U \left| \forall a \in P, f\left(u,a\right) = f\left(v,a\right) \vee f\left(u,a\right) = {}'{*}' \vee f\left(v,a\right) = {}'{*}' \right.\right\}. \end{aligned}$$

$SIM(P)$ is a tolerance relation on U. It can be easily shown that $SIM(P)$ $= \cap_{a\in P} SIM(\{a\})$. Let $U/SIM(P)$ denote the family sets $\{S_P(u) \,|u \in U\}$ where $S_P(u) = \{v \in U \,|(u,v) \in SIM(P)\}$ is the maximal set of objects which are possibly indistinguishable by P with u. A member $S_P(u)$in $U/SIM(P)$ is called a tolerance class or a granule of information. It is clear that the tolerance classes in $U/SIM(P)$ do not constitute a partition of U in general. They constitute a covering of U, i.e., $S_P(u) \neq \emptyset$ for every $u \in U$, and $\cup_{u\in U} S_P(u) = U$.

For any $B \subseteq A$, $X \subseteq U$, *B-lower approximation* of$X$ is the set $\underline{B}X = \{u \in U \,|S_B(u) \subseteq X\} = \{u \in X \,|S_B(u) \subseteq X\}$, *B-upper approximation of X* is the set $\overline{B}X = \{u \in U \,|S_B(u) \cap X \neq \emptyset\} = \cup\{S_B(u) \,|u \in U\}$, *B-boundary region* of X is the set $BN_P(X) = \overline{P}X - \underline{P}X$. For such approximation set, *B-positive region* with respect to *{d}* is defined as

$$POS_B(\{d\}) = \bigcup_{X\in U/\{d\}} (\underline{B}X).$$

Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table. For any $B \subseteq A$ and $u \in U$, $\partial_B(u) = \{f_d(v) \,|v \in S_B(u)\}$ is called the *generalized decision* in IDS. If $|\partial_C(u)| = 1$ for any $u \in U$ then *IDS* is *consistent*, otherwise it is *inconsistent.* According to the definition of positive region, IDS is consistent if and only if $POS_A(\{d\}) = U$, otherwise it is inconsistent.

It has been shown that one of the crucial concepts in rough set theory is reduct or decision reduct. In general, reducts are minimal subsets (with respect to the set inclusion relation) of attributes which contain a necessary portion of information about the set of all attributes. In the sequel, we present some concepts about reducts of incomplete decision tables which are related to this paper.

According to Kryszkiewicz [4], a reduct of an incomplete decision table is a minimal subset of the conditional attribute set which preserves the generalized decision for all objects. The reduct is defined as follows:

Definition 1 [4]**.** *Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table. If $R \subseteq A$ satisfies:*

(1) $\partial_R(u) = \partial_A(u)$ *for any* $u \in U$*,*

(2) $\forall r \in R,\ R' = R - \{r\}$ *is not satisfied* (1)*,*

then R is called a reduct of IDS based on generalized decision.

Example 1. Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table where $U = \{u_1, u_2, u_3, u_4, u_5\}$, $A = \{a_1, a_2, a_3\}$ as Table 1 with a_1 (Price), a_2 (Colours), a_3 (Size), a_4 (Resolution).

With $u_1 \in U$ we have that $S_{a_1}(u_1) = \{u_1, u_3, u_4, u_5\}$, $S_{a_2}(u_1) = \{u_1, u_2, u_3, u_5\}$, $S_{a_3}(u_1) = \{u_1, u_2, u_4, u_5, u_6\}$, $S_{a_4}(u_1) = \{u_1, u_2, u_4, u_6\}$.

Table 1. An example incomplete decision table

Television	Price	Colour	Size	Resolution	Quality(d)
u_1	High	Black	Large	Low	Good
u_2	Low	$*$	Large	Low	Good
u_3	$*$	$*$	Small	High	Bad
u_4	High	Brown	Large	Low	Good
u_5	$*$	$*$	Large	High	Excellent
u_6	Low	Brown	Large	$*$	Good

Hence, $S_A(u_1) = S_{a_1}(u_1) \cap S_{a_2}(u_1) \cap S_{a_3}(u_1) \cap S_{a_4}(u_1) = \{u_1\}$.

Similar, $S_A(u_2) = \{u_2, u_6\}$, $S_A(u_3) = \{u_3\}$, $S_A(u_4) = \{u_4\}$, $S_A(u_5) = \{u_5, u_6\}$, $S_A(u_6) = \{u_2, u_5, u_6\}$.

Consequently, $\partial_A(u_1) = \partial_A(u_2) = \partial_A(u_4) = \{\text{Good}\}$, $\partial_A(u_3) = \{\text{Bad}\}$, $\partial_A(u_5) = \partial_A(u_6) = \{\text{Good, Excellent}\}$. So IDS is inconsistent.

3. Attribute reduction in incomplete decision tables based on generalized discernibility function. Attribute reduction in decision systems is the process of selecting the smallest subset of the attribute set conditions that preserve the classification information of decision tables. In traditional rough set theory, Skowron [8] has introduced discernibility matrix and discernibility function to find reduct. Based on this approach, we propose generalized discernibility matrix and generalized discernibility function to find reduct of incomplete decision systems.

Definition 2. *Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table where $R \subseteq A$ and $|U| = n$. Generalized discernibility matrix on the attribute set R is $M_R = (m_{ij})_{n\times n}$, element m_{ij} is defined as*

(1) $m_{ij} = 1 \quad d(u_j) \notin \partial_R(u_i)$,

(2) $m_{ij} = 0 \quad d(u_j) \in \partial_R(u_i)$.

Note. If $R = \phi$ then $m_{ij} = 0$. In general, M_R is not a symmetric matrix because there exists $u_i, u_j \in U$ such that $d(u_j) \notin \partial_R(u_i)$ and $d(u_i) \in \partial_R(u_j)$.

Example 2. From Example 1, generalized discernibility matrix on the attribute set R is

$$M_A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Definition 3. *Let $X = (x_{ij})_{m\times n}$ and $Y = (y_{ij})_{m\times n}$. Relations " $\preceq$ " and " $\succeq$ " are defined as:*
(1) $X \preceq Y$ *if and only if* $x_{ij} \leq y_{ij}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$,
(2) $X \succeq Y$ *if and only if* $x_{ij} \geq y_{ij}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Proposition 1. *Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table and $P, Q \subseteq A$. If $P \subseteq Q$ then $M_P \preceq M_Q$.*

Example 3. From Example 2, assume that $R = \{a_1, a_2, a_3\}$, then

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

from Example 2 we have $M_R \prec M_A$.

Definition 4. *Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table, $R \subseteq A$ and $M_R = (m_{i,j})_{n\times n}$ is generalized discernibility matrix on the attribute set R. Then generalized discernibility function on the attribute set R, denoted by $DIS\,(R)$, is defined as:*

$$DIS\,(R) = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \quad \textit{for any} \quad 1 \leq i \leq n,\ 1 \leq j \leq n.$$

Example 4. For generalized discernibility matrix M_A as Example 2, the generalized discernibility function is:

$$DIS\,(A) = 2 + 2 + 5 + 2 + 1 + 1 = 13.$$

From Definition 4 and Proposition 1, we have the following Proposition:

Proposition 2. *Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table and $P, Q \subseteq A$. If $P \subseteq Q$ then $DIS\,(Q) \geq DIS\,(P)$.*

Proposition 3. *Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table, M_A is generalized discernibility matrix and $DIS\,(A)$ is generalized discernibility function. Then $DIS\,(R) = DIS(A)$ if and only if $\partial_R\,(u) = \partial_A\,(u)$ for $u \in U$.*

P r o o f. i) Suppose that there exists $u_{i_0} \in U$ such that $\partial_R\,(u_{i_0}) \neq \partial_A\,(u_{i_0})$. Let $\partial_A\,(u_{i_0}) \subseteq \partial_R\,(u_{i_0})$. Then there exists $d\,(u_{j_0})$ such that $d\,(u_{j_0}) \in \partial_R\,(u_{i_0}) \wedge d\,(u_{j_0}) \notin \partial_A\,(u_{i_0})$. Since $d\,(u_{j_0}) \notin \partial_A\,(u_{i_0})$ we have

$$m_{i_0 j_0} = 1, \quad m_{i_0 j_0} \in M_A. \tag{1}$$

Since

$$m_{i_0 j_0} = 0, \quad m_{i_0 j_0} \in M_R. \tag{2}$$

Since $R \subseteq A$ we have $M_R \prec M_A$, From (1) and (2) it follows that $DIS\,(R) \neq DIS(A)$, which contradicts $DIS\,(R) = DIS(A)$. Consequently, the assumption is not true and we can conclude that if $DIS\,(R) = DIS(A)$ then $\partial_R\,(u) = \partial_A\,(u)$ for $\forall u \in U$.

ii) Conversely, suppose that $DIS\,(R) \neq DIS(A)$. According to Proposition 1, from $R \subseteq A$ we have $M_R \prec M_A$. Combined with $DIS\,(R) \neq DIS(A)$, it follows that $M_R \neq M_A$. Then there exist i_0 and j_0 such that

$$m_{i_0 j_0} \in M_R, \quad m_{i_0 j_0} = 0 \tag{3}$$

and

$$m_{i_0 j_0} \in M_A, \quad m_{i_0 j_0} = 1. \tag{4}$$

Since (4) we have $d\,(u_{j_0}) \notin \partial_A\,(u_{i_0})$. Since (3) we have $d\,(u_{j_0}) \in \partial_R\,(u_{i_0})$. It follows that $\partial_R\,(u_{i_0}) \neq \partial_A\,(u_{i_0})$,which contradicts $\partial_R\,(u) = \partial_A\,(u)$ for $\forall u \in U$. Consequently, the assumption is not true and we can conclude that if $\partial_R\,(u) = \partial_A\,(u)$ for $\forall u \in U$ then $DIS\,(R) = DIS(A)$.

From i) and ii) we can conclude that $DIS\,(R) = DIS(A)$ if and only if $\partial_R\,(u) = \partial_A\,(u)$ for $\forall u \in U$. $\square$

In the sequel, we present a method to find a reduct of an incomplete decision system using the generalized discernibility function. As the method of finding reduct in traditional rough set theory, our method includes steps: definition of reduct, definition of the importance of attributes and building an heuristic algorithm to find the best reduct based on the importance of attributes. Generalized discernibility function is used as selection criterion in an heuristic algorithm to find the best reduct.

Definition 5. *Let* $IDS = (U, A \cup \{d\})$ *be an incomplete decision table. If* $R \subseteq A$ *satisfies*

(1) $DIS(R) = DIS(A)$,

(2) $\forall R' \subset R$, $DIS(R') \neq DIS(A)$,

then R *is called a* reduct of IDS based on generalized discernibility function.

Proposition 3 shows that reduct based on generalized discernibility function is equivalent to reduct based on generalized decision function.

Definition 6. *Let* $IDS = (U, A \cup \{d\})$ *be an incomplete decision table,* $R \subseteq A$ *and* $a \in A - R$*. The importance of the attributes* a *with respect to the attribute set* R *is defined as:*

$$SIG_R^{out}(a) = DIS(R \cup \{a\}) - DIS(R)$$

Definition 7. *Let* $IDS = (U, A \cup \{d\})$ *be an incomplete decision table,* $R \subseteq A$ *and* $a \in R$.

The importance of the attributes a *with respect to the attribute set* R *is defined as:*

$$SIG_R^{in}(a) = DIS(R) - DIS(R - \{a\})$$

From Proposition 2 we have $SIG_R^{out}(a) \geq 0$ and $SIG_R^{in}(a) \geq 0$.

Next, we propose a heuristic algorithm to find the best reduct based on the importance of the attributes.

Algorithm 1. A heuristic algorithm to find a best reduct using generalized discernibility function.

Input: An incomplete decision table $IDS = (U, A \cup \{d\})$.

Output: The best reduct R.

1. $R = \emptyset$;

   // Add gradually to R attributes that have the greatest importance;

2. While $DIS(R) \neq DIS(A)$ do
3. Begin
4. For each $a \in A - R$ calculation $SIG_R^{out}(a) = DIS(R \cup \{a\}) - DIS(R)$;
5. Select $a_m \in A - R$ such that $SIG_R^{out}(a_m) = \underset{a \in A - R}{\text{Max}} \left\{ SIG_R^{out}(a) \right\}$;

6. $R = R \cup \{a_m\}$;

7. End;

//Remove redundant attributes in R, if any;

8. For each $a \in R$

9. If $DIS\left(R - \{a\}\right) = DIS\left(R\right)$ then $R = R - \{a\}$;

10. Return R;

Suppose that k is the number of condition attributes and n is the number of objects. The time complexity of M_A is $O\left(kn^2\right)$; it follows that the time complexity of $DIS\left(A\right)$ is $O\left(kn^2\right)$. At the while loop from line 2 to line 7, the time complexity of computing all of $SIG_R\left(a\right)$ is $\left(k + \left(k-1\right) + \cdots + 1\right) * kn^2 = \left(k * \left(k-1\right)/2\right) * kn^2 = O\left(k^3n^2\right)$. The time complexity of selecting the attribute with the greatest importance is $k + \left(k-1\right) + \cdots + 1 = k * \left(k-1\right)/2 = O\left(k^2\right)$, so the time complexity of the while loop is $O\left(k^3n^2\right)$. Similarly, the time complexity of the for loop is $O\left(k^2n^2\right)$. Consequently, the time complexity of Algorithm 1 is $O\left(k^3n^2\right)$.

Example 5. Let $IDS = \left(U, A \cup \{d\}\right)$ be an incomplete decision table in Example 1. By using Algorithm 1, initialization $R = \emptyset$ and calculating:

$$\begin{aligned}
SIG_{\emptyset}^{out}\left(a_1\right) &= DIS\left(\{a_1\}\right) - DIS\left(\emptyset\right) = DIS\left(\{a_1\}\right) = 0 \\
SIG_{\emptyset}^{out}\left(a_2\right) &= DIS\left(\{a_2\}\right) - DIS\left(\emptyset\right) = DIS\left(\{a_2\}\right) = 0 \\
SIG_{\phi}^{out}\left(a_3\right) &= DIS\left(\{a_3\}\right) - DIS\left(\phi\right) = DIS\left(\{a_3\}\right) = 10 \\
SIG_{\phi}^{out}\left(a_4\right) &= DIS\left(\{a_4\}\right) - DIS\left(\phi\right) = DIS\left(\{a_4\}\right) = 6
\end{aligned}$$

Select a_3 attribute with the greatest importance and $R = \{a_3\}$. From Example 4 we have $DIS\left(A\right) = 13$, so $DIS\left(R\right) \neq DIS(A)$. Go to the 2^{nd} loop and calculate:

$$\begin{aligned}
SIG_{a_3}^{out}\left(a_1\right) &= DIS\left(\{a_1, a_3\}\right) - DIS\left(\{a_3\}\right) = 10 - 10 = 0 \\
SIG_{a_3}^{out}\left(a_2\right) &= DIS\left(\{a_2, a_3\}\right) - DIS\left(\{a_3\}\right) = 10 - 10 = 0 \\
SIG_{a_3}^{out}\left(a_4\right) &= DIS\left(\{a_3, a_4\}\right) - DIS\left(\{a_3\}\right) = 13 - 10 = 3
\end{aligned}$$

Select a_4 attribute with the greatest importance and $R = \{a_3, a_4\}$.

We have $DIS\left(\{a_3, a_4\}\right) = DIS(A) = 13$, go to the for loop.

Similarly, we have $DIS\left(\{a_4\}\right) \neq DIS(A)$ and $DIS\left(\{a_3\}\right) \neq DIS(A)$. Consequently, the algorithm ends and $R = \{a_3, a_4\}$ is a best reduct of A.

4. Attribute reduction in incomplete decision tables when adding and removing an attribute set.

Proposition 4. *Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table. For $P, Q \subseteq A$, $P \cap Q = \phi$ and $U = \{u_1, \ldots, u_n\}$, suppose that $S_{P \cup Q}(u)$, $S_P(u)$ and $S_Q(u)$ are respectively the tolerance class on $P \cup Q$, P and Q. Then, we have $S_{P \cup Q}(u) = S_P(u) \cap S_Q(u)$.*

Example 6. Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table in Example 1. We add the attribute set $\{a_5, a_6\}$ where a_5 (energy savings), a_6 (Internet).

Table 2. An example of an incomplete decision table

Television	Price	Colours	Size	Resolution	Energy savings	Internet	Quality(d)
u_1	High	Black	Large	Low	No	No	Good
u_2	Low	$*$	Large	Low	Yes	Yes	Good
u_3	$*$	$*$	Small	High	No	No	Bad
u_4	High	Brown	Large	Low	$*$	No	Good
u_5	$*$	$*$	Large	High	Yes	Yes	Excellent
u_6	Low	Brown	Large	$*$	No	No	Good

Let $P = \{a_1, a_2, a_3, a_4\}$, $Q = \{a_5, a_6\}$. From Example 2, generalized discernibility matrix of IDS on P is:

$$M_P = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

We calculate $M_{P \cup Q}$ from Proposition 4. With objects $u_1 \in U$ we have $S_{a_5}(u_1) = \{u_1, u_3, u_4, u_6\}$, $S_{a_6}(u_1) = \{u_1, u_3, u_4, u_6\}$, so $S_Q(u_1) = \{u_1, u_3, u_4, u_6\}$. Otherwise, $S_P(u_1) = \{u_1\}$, so $S_{P \cup Q}(u_1) = S_P(u_1) \cap S_Q(u_1) = \{u_1\}$. Similarly, we calculate $S_{P \cup Q}(u_i)$ for $i = 2, \ldots, 6$.

From Definition 2, generalized discernibility matrix of IDS on $P \cup Q$ is:

$$M_{P\cup Q} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & \underline{1} & 0 \end{bmatrix}.$$

Proposition 5. *Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table. For $Q \subset P \subseteq A$ and $U = \{u_1, \ldots, u_n\}$, suppose that $M_{P-Q} = \left(m_{ij}^{P-Q}\right)_{n\times n}$ and $M_p = \left(m_{ij}^{P}\right)_{n\times n}$ is generalized discernibility matrix of IDS on $P - Q$ and P. Then, elements of $M_{P-Q} = \left(m_{ij}^{P-Q}\right)_{n\times n}$ are calculated based on elements of $M_p = \left(m_{ij}^{P}\right)_{n\times n}$ as follow:*

(1) $m_{ij}^{P-Q} = 1$ if $m_{ij}^{P} = 1$ and $d(u_j) \notin \partial_{P-Q}(u_i)$,

(2) $m_{ij}^{P-Q} = 0$ if $m_{ij}^{P} = 0$ or $d(u_j) \in \partial_{P-Q}(u_i)$.

Example 7. Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table in Example 6, with $P = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $Q = \{a_2, a_4\}$. From Example 6 we have:

$$M_P = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

We calculate M_{P-Q} from Proposition 5. With the object u_2 we have $\partial_{P-Q}(u_2) = \{\text{Good}, \text{Excellent}\}$. From Proposition 5, $m_{21}^{P-Q} = m_{22}^{P-Q} = m_{24}^{P-Q} = m_{26}^{P-Q} = 0$. From $d(u_5) = \text{Excellent} \in \{\text{Good}, \text{Excellent}\}$ we have $m_{25}^{P-Q} = 0$. From $d(u_3) = \text{Bad} \notin \{\text{Good}, \text{Excellent}\}$ we have $m_{23}^{P-Q} = 1$. Consequently, we have

$$M_{P-Q} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \underline{0} & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Algorithm 2. A heuristic algorithm to find the best reduct when adding an attribute set

Input: An incomplete decision table $IDS = (U, A \cup \{d\})$, the best reduct R_A and an attribute set P where $P \cap A = \phi$.

Output: A best reduct $R_{A \cup P}$ of the attribute set $A \cup P$.

1. $R = R_A$;
2. Calculate $M_{A \cup P}$ from Proposition 4; Calculate $DIS\,(A \cup P)$;
3. While $DIS\,(R) \neq DIS\,(A \cup P)$ do
4. Begin

   For each $a \in P - R$ we calculate $SIG_R^{out}\,(a) = DIS\,(R \cup \{a\}) - DIS\,(R)$.

   $DIS\,(R \cup \{a\})$ is calculated from Proposition 4;
5. Select $a_m \in P - R$ such that $SIG_R^{out}\,(a_m) = \underset{a \in P-R}{\text{Max}} \left\{ SIG_R^{out}\,(a) \right\}$;
6. $R = R \cup \{a_m\}$;
7. End;
8. For each $a \in R$
9. If $DIS\,(R - \{a\}) = DIS\,(A \cup P)$ then $R = R - \{a\}$;
10. Return R;

Suppose that p is the number of attribute of P and n is the number of objects. From Proposition 4, the time complexity of $M_{R \cup \{a\}}$ when M_R is calculated is $O\left(n^2\right)$. So the time complexity of $DIS\,(R \cup \{a\})$ when $DIS\,(R)$ is calculated is $O\left(n^2\right)$. At the while loop from line 3 to line 7, the time complexity to compute all of $SIG_R^{out}\,(a)$ is $(p + (p-1) + \cdots + 1) * n^2 = (p * (p-1)/2) * n^2 = O\left(p^2 n^2\right)$. The time complexity of selecting the properties that are most important is $p + (p-1) + \cdots + 1 = p * (p-1)/2 = O\left(p^2\right)$. So the time complexity of the While loop is $O\left(k^3 n^2\right)$. Similarly, the time complexity of the For loop is $O\left(pn^2\right)$. So the time complexity of Algorithm 1 is $O\left(p^2 n^2\right)$. If we use Algorithm 1 to find reduct then the time complexity is $O\left((k+p)^3 n^2\right)$. So Algorithm 2 to find a best reduct when adding an attribute set will reduce the time complexity.

Example 8. From Example 5, $\{a_3, a_4\}$ is the best reduct of incomplete decision table in example 1. Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table in Example 6 (Table 2) with $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, by using Algorithm 2 we have:

With $R = \{a_3, a_4\}$, from example 6 we calculate $DIS\,(A) = 18$

$$\begin{aligned} SIG^{out}_{\{a_3,a_4\}}(a_5) &= DIS\,(\{a_3, a_4, a_5\}) - DIS\,(\{a_3, a_4\}) = 18 - 13 = 5, \\ SIG^{out}_{\{a_3,a_4\}}(a_6) &= DIS\,(\{a_3, a_4, a_6\}) - DIS\,(\{a_3, a_4\}) = 18 - 13 = 5. \end{aligned}$$

Select the attribute a_4 with the greatest importance and $R = \{a_3, a_4, a_6\}$. Since $DIS\,(\{a_3, a_4, a_6\}) = 18$ we have $DIS\,(\{a_3, a_4, a_6\}) = DIS\,(A)$. Go to For loop, test the attribute set R.

We have $DIS\,(\{a_3, a_6\}) = 13$, so $DIS\,(\{a_3, a_6\}) \neq DIS\,(A)$
We have $DIS\,(\{a_4, a_6\}) = 14$, so $DIS\,(\{a_4, a_6\}) \neq DIS\,(A)$
We have $DIS\,(\{a_3, a_4\}) = 13$, so $DIS\,(\{a_3, a_4\}) \neq DIS\,(A)$
The Algorithm ends and $R = \{a_3, a_4, a_6\}$ is the best reduct of A.

Algorithm 3. A heuristic algorithm to find the best reduct when removing the attribute.

Input: $IDS = (U, A \cup \{d\})$ is an incomplete decision table, the best reduct R_A and an attribute set P where $P \subset A$.

Output: The best reduct R_{A-P} of the attribute set $A - P$.

1. $R = R_A - P$;
2. Calculate M_{A-P} from Proposition 5; Calculate $DIS\,(A - P)$;
3. While $DIS\,(R) \neq DIS\,(A - P)$ do
4. Begin
   For each $a \in R$ Calculate $SIG^{in}_R(a) = DIS\,(R) - DIS\,(R - \{a\})$ where $DIS\,(R - \{a\})$ is calculated from Proposition 5;
5. Select $a_m \in R$ such that $SIG^{in}_R(a_m) = \underset{a \in R}{\text{Min}} \left\{ SIG^{in}_R(a) \right\}$;
6. $R = R - \{a_m\}$;
7. End;
8. For each $a \in R$
9. If $DIS\,(R - \{a\}) = DIS\,(A - P)$ then $R = R - \{a\}$;

10. Return R;

Similarly to Algorithm 2, the complexity of Algorithm 3 is $O\left(|R_A - P|^2 n^2\right)$ with $|R_A - P|$ is the number of attribute of $R_A - P$.

Example 9. Let $IDS = (U, A \cup \{d\})$ be an incomplete decision table in Example 6 (Table 2) where $R = \{a_3, a_4, a_6\}$ is the best reduct. By using Algorithm 3 to calculate reduct, we have:

For $R = \{a_3, a_4, a_6\} - \{a_2, a_4\} = \{a_3, a_6\}$, from example 7 we have $DIS\,(A - P) = 13$. From example 8 we have$DIS\,(\{a_3, a_6\}) = 13$, so $DIS\,(\{a_3, a_6\})$ $= DIS\,(A - P)$. Go to For loop, test the attribute set R.

Calculate $DIS\,(\{a_3\}) = 10$, so $DIS\,(\{a_3\}) \neq DIS\,(A - P)$.

Calculate $DIS\,(\{a_6\}) = 6$, so $DIS\,(\{a_6\}) \neq DIS\,(A - P)$.

The algorithm ends and $R = \{a_3, a_6\}$ is the best reduct of $A - P$.

5. Conclusion. Based on the idea of discernibility matrix and function [8] in traditional rough set theory, in this paper we propose generalized discernibility matrix and function to find reduct in incomplete decision systems. We have developed attribute reduction algorithms in two cases: adding an attribute set and deleting an attribute set. The methods significantly reduce the time complexity. Further research is building increasing algorisms with dynamically increasing or decreasing set-object in order to find reduct in dynamic incomplete decision systems.

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