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SUB- AND SUPER-SOLUTIONS OF A NONLINEAR PDE, AND APPLICATION TO A SEMILINEAR SPDE

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Abstract. We obtain upper and lower bounds for the explosion time of a semi-linear heat equation on a bounded d-dimensional domain, perturbed by white noise. The bounds we get are expressed in terms of exponential functionals of one-dimensional Brownian motion, whose density function can be explicitly calculated.

1. Introduction and backgrownd. Consider the semilinear stochastic partial differential equation

$$
\begin{aligned}
du(t,x) &= \Big[\Delta u(t,x) + u^{1+\beta}(t,x)\Big]\,dt + \kappa u(t,x)\,dW_t, \\
u(0,x) &= f(x) \geq 0, \quad x \in D, \\
u(t,x) &= 0, \quad t \geq 0, \quad x \in \partial D,
\end{aligned}
\tag{1}
$$

where $D \subset \mathbb{R}^d$ is a bounded smooth domain, $\beta > 0$ is a constant, κ is a given positive number, $\{W_t,\ t \geq 0\}$ is a standard one-dimensional Brownian motion and $f : D \to \mathbb{R}_+$ is of class C^2 and not identically zero. The nonlinear term in Equation (1) is only locally Lipschitz, and therefore its solutions can exhibit blow up in finite time. Our goal in this note is to obtain useful upper and lower bounds for the explosion times of (1).

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Let $\Lambda > 0$ be the first eigenvalue of the Laplacian on D, and ψ the corresponding eigenfunction normalized so that $\|\psi\|_{L^1} = 1$. We are going to show that for initial values of the form $u(0,x) = k\psi(x)$, $x \in D$, where $k > 0$ is a parameter, the explosion time ϱ of (1) satisfies $\sigma_* \le \varrho \le \sigma^*$, where

$$\begin{aligned}\sigma_* &= \inf\left\{ t \ge 0 : \int_0^t \exp\{\kappa\beta W_r - \beta(\Lambda + \kappa^2/2)r\}\, dr \ge 1/(\beta k^\beta \|\psi\|_\infty^\beta) \right\},\\ \sigma^* &= \inf\left\{ t \ge 0 : \int_0^t \exp\{\kappa\beta W_r - \beta(\Lambda + \kappa^2/2)r\}\, dr \ge 1/\left(\beta k^\beta \left(\textstyle\int \psi^2(x)\, dx\right)^\beta\right) \right\}.\end{aligned}$$

A remarkable fact is that the density of the Brownian functional

$$\int_0^t \exp\{\kappa\beta W_r - \beta(\Lambda + \kappa^2/2)r\}\, dr, \quad t > 0,$$

appearing in the above expressions, can be obtained from Yor's formula ([5], Ch. 4 and [4]). This is a reason to consider initial values of the form given above. Moreover, the random times σ_* and σ^* arise, respectively, as the explosion times of a super- and a sub-solution of the nonlinear equation

$$\begin{aligned}(2)\qquad \frac{\partial v}{\partial t}(t,x) &= \Delta v(t,x) - \frac{1}{2}\kappa^2 v(t,x) + e^{\kappa\beta W_t} v^{1+\beta}(t,x), \quad t > 0,\ x \in D,\\ v(0,x) &= f(x), \quad x \in D,\\ v(t,x) &= 0, \quad x \in \partial D,\end{aligned}$$

where $f = k\psi$ with $k > 0$. Recall [1] that ψ is nonnegative, with $\psi > 0$ on D, and that

$$T_t\psi = e^{-\Lambda t}\psi, \quad t \ge 0.$$

Here $\{T_t,\, t \ge 0\}$ is the semigroup of bounded linear operators defined by

$$T_t f(x) = \mathbb{E}\left[f(X_t),\ t < \tau_D \,|\, X_0 = x \right],\ x \in D,$$

for all bounded and measurable $f : D \to \mathbb{R}$, where $\{X_t\}_{t\ge 0}$ is the d-dimensional Brownian motion with variance parameter 2, killed at the time τ_D at which it hits ∂D. As before, we are going to assume that ψ is normalized so that $\int_D \psi(x)\, dx = 1$.

Solutions of our equation are to be understood in the weak sense, namely, letting $\tau \ge 0$ be a random time, we say that a continous adapted random field

$u = \{u(t,x),\, t \geq 0,\, x \in D\}$ is a *weak solution* of (1) on the interval $]0,\tau[$ if, for every $\varphi \in C^2(D)$ vanishing on ∂D, the equality

$$\int_D u(t,x)\varphi(x)\,dx = \int_D f(x)\varphi(x)\,dx + \int_0^t \int_D [u(s,x)\Delta\varphi(x) + G(u(s,x))\varphi(x)]\,dx\,ds$$
$$+\kappa \int_0^t \int_D u(t,x)\varphi(x)\,dx\,dW_s \quad P-\text{a.s.}$$

holds for all $t \in [0,\tau[$. A (random) time T is called blowup time of u if

$$\lim_{t \nearrow T} \sup_{x \in D} |u(t,x)| = +\infty \quad P-\text{a.s. on } \{T < +\infty\}.$$

2. Weak solutions of a random PDE. Our approach follows closely the methods used in [2], where estimates for the probabilities of existence of global and of non-global solutions of (1) were obtained. Here, however, the emphasis is rather on getting information about blowup times.

Proposition 1. *For any given continuous Brownian path $W.$, let us define*

$$v(t,x) = \exp\{-\kappa W_t\}u(t,x), \quad t \geq 0,\ x \in D, \tag{3}$$

where u is a weak solution of Eq. (1)*. Then v is a weak solution of Equation* (2)*.*

P r o o f. In order to prove this assertion, let us recall that, according to Itô's formula for Brownian motion, for all $g \in C^2(\mathbb{R})$,

$$g(W_t) = g(0) + \int_0^t g'(W_s)\,dW_s + \frac{1}{2}\int_0^t g''(W_s)\,ds, \quad t \geq 0.$$

Hence, for a g of the form $g(W_t) = e^{-\kappa W_t}$ we get, by Ito's formula,

$$e^{-\kappa W_t} = 1 - \kappa \int_0^t e^{-\kappa W_s} dW_s + \frac{\kappa^2}{2}\int_0^t e^{-\kappa W_s} ds$$

or, in differential form,

$$de^{-\kappa W_t} = -\kappa e^{-\kappa W_t} dW_t + \frac{\kappa^2}{2} e^{-\kappa W_t}\,dt. \tag{4}$$

Recall also that, if $\{X(t),\, t \geq 0\}$ and $\{Y(t),\, t \geq 0\}$ are real-valued stochastic processes admitting stochastic differentials

$$\begin{aligned} dX(t) &= \mu_X(t, X(t))\,dt + \sigma_X(t, X(t))\,dW_t, \\ dY(t) &= \mu_Y(t, Y(t))\,dt + \sigma_Y(t, Y(t))\,dW_t, \end{aligned}$$

with continuous coefficients, then the integration-by-parts formula holds, i.e.

$$X(t)Y(y) = X(0)Y(0) + \int_0^t X(s)\,dY(s) + \int_0^t Y(s)\,dX(s) + [X,Y](t), \tag{5}$$

where $[X,Y](t) = \int_0^t \sigma_X(s,X(s))\sigma_Y(s,Y(s))\,ds$. In our setting, the processes $X(t) = e^{-\kappa W_t}$ and $Y(t) = u(t,x)$ have, respectively, the stochastic differentials (4) and (1). Hence we get

$$\left[e^{-\kappa W_\cdot}, u_i(\cdot,x)\right](t) = -\kappa^2 \int_0^t e^{-\kappa W_s} u(s,x)\,ds, \quad t \geq 0.$$

Putting $u(t,\varphi) = \int_D u(t,x)\varphi(x)\,dx$, where φ is any smooth function with compact support, we see that a weak solution of (1) is given by

$$u(t,\varphi) = u(0,\varphi) + \int_0^t u(s,\Delta\varphi)\,ds + \int_0^t u^{1+\beta}(s,\varphi)\,ds + \kappa \int_0^t u(s,\varphi)\,dW_s. \tag{6}$$

By applying the integration by parts formula we further obtain that

$$\begin{aligned} v(t,\varphi) &:= \int_D v(t,x)\varphi(x)\,dx \\ &= v(0,\varphi) + \int_0^t e^{-\kappa W_s}\,du(s,\varphi) \\ &+ \int_0^t u(s,\varphi)\left(-\kappa e^{-\kappa W_s} dW_s + \frac{\kappa^2}{2} e^{-\kappa W_s}\,ds\right) + \left[e^{-\kappa W_\cdot}, u(\cdot,\varphi)\right](t), \end{aligned}$$

where the quadratic variation in the last line is given by

$$\left[e^{-\kappa W_\cdot}, u(\cdot,\varphi)\right](t) = -\kappa^2 \int_0^t e^{-\kappa W_s} u(s,\varphi)\,ds, \quad t \geq 0.$$

Therefore,

$$\begin{aligned} v(t,\varphi) &= v(0,\varphi) + \int_0^t v(s,\Delta\varphi)\,ds + \int_0^t e^{-\kappa W_s}\left(e^{\kappa W_s} v\right)^{1+\beta}(s,\varphi)\,ds \\ &\quad -\frac{\kappa^2}{2}\int_0^t e^{-\kappa W_s} u(s,\varphi)\,ds \\ &= v(0,\varphi) + \int_0^t \left[v(s,\Delta\varphi) - \frac{\kappa^2}{2} v(s,\varphi)\right] ds + \int_0^t e^{\kappa\beta W_s} v^{1+\beta}(s,\varphi)\,ds. \end{aligned} \tag{7}$$

Hence, the function $v(t,x)$ is a weak solution of the random PDE (2). □

Remark. Notice that the integral form of Equation (2) is given by

(8)
$$v(t,x) = e^{-\kappa^2 t/2} T_t f(x) + \int_0^t e^{\beta\kappa W_{t-r} - \kappa^2 r/2} T_r \left[v(t-r,\cdot)^{1+\beta} \right](x)\, dr, \quad t \geq 0.$$

In what follows, we write ϱ for the blow up time of Equation (8) when the initial value is of the form $f = k\psi$ for some constant $k > 0$. Due to (3) and to the a.s. continuity of Brownian paths, ϱ is also the explosion time of Equation (1) with initial value of the above form.

3. An upper bound for ϱ. Due to $\Delta\psi = -\Lambda\psi$,

$$v(s,\Delta\psi) = \int_D v(s,x)\Delta\psi(x)\, dx = -\Lambda v(s,\psi).$$

From Proposition 1 we know that

$$v(t,\psi) = v(0,\psi) + \int_0^t \left[v(s,\Delta\psi) \ - \frac{\kappa^2}{2} v(s,\psi) \right] ds + \int_0^t e^{\kappa\beta W_s} v^{1+\beta}(s,\psi)\, ds,$$

where, by Jensen's inequality,

$$v^{1+\beta}(s,\psi) = \int_D v^{1+\beta}(s,x)\psi(x)\, dx \ \geq \ \left[\int_D v(s,x)\psi(x)\, dx \right]^{1+\beta} \ = \ v(s,\psi)^{1+\beta}.$$

Therefore

$$\frac{d}{dt} v(t,\psi) \geq -\left(\Lambda + \frac{\kappa^2}{2} \right) v(t,\psi) + e^{\kappa\beta W_t} v(t,\psi)^{1+\beta}.$$

Hence $v(t,\psi) \geq I(t)$ for all $t \geq 0$, where $I(\cdot)$ solves

$$\frac{d}{dt} I(t) = -\left(\Lambda + \frac{\kappa^2}{2} \right) I(t) \ + e^{\kappa\beta W_s} I(t)^{1+\beta}, \quad I(0) = v(0,\psi),$$

and is given by

$$I(t) = e^{-(\Lambda+\kappa^2/2)t} \left[v(0,\psi)^{-\beta} - \beta \int_0^t e^{-(\Lambda+\kappa^2/2)\beta s + \kappa\beta W_s}\, ds \right]^{-\frac{1}{\beta}}, \quad 0 \leq t < \tau,$$

with

(9)
$$\tau := \inf \left\{ t \geq 0 \,\middle|\, \int_0^t e^{-(\Lambda+\kappa^2/2)\beta s + \kappa\beta W_s}\, ds \geq \frac{1}{\beta} v(0,\psi)^{-\beta} \right\}.$$

It follows that I exhibits finite time blowup on the event $[\tau < \infty]$. Since $I \leq v(\cdot,\psi)$, τ is an upper bound for the blow-up time of $v(\cdot,\psi)$, and therefore for the blowup times of v and u.

Corollary 2. *The function*

$$v(t,\psi)=\int_D v(t,x)\psi(x)\,dx$$

explodes in finite time on the event $[\tau<\infty]$. *Moreover, since by assumption* $\int_D \psi(x)\,dx=1$, $v(t,x)$ *cannot be bounded on* $[\tau<\infty]$. *Therefore*

$$\limsup_{t\uparrow\tau}\sup_{x\in D} v(t,x)=+\infty \text{ a.s. on } [\tau<\infty].$$

Hence $u(t,x)=e^{\kappa W_t}v(t,x)$ *also explodes in finite time if* $\tau<\infty$.

Remark. In the deterministic case $\kappa=0$, with a nonnegative $f\in L^2(D)$, Fujita proved that the condition

$$\int_D f(x)\psi(x)\,dx>\Lambda^{1/\beta} \tag{10}$$

implies finite time blowup of (1); see [3]. For $\kappa\neq 0$ we have shown above that

$$P[\tau=+\infty]=P\left[\int_0^\infty \exp(-(\Lambda+\kappa^2/2)\beta s+\kappa\beta W_s)\,ds\le\frac{1}{\beta}v(0,\psi)^{-\beta}\right].$$

Putting $\kappa=0$ in this equality we recover Fujita's result, namely

$$P[\tau<+\infty]=\begin{cases}1 & \\ & \text{according to} \\ 0 & \end{cases}\quad\begin{matrix}\int_D f(x)\psi(x)\,dx>\Lambda^{1/\beta}\\ \\ \int_D f(x)\psi(x)\,dx\le\Lambda^{1/\beta}.\end{matrix}$$

Remark. Setting the initial value $f=k\psi$ with $k>0$, gives

$$v(0,\psi)=k\int_D \psi^2(x)\,dx.$$

Plugging this into (9) yields the random time σ^* defined in Section 1.

4. A lower bound for ϱ. In this section we are going to obtain a random time σ_* which satisfies $\sigma_*\le\varrho$.

Let v solve Equation (8). We define the operator $\mathfrak{R}$ by

$$\begin{aligned}\mathfrak{R}w(t,x)\;&=\;e^{-t\kappa^2/2}T_tf(x)+\int_0^t e^{\kappa\beta W_r-\kappa^2(t-r)/2}T_{t-r}\left(w(r,\cdot)^{1+\beta}\right)(x)\,dr,\\ &\quad\; x\in D,\ t\ge 0,\end{aligned}$$

where w is any nonnegative, bounded and measurable function. Moreover, we put

$$\mathfrak{B}(t) = \left[1 - \beta \int_0^t e^{\kappa\beta W_r} \|e^{-r\kappa^2/2} T_r f\|_\infty^\beta dr\right]^{-1/\beta}, \quad 0 \le t < \tau,$$

where

$$\tau = \inf\left\{t > 0 : \int_0^t e^{\kappa\beta W_r} \|e^{-r\kappa^2/2} T_r f\|_\infty^\beta \, dr \ge 1/\beta\right\}.$$

Then we have

$$\frac{d\mathfrak{B}(t)}{dt} = e^{\kappa\beta W_t} \|e^{-t\kappa^2/2} T_t f\|_\infty^\beta \mathfrak{B}^{1+\beta}(t), \quad \mathfrak{B}(0) = 1,$$

hence

$$\mathfrak{B}(t) = 1 + \int_0^t e^{\kappa\beta W_r} \|e^{-r\kappa^2/2} T_r f\|_\infty^\beta \mathfrak{B}(r)^{1+\beta} dr.$$

Choose $w \ge 0$ such that $w(t,x) \le e^{-t\kappa^2/2} T_t f(x) \mathfrak{B}(t)$ for $x \in D$ and $t < \tau$. Then $e^{-t\kappa^2/2} T_t f(x) \le \mathfrak{R}w(t,x)$ and

$$\begin{aligned}
&\mathfrak{R}w(t,x) \\
&= e^{-t\kappa^2/2} T_t f(x) + \int_0^t e^{\kappa\beta W_r - \kappa^2(t-r)/2} T_{t-r}\left(w(r,\cdot)^{1+\beta}\right)(x)\, dr \\
&\le e^{-t\kappa^2/2} T_t f(x) \\
&\quad + \int_0^t e^{\kappa\beta W_r - \kappa^2(t-r)/2} \mathfrak{B}(r)^\beta e^{-r\beta\kappa^2/2} \|T_r f\|_\infty^\beta \mathfrak{B}(r) e^{-r\kappa^2/2} T_{t-r}(T_r f)(x)\, dr \\
&= e^{-t\kappa^2/2} T_t f(x) \left[1 + \int_0^t e^{\kappa\beta W_r} \left\|e^{-r\kappa^2/2} T_r f\right\|_\infty^\beta \mathfrak{B}^{1+\beta}(r)\, dr\right] \\
&= e^{-t\kappa^2/2} T_t f(x) \mathfrak{B}(t).
\end{aligned} \tag{11}$$

For $x \in D$ and $0 \le t \le \tau$, let

$$v^{(0)}(t,x) = e^{-t\kappa^2/2} T_t f(x) \quad \text{and} \quad v^{(n)}(t,x) = \mathfrak{R}v^{(n-1)}(t,x), \quad n \ge 1.$$

Using induction, one can easily prove that the function sequence $\{v^{(n)}\}$ is increasing, and therefore the limit

$$v(t,x) = \lim_{n\to\infty} v^{(n)}(t.x)$$

exists for all $x \in D$ and $0 \le t < \tau$. The monotone convergence theorem implies that

$$v(t,x) = \Re v(t,x) \text{ for } x \in D \text{ and } 0 \le t < \tau,$$

i.e. the function $v(t,x)$ solves (8) on $[0,\tau) \times D$. Moreover, because of (11),

$$v(t,x) \le \frac{e^{-t\kappa^2/2} T_t f(x)}{\left[1 - \beta \int_0^t e^{\kappa\beta W_r} \| e^{-r\kappa^2/2} T_r f\|_\infty^\beta \, dr\right]^{1/\beta}} \ < \ \infty$$

as long as $\int_0^t e^{\kappa\beta W_r} \|e^{-r\kappa^2/2} T_r f\|_\infty^\beta \, dr < 1/\beta$. It follows that the blowup time of (8) is lower bounded by τ. Setting the initial value of (8) of the form $f(x) = k\psi(x)$ with $k > 0$, we obtain $\tau = \sigma_*$.

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