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ON THE POISSON PROCESS OF ORDER *k*

Krasimira Y. Kostadinova* and Leda D. Minkova

Abstract. In this notes, the Poisson process of order k as a compound Poisson process is analyzed. We give a brief review of the distributions of order k. Then, some properties of the Poisson process of order k are given as well as probability mass function and recursion formulas. We then describe the defined process as a compound birth process. As application we consider the standard risk model which counting process is the Poisson process of order k. For the Poisson of order k risk model we derive the joint distribution of the time to ruin and the deficit at ruin. As a limiting case we obtain an equation for the ruin probability. We discuss in detail the particular case of exponentially distributed claims.

1. Introduction. The homogeneous Poisson process with constant intensity is the main counting process in the classical risk model, see for example, Grandell (1991), [6] and Rolski et all. (1999), [14]. Having realized the necessity of introducing more realistic model, we are faced with the problem of finding suitable way to describe the counting process. In this paper we suppose that the counting process is the Poisson process of order k, defined by Philippou A. (1983), [11] and Charalambides (1986), [3]. The probability distributions of order k are introduced by Philippou A., et all. (1983), [12] and Philippou and Makri (1986), [13], see Philippou (1983), [11] and Hirano K. (1986) also, [7]. The geometric distribution of order k, $(Ge_k(p))$ is defined by the number of trials until

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the first occurrence of k consecutive successes in a sequence of independent trials with success probability p. The negative binomial $(NB_k(r,p))$ distribution of order k is the distribution of the sum of r independent, identically $Ge_k(p)$ distributed random variables. Then, the Poisson distribution of order k, $(Po_k(\lambda))$, was obtained as a limiting distribution of a sequence of shifted NB_k distributed random variables. A good reference for the distributions of order k is the book of Balakrishnan and Koutras (2002), [2].

Charalambides (1986), [3] proved that the discrete distributions of order k can be represented as Compound Generalized Powers Series Distributions, see Aki et all. (1984) also, [1]. The compounding distribution is a discrete distribution over $k \geq 1$ points.

Our interest is related to the application of a compound Poisson process in risk models. Minkova (2010), [9], considered the case of truncated geometric compounding distribution and the resulting process is called a Pólya-Aeppli of order k process. In Chukova and Minkova (2013), [4], the Pólya-Aeppli of order k risk model is defined and ruin probability is estimated.

In this study, in the next Section 2, we discuss the Poisson process of order k as a compound Poisson process with discrete uniform compounding distribution. Then, we consider this process as a pure birth process. In Section 3 we consider the Poisson risk model of order k and derive a differential equation for the joint distribution of the time to ruin and the deficit at the time of ruin and an expression for the ruin probability. The results are illustrated for the particular case of exponentially distributed claims.

2. Poisson process of order *k*. In this section we consider some properties of the Poisson process of order k as a compound Poisson process and later we give a second definition for this process as a compound birth process.

We consider the stochastic process $N(t),\ t > 0$ defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ and given by

$$N(t) = X_1 + X_2 + \ldots + X_{N_1(t)}, \tag{1}$$

where $X_i,\ i = 1, 2, \ldots$ are independent, identically distributed (iid) as X random variables, independent of $N_1(t)$. We suppose that the counting process $N_1(t)$ is a Poisson process with intensity $k\lambda > 0$ $(N_1(t) \sim Po(k\lambda t))$. In this case $N(t)$ is a compound Poisson process. The probability mass function (PMF) and probability generating function (PGF) of $N_1(t)$ are given by

$$P(N_1(t) = i) = \frac{(k\lambda t)^i e^{-k\lambda t}}{i!}, \quad i = 0, 1, \ldots \tag{2}$$

and

$$\psi_{N_1(t)}(s) = e^{-k\lambda t(1-s)}. \tag{3}$$

Let the compounding random variable X be a discrete uniformly distributed over $k > 1$ points with PMF

$$P(X = i) = \frac{1}{k}, \quad i = 1, 2, \dots, k. \tag{4}$$

The PGF of X, $\psi_X(s) = E(s^X)$, $s \in (0, 1)$ is then given by

$$\psi_X(s) = \frac{s}{k}\frac{1 - s^k}{1 - s}. \tag{5}$$

For the PGF of the process $N(t)$, given in (1) we get

$$\psi_{N(t)}(s) = e^{-k\lambda t(1-\psi_X(s))}, \tag{6}$$

where $\psi_X(s)$ is the PGF of the compounding distribution, given by (5).

Definition 1. *The stochastic process, defined by the PGF* (6) *and compounding distribution, defined by* (5) *is called a Poisson process of order k with parameter λ,* $(N(t) \sim Po_k(\lambda t))$.

Remark 1. If $k = 1$, the discrete uniform distribution in (4) degenerates at point one, and the process $N(t)$ is a homogeneous Poisson process.

Theorem 1. *The probability mass function of the $Po_k(\lambda t)$ process is given by:*

$$\begin{aligned}
p_0 &= e^{-k\lambda t}, \\
p_i &= e^{-k\lambda t}\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{(\lambda t)^j}{j!}, \quad i = 1, 2, \dots, k \\
p_i &= e^{-k\lambda t}\left[\sum_{j=1}^{i}\binom{i-1}{j-1}\frac{(\lambda t)^j}{j!} - \right. \\
&\left. - \sum_{n=1}^{l}(-1)^{n-1}\frac{(\lambda t)^n}{n!}\sum_{j=0}^{i-n(k+1)}\binom{i-n(k+1)+n-1}{j+n-1}\frac{(\lambda t)^j}{j!}\right], \\
&\qquad i = l(k+1) + m, \ m = 0, 1, \dots k, \ l = 1, 2, \dots \infty.
\end{aligned}$$

P r o o f. The PMF is obtained by equating the coefficients of s^i on both sides of the Taylor expansion of the PGF in (6). □

Remark 2. The mean and the variance of the Poisson process of order k are given by

$$E(N(t)) = \frac{1+k}{2}k\lambda t \quad \text{and} \quad Var(N(t)) = \frac{(k+1)(2k+1)}{6}k\lambda t.$$

For the Fisher index of dispersion we get

$$FI(N(t)) = \frac{Var(N(t))}{E(N(t))} = 1 + \frac{2}{3}(k-1) > 1,$$

i. e. the Poisson process of order k is over-dispersed.

The following proposition gives an extension of the Panjer recursion formulas, (see Panjer (1981), [10]).

Proposition 1. *The PMF of the Poisson process of order k satisfies the following recursions:*

$$p_i = \begin{cases} \lambda t p_0, & i = 1, \\ \left(2 + \frac{\lambda t - 2}{i}\right) p_{i-1} - \left(1 - \frac{2}{i}\right) p_{i-2}, & i = 2, 3, \dots, k \\ \left(2 + \frac{\lambda t - 2}{i}\right) p_{i-1} - \left(1 - \frac{2}{i}\right) p_{i-2} - \frac{k+1}{i}\lambda t p_{i-k-1} + \frac{k}{i}\lambda t p_{i-k-2}, & \\ & i = k+1, k+2, \dots \end{cases}$$

P r o o f. Differentiation in (6) leads to

$$\psi'_{N(t)}(s) = k\lambda t \psi'_X(s)\psi_{N(t)}(s), \tag{7}$$

where $\psi_{N(t)}(s) = \sum_{i=0}^{\infty} p_i s^i$, $\psi'_{N(t)}(s) = \sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$, and

$$\psi'_X(s) = \frac{1-(k+1)s^k + ks^{k+1}}{k(1-s)^2}$$

is the derivative of (5). So, the equation (7) has the form

$$(1-s)^2 \sum_{i=0}^{\infty}(i+1)p_{i+1}s^i = \lambda t(1-(k+1)s^k + ks^{k+1}) \sum_{i=0}^{\infty} p_i s^i.$$

The recursions are obtained by equating the coefficients of s^i on both sides for fixed $i = 0, 1, 2, \ldots$. $\square$

In a similar way, substituting $\psi'_X(s) = \dfrac{1}{k}\displaystyle\sum_{j=1}^{k} js^{j-1}$ in (7), we obtain the recursions in the following proposition.

Proposition 2. *The PMF of the Poisson process of order k satisfies the recursions:*

$$p_i = \begin{cases} \lambda t \displaystyle\sum_{j=0}^{i-1}\left(1-\frac{j}{i}\right)p_j, & i = 1, 2, \ldots, k \\ \lambda t \displaystyle\sum_{j=i-k}^{i-1}\left(1-\frac{j}{i}\right)p_j, & i = k+1, k+2, \ldots \end{cases} \tag{8}$$

Remark 3. It is easy to check that the recursions (8) are equivalent to

$$p_i = \begin{cases} \lambda t \displaystyle\sum_{j=1}^{i}\frac{j}{i}p_{i-j}, & i = 1, 2, \ldots, k \\ \lambda t \displaystyle\sum_{j=1}^{k}\frac{j}{i}p_{i-j}, & i = k+1, k+2, \ldots \end{cases}$$

2.1. Poisson process of order *k* as a pure birth process. Let $\{N(t), t \geq 0\}$ be the number of times a certain event occurs in time interval $(0, t]$. The transition probabilities of the counting process $N(t)$, for every $m = 0, 1, \ldots$ are specified by the following postulates:

$$P(N(t+h) = n \mid N(t) = m) = \begin{cases} 1 - k\lambda h + o(h), & n = m, \\ \lambda h + o(h), & n = m+i,\ i = 1, 2, \ldots, k, \end{cases}$$

where $o(h) \to 0$ as $h \to 0$. Note that the postulates imply that for $i = k+1, k+2, \ldots, P(N(t+h) = m+i \mid N(t) = m) = o(h)$.

Let $P_m(t) = P(N(t) = m),\ m = 0, 1, 2, \ldots$. Then the above postulates yield the following Kolmogorov forward equations:

$$P'_0(t) = -k\lambda P_0(t),$$

(9)
$$P'_m(t) = -k\lambda P_m(t) + \lambda \sum_{j=1}^{m\wedge k} P_{m-j}(t),\ m = 1, 2, \ldots .$$

with initial conditions

(10) $$P_0(0) = 1 \quad \text{and} \quad P_m(0) = 0, \quad m = 1, 2, \ldots .$$

Multiplying the mth equation of (9) by s^m and summing for all $m = 0, 1, 2, \ldots$ we get the following differential equation for the PGF

(11) $$\frac{\partial \psi_{N(t)}(s)}{\partial t} = -k\lambda[1 - \psi_X(s)]\psi_{N(t)}(s).$$

The solution of (11) with the initial condition $\psi_{N(t)}(0) = 1$ is given by

(12) $$\psi_{N(t)}(s) = e^{-k\lambda t(1-\psi_X(s))},$$

where $\psi_X(s)$ is the PGF of the discrete uniform distribution over k points. (12) is the PGF of the $Po_k(\lambda t)$ process. This leads to the second definition.

Definition 2. *The stochastic process, defined by* (9) *and* (10) *is called a Poisson process of order k.*

Remark 4. For $k = 1$, the process is a homogeneous Poisson process with intensity λ.

2.2. Compound Poisson decomposition. Let us rewrite the PGF of $N(t) \sim Po_k(\lambda t)$ in the following way:

$$\begin{aligned} \psi_{N(t)}(s) &= \exp\left(-k\lambda t\left[1 - \frac{1}{k}(s + s^2 + \ldots + s^k)\right]\right) \\ &= \prod_{i=1}^{k} e^{-\lambda t(1-s^i)}. \end{aligned}$$

This means that $N(t)$ can be represented as a sum of k independent compound Poisson processes $M_1(t), \ldots, M_k(t)$ with means $EM_i(t) = i\lambda t$ and PGFs $\psi_{M_i(t)}(s) = e^{-\lambda t(1-s^i)},\ i = 1, 2, \ldots, k$.

3. Application to Risk Theory. Consider the standard risk model $\{X(t),\ t \geq 0\}$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and given by

$$X(t) = ct - \sum_{i=1}^{N(t)} Z_i, \quad \left(\sum_{1}^{0} = 0 \right). \tag{13}$$

Here c is a positive real constant representing the risk premium rate. The sequence $\{Z_i\}_{i=1}^{\infty}$ of non-negative iid random variables is independent of the counting process $N(t),\ t \geq 0$. The claim sizes $\{Z_i\}_{i=1}^{\infty}$ are distributed as the random variable Z with distribution function $F,\ F(0) = 0$ and mean value μ.

We consider the risk model (13), where $N(t)$ is a Poisson process of order k and we call this process Poisson of order k risk model. The interpretation of the counting process is the following. If the insurance policies are separated in independent groups, then the number of groups has a Poisson distribution. We suppose that the groups are homogeneous, identically distributed. The number of policies in each of the groups has a discrete uniform distribution over k points.

The relative safety loading θ for the Poisson of order k risk model in (13), is given by

$$\theta = \frac{EX(t)}{E\sum_{i=1}^{N(t)} Z_i} = \frac{2c}{k(k+1)\lambda\mu} - 1.$$

In the case of positive safety loading $\theta > 0$, the premium income per unit time c should satisfy the following inequality

$$c > \frac{k(k+1)}{2}\lambda\mu.$$

Let $\tau = \inf\{t : X(t) < -u\}$ with the convention of $\inf \emptyset = \infty$ be the time to ruin of an insurance company having initial capital $u \geq 0$. We denote by $\Psi(u) = P(\tau < \infty)$ the ruin probability and by $\Phi(u) = 1 - \Psi(u)$ the non-ruin probability. The main in the application is to analyze for this model the joint probability distribution $G(u, y)$ of the time to ruin τ and the deficit at the time of ruin $D = |u + X(t)|$. The function $G(u, y)$ was introduce by Gerber et al. (1987), [5], see Klugman et al. (2004) also, [8], and is given by

$$G(u, y) = P(\tau < \infty, D \leq y), \quad y \geq 0. \tag{14}$$

It is clear that

$$\lim_{y \longrightarrow \infty} G(u, y) = \Psi(u). \tag{15}$$

Using the postulates, we get

$$G(u,y) = (1-k\lambda h)\, G(u+ch,y)$$

$$+\lambda h\left[\int\limits_0^{u+ch} G(u+ch-x,y)dF(x) + (F(u+ch+y)-F(u+ch))\right]+$$

$$+\lambda h\left[\int\limits_0^{u+ch} G(u+ch-x,y)dF^{\star 2}(x) + \big(F^{\star 2}(u+ch+y)-F^{\star 2}(u+ch)\big)\right]$$

$$+\cdots$$

$$+\lambda h\left[\int\limits_0^{u+ch} G(u+ch-x,y)dF^{\star k}(x) + \Big(F^{\star k}(u+ch+y)-F^{\star k}(u+ch)\Big)\right]+o(h),$$

where $F^{\star i}(x),\quad i=1,2,\ldots,k$ is the distribution function of $Z_1+Z_2+\ldots+Z_i$. Rearranging the terms leads to

$$(16)\quad \frac{G(u+ch,y)-G(u,y)}{ch} = \frac{k\lambda}{c}G(u+ch,y) - \frac{\lambda}{c}\sum_{i=1}^{k}\left[\int\limits_0^{u+ch} G(u+ch-x,y)dF^{\star i}(x) + F^{\star i}(u+ch+y)-F^{\star i}(u+ch)\right]+\frac{o(h)}{h}.$$

Let us denote by

$$(17)\qquad H(x) = \frac{\lambda}{k}\sum_{i=1}^{k} F^{*i}(x),$$

the probability distribution function of the aggregated claims. It follows from (17), that $H(0)=0$ and $H(\infty)=\lambda$, i.e., $H(x)$ is a defective distribution function. By letting $h\to 0$ in (16), we obtain the following differential equation

$$(18)\quad \frac{\partial G(u,y)}{\partial u} = \frac{k\lambda}{c}G(u,y) - \frac{k}{c}\left[\int_0^u G(u-x,y)dH(x) + [H(u+y)-H(u)]\right].$$

Let

$$H_1(x) = \frac{H(x)}{\lambda} \tag{19}$$

be the proper distribution function of the aggregated claims. In terms of $H_1(x)$, the equation (18) has the form

$$\frac{\partial G(u,y)}{\partial u} = \frac{k\lambda}{c}\left[G(u,y) - \int_0^u G(u-x,y)dH_1(x) - [H_1(u+y) - H_1(u)]\right]. \tag{20}$$

Theorem 2. *The function $G(0,y)$ is given by*

$$G(0,y) = \frac{k\lambda}{c}\int_0^y [1 - H_1(u)]du. \tag{21}$$

Proof. Integrating (20) from 0 to ∞ with $G(\infty, y) = 0$ leads to

$$-G(0,y) = \frac{k\lambda}{c}\left[\int_0^\infty G(u,y)du - \int_0^\infty\int_0^u G(u-x,y)dH_1(x)du - \int_0^\infty (H_1(u+y) - H_1(u))du\right]$$

The change of variables in the double integral and simple calculations yield

$$G(0,y) = \frac{k\lambda}{c}\int_0^\infty [H_1(u+y) - H_1(u)]du$$

and then (21). □

3.1. Ruin probability.

Theorem 3. *For $u \geq 0$, the ruin probability $\Psi(u)$ satisfies the equation*

$$\frac{\partial \Psi(u)}{\partial u} = \frac{k\lambda}{c}\left[\Psi(u) - \int_0^u \Psi(u-x)dH_1(x) - [1 - H_1(u)]\right], \tag{22}$$

Proof. The result follows from (20) and (15). □

Remark 5. The nonruin probability $\Phi(u)$ satisfies the equation

$$\frac{\partial \Phi(u)}{\partial u} = \frac{k\lambda}{c}\left[\Phi(u) - \int_0^u \Phi(u-x)dH_1(x)\right]. \tag{23}$$

Theorem 4. *The ruin probability with no initial capital satisfies*

$$\Psi(0) = \frac{k(k+1)\lambda\mu}{2c}. \tag{24}$$

P r o o f. According to (15) and (21)

$$\Psi(0) = \lim_{y\to\infty} G(0,y) = \frac{k\lambda}{c}\int_0^\infty [1-H_1(u)]du.$$

If X is a random variable with distribution function $H_1(x)$, then, by the definition of $H_1(x)$ and $EZ = \mu$ we obtain

$$EX = \frac{1}{k}(\mu + 2\mu + \ldots + k\mu) = \frac{(k+1)\mu}{2}.$$

Using the fact that $EX = \int_0^\infty [1-H_1(x)]dx$ we obtain (24). □

Remark 6. Based on (24), it is easy to see that the ruin probability with no initial capital does not depend on t.

3.2. Exponentially distributed claims. Let us consider the case of exponentially distributed claim sizes, i.e. $F(x) = 1 - e^{-\frac{x}{\mu}},\ x \geq 0,\ \mu > 0$. In this case, the function $H_1(x)$ is a mixture of Erlang distribution functions:

$$H_1(x) = \frac{1}{k}\sum_{i=1}^{k} F^{*i}(x),$$

where $F^{*i}(x)$ is the distribution function of $Erlang(i,\mu)$ distributed random variable. The corresponding density function is given by

$$h_1(x) = \frac{1}{\mu k}\sum_{i=1}^{k}\frac{\left(\frac{x}{\mu}\right)^{i-1}}{(i-1)!}e^{-\frac{x}{\mu}},\ x > 0.$$

Denote by $\Gamma(x,\alpha) = \int_x^\infty t^{\alpha-1}e^{-t}dt$ and $\gamma(x,\alpha) = \int_0^x t^{\alpha-1}e^{-t}dt$ the incomplete Gamma functions. Then, for the survival function we obtain

$$\overline{H}_1(x) = \frac{1}{k}\sum_{i=1}^{k}\frac{\Gamma\left(\frac{x}{\mu}, i\right)}{\Gamma(i)},\ x > 0. \tag{25}$$

The initial condition (21), in the case of exponentially distributed claims is given by

$$G(0,y) = \frac{\lambda}{c}\sum_{i=1}^{k}\frac{1}{\Gamma(i)}\left[\gamma\left(\frac{y}{\mu}, i+1\right) + y\Gamma\left(\frac{y}{\mu}, i\right)\right].$$

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Krasimira Y. Kostadinova
Faculty of Mathematics and Informatics
Shumen University "Konstantin Preslavsky"
Shunem, Bulgaria
e-mail: `kostadinova@shu-bg.net`

Leda D. Minkova
Faculty of Mathematics and Informatics
Sofia University "St. Kl. Ohridski"
Sofia, Bulgaria
e-mail: `leda@fmi.uni-sofia.bg`