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MULTIPLIERS ON SPACES OF FUNCTIONS ON A LOCALLY COMPACT ABELIAN GROUP WITH VALUES IN A HILBERT SPACE

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We prove a representation theorem for bounded operators commuting with translations on $L^2_\omega(G, H)$, where G is a locally compact abelian group, H is a Hilbert space and ω is a weight on G. Moreover, in the particular case when $G = \mathbb{R}$, we characterize completely the spectrum of the shift operator $S_{1,\omega}$ on $L^2_\omega(\mathbb{R}, H)$.

1. Introduction. Let G be a locally compact abelian group. Denote by $\widehat{G}$ the dual group of G. The groups G and $\widehat{G}$ are equipped with the Haar measure. Let H be a separable Hilbert space and denote by $\langle u, v\rangle$ the scalar product of two elements u and v in H. Let ω be a weight on G i.e. ω is a continuous, positive, measurable function on G such that

$$0 < \sup_{x\in G} \frac{\omega(x+y)}{\omega(x)} < +\infty,\ \forall y \in G.$$

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For $1 \leq p < +\infty$, we denote by $L^p_\omega(G,H)$ the space of the functions f on G with values in H such that

$$G \ni x \longrightarrow \|f(x)\| \in \mathbb{R}^+$$

is a function in $L^p_\omega(G)$, where $L^p_\omega(G)$ is the set of the measurable functions g on G such that

$$\int_G |g(x)|^p \omega(x)^p dx < +\infty.$$

Let $C_c(G)$ be the space of the continuous functions from G into $\mathbb{C}$ with compact support. Denote by $C_c(G,H)$ the space of the functions f on G with values in H such that $\|f(.)\| \in C_c(G)$. For $a \in G$, define

$$S_{a,\omega} : L^p_\omega(G,H) \longrightarrow L^p_\omega(G,H)$$

by the formula

$$(S_{a,\omega} f)(x) = f(x-a),\ \forall f \in L^p_\omega(G,H),\ \ a.e.$$

and let

$$\mathbf{S}_{a,\omega} : L^p_\omega(G) \longrightarrow L^p_\omega(G)$$

be the operator defined by the formula

$$(\mathbf{S}_{a,\omega} g)(x) = g(x-a),\ \forall g \in L^p_\omega(G),\ a.e.$$

Notice that we have

$$\|S_{a,\omega}\| = \|\mathbf{S}_{a,\omega}\| = \sup_{x\in G} \frac{\omega(x+a)}{\omega(x)},\ \forall a \in G,$$

and consequently

$$\rho(S_{a,\omega}) = \rho(\mathbf{S}_{a,\omega}),\ \forall a \in G.$$

Here $\rho(S_{a,\omega})$ (resp. $\rho(\mathbf{S}_{a,\omega})$) denotes the spectral radius of $S_{a,\omega}$ (resp. $\mathbf{S}_{a,\omega}$). Denote by $C_c(G) \otimes H$ the closed vector space generated by functions

$$fu : G \ni x \longrightarrow f(x)u \in H$$

with $f \in C_c(G)$ and $u \in H$. The space $C_c(G) \otimes H$ is dense in $L^p_\omega(G,H)$, for $1 \leq p < +\infty$. We say that M is a multiplier on $L^p_\omega(G,H)$ if M is a bounded operator from $L^p_\omega(G,H)$ into $L^p_\omega(G,H)$ such that

$$MS_a = S_a M,\ \forall a \in G.$$

Define $\mathcal{M}^p_\omega$ the algebra of the multipliers on $L^p_\omega(G,H)$. We denote by $\mathcal{F}$ (resp. $\mathcal{F}$) the usual Fourier transformation from $L^2(G,H)$ (resp. $L^2(G)$) into $L^2(\widehat{G},H)$ (resp. $L^2(\widehat{G})$). We have the following representation theorem for the multipliers on $L^p(G,H)$.

Theorem 1 ([2]). *For every M multiplier on $L^p(G,H)$, $1 \le p < +\infty$, there exists a measurable function*

$$\Phi_M : \widehat{G} \longrightarrow \mathcal{L}(H),$$

which is essentially bounded for the operator norm of $\mathcal{L}(H)$ such that

$$\mathcal{F}(Mf)(\chi) = \Phi_M(\chi)[\mathcal{F}(f)(\chi)], \ a.e. \text{ on } \widehat{G},$$

for every $f \in L^p(G,H) \cap L^2(G,H)$. Moreover,

$$\operatorname*{ess\,sup}_{\chi \in \widehat{G}} \|\Phi_M(\chi)\| \le \|M\|.$$

The proof of this theorem is based on the well-known result about the multipliers on $L^p(G)$. Indeed, for every bounded operator M commuting with the translations on $L^p(G)$ there exists a function $h \in L^\infty(\widehat{G})$ (see [3]) such that

$$\widehat{Mf} = h\hat{f}, \ \forall f \in C_c(G) \tag{1.1}$$

and $\|h\|_\infty \le \|M\|$. This paper is motivated by a recent result generalizing the representation (1.1) for a more general class of Banach spaces of functions on G. The spaces $L^p_\omega(G)$ are included in this class. Denote by $\widetilde{G^p_\omega}$ the set of the continuous morphisms θ from G into $\mathbb{C}^*$ such that

$$\Big| \int_G f(x)\theta^{-1}(x)dx \Big| \le \|M_f\|_{\mathcal{L}(L^p_\omega(G))}, \tag{1.2}$$

where M_f is the operator of convolution by f on $L^p_\omega(G)$. Define

$$\widetilde{G^{p+}_\omega} = \{|\theta|, \ \theta \in \widetilde{G^p_\omega}\}.$$

It was proved in [6] that the set $\widetilde{G^{p+}_\omega}$ is not empty, log-convex and compact for the topology of the uniform convergence on every compact set of G. It is clear that $\widetilde{G^p_\omega} = \widetilde{G^{p+}_\omega}\widehat{G}$. Let $\widetilde{G}$ be the set of the continuous morphisms from G into $\mathbb{C}^*$. We have the following proposition.

Proposition 1 (see [6], [7]). *If G is either a discrete group or a compact group, we have*

$$\widetilde{G^p_\omega} = \{\theta \in \widetilde{G} \mid |\theta^{-1}(x)| \leq \rho(S_{x,\omega}),\ \forall x \in G\}$$

and $\widetilde{G^p_\omega}$ is isomorphic to the joint spectrum of $\{S_{x,\omega}\}_{x\in G}$. The same result holds for $G = \mathbb{R}$.

Also in [6], was proved the following result, which we will use.

Theorem 2 ([6], [7]). *Fix $\theta \in \widetilde{G^p_\omega}$. For every bounded operator M commuting with the translations on $L^p_\omega(G)$, we have $(Mg)\theta^{-1} \in L^2(G)$, $\forall g \in C_c(G)$. There exists a function $h_{M,\theta} \in L^\infty(\widehat{G})$ such that*

$$\widehat{(Mg)\theta^{-1}} = h_{M,\theta}\widehat{(g\theta^{-1})},\ \forall g \in C_c(G)$$

and $\|h_{M,\theta}\|_\infty \leq C_\omega \|M\|$, where C_ω is a constant independent of M.

The main result in this paper is the following.

Theorem 3. *For $M \in \mathcal{M}^p_\omega$ and $\theta \in \widetilde{G^p_\omega}$, we have:*
1) $(Mg)\theta^{-1} \in L^2(G,H)$, $\forall g \in C_c(G) \otimes H$.
2) *There exists $\Phi_\theta \in L^\infty(\widehat{G}, \mathcal{L}(H))$ such that*

$$\mathcal{F}((Mg)\theta^{-1})(\chi) = \Phi_\theta(\chi)[\mathcal{F}(g\theta^{-1})(\chi)],\ \forall g \in C_c(G) \otimes H,\ a.e.$$

Moreover, $\operatorname{ess\,sup}_{\chi\in\widehat{G}} \|\Phi_\theta(\chi)\| \leq C_\omega \|M\|$.

2. Proof of Theorem 3. Since Theorem 2 plays an important role in the proof of Theorem 3, for the convenience of the reader we give a sketch of it's proof. The full proof is exposed in [7] and [6].

P r o o f o f T h e o r e m 2. First, every multiplier M in $L^p_\omega(G)$ is the limit for the strong operators topology of a net (M_{ϕ_α}) where $\phi_\alpha \in C_c(G)$ and $\|M_{\phi_\alpha}\| \leq C_\omega \|M\|$. This may be proved using the fact that the restriction of every multiplier on $C_c(G)$ is a convolution with a quasimeasure (see [2]). Fix $M \in \mathcal{M}^p_\omega$ and let (M_{ϕ_α}) be a net satisfying the above property. Fix $\theta \in \widetilde{G^p_\omega}$. From the definition of $\widehat{G}$, it follows that

$$\left|\widehat{\phi_\alpha\theta^{-1}}(\chi)\right| \leq \|M_{\phi_\alpha}\| \leq C_\omega \|M\|,\ \forall \chi \in \widetilde{G^p_\omega}.$$

If we replace (ϕ_α) by a suitable subnet, we obtain that $(\widehat{\phi_\alpha\theta^{-1}})$ converges to a function $h_{M,\theta} \in L^\infty(\widehat{G})$ for the weak* topology $\sigma(L^\infty(\widehat{G}), L^1(\widehat{G}))$. This implies that for each $f \in C_c(G)$, the net

$$\Big(\mathcal{F}((M_{\phi_\alpha} f)\theta^{-1})\Big) = \Big(\mathcal{F}((\phi_\alpha * f)\theta^{-1})\Big) = \Big(\widehat{\phi_\alpha\theta^{-1}}\,\widehat{f\theta^{-1}}\Big)$$

converges to $h_{M,\theta}\widehat{f\theta^{-1}}$ with respect to the weak topology of $L^2(\widehat{G})$. Consequently,

$$\lim_\alpha (M_{\phi_\alpha} f)\theta^{-1} = \mathcal{F}^{-1}(h_{M,\theta}\widehat{f\theta^{-1}}),\ \forall f \in C_c(G)$$

with respect to the weak topology of $L^2(G)$. On the other hand, since $L^p_\omega(G) \subset L^1_{loc}(G)$ and the inclusion is continuous, for $g \in C_c(G)$, we get

$$\lim_\alpha \Big| \int_G g(y)\theta^{-1}(y)\Big(M_{\phi_\alpha} f(y) - Mf(y)\Big)dy\Big| = 0.$$

We conclude that for every $f \in C_c(G)$ the functions $(Mf)\theta^{-1}$ and $\mathcal{F}^{-1}(h_{M,\theta}\widehat{f\theta^{-1}})$ define the same linear functional on $C_c(G)$ and so $(Mf)\theta^{-1}(x) = \mathcal{F}^{-1}(h_{M,\theta}\widehat{f\theta^{-1}})(x)$, for almost every $x \in G$. We conclude that $(Mf)\theta^{-1} \in L^2(G)$ and

$$\widehat{(Mg)\theta^{-1}} = h_{M,\theta}\widehat{(g\theta^{-1})},\ \forall g \in C_c(G). \qquad \square$$

In order to proof Theorem 3, we need the following lemma.

Lemma 1. *Let $g \in L^2(G, H)$ and $v \in H$. Then we have*

$$\mathcal{F}(\langle g(.), v\rangle)(\chi) = \langle \mathcal{F}(g)(\chi), v\rangle,$$

for almost every $\chi \in \widehat{G}$.

Proof. Let $g \in L^2(G, H)$ and $(g_n)_{n\in\mathbb{N}} \subset C_c(G, H)$ be a sequence converging to g in $L^2(G, H)$. Then, we have

$$\mathcal{F}(\langle g(.), v\rangle) = \lim_{n\to+\infty} \mathcal{F}(\langle g_n(.), v\rangle),$$

with respect to the norm of $L^2(\widehat{G})$. For fixed $\chi \in \widehat{G}$ and $v \in H$, the map

$$C_c(\widehat{G}, H) \ni h \longrightarrow \langle h(\chi), v\rangle \in \mathbb{C}$$

is a continuous linear form. For given $\phi \in C_c(\widehat{G})$, the integral

$$\int_G g_n(x)\phi(\chi)\chi^{-1}(x)dx$$

is a convergent Bochner integral with values in $C_c(\widehat{G}, H)$. Indeed, we have

$$\int_G \sup_{\chi\in\widehat{G}} \|g_n(x)\phi(\chi)\chi^{-1}(x)\|dx$$

$$\leq \|\phi\|_\infty \int_G \|g_n(x)\|dx < +\infty.$$

Since the Bochner integral commutes with continuous linear maps, we have for almost every $\chi \in \widehat{G}$,

$$\phi(\chi)\mathcal{F}(\langle g_n(.), v\rangle)(\chi) = \phi(\chi)\langle \mathcal{F}(g_n)(\chi), v\rangle.$$

Since $\lim\limits_{n\to+\infty} \mathcal{F}(g_n) = \mathcal{F}(g)$ with respect to the norm of $L^2(\widehat{G}, H)$, if we replace $(g_n)_{n\in\mathbb{N}}$ by a suitable subsequence, we get

$$\lim_{n\to+\infty} \|\mathcal{F}(g_n)(\chi) - \mathcal{F}(g)(\chi)\| = 0, \ a.e.$$

and hence

$$\lim_{n\to+\infty} \langle \mathcal{F}(g_n)(\chi), v\rangle = \langle \mathcal{F}(g)(\chi), v\rangle, \ a.e.$$

We conclude that

$$\mathcal{F}(\langle g(.), v\rangle)(\chi) = \langle \mathcal{F}(g)(\chi), v\rangle,$$

for almost every $\chi \in \widehat{G}$. □

Proof of Theorem 3. Fix $M \in \mathcal{M}^p_\omega$ and fix u and $v \in H$. Introduce the operator $M_{u,v}$ defined for $f \in L^p_\omega(G)$ by the formula

$$M_{u,v}(f)(x) = \langle M(fu)(x), v\rangle, \ a.e. \tag{2.1}$$

Notice that $M_{u,v}(f) \in L^p_\omega(G)$ for every $f \in L^p_\omega(G)$. Indeed, since $M(fu) \in L^p_\omega(G, H)$, we have

$$\int_G |\langle M(fu)(x), v\rangle|^p \omega(x)^p dx$$

$$\leq \int_G \|M(fu)(x)\|^p \|v\|^p \omega(x)^p dx < +\infty.$$

Moreover, notice that

$$\|M_{u,v}\| \leq \|M\|\|u\|\|v\|.$$

It is clear that

$$\langle M(S_a(fu))(x), v\rangle = \langle M(fu)(x-a), v\rangle, \ a.e.$$

hence $M_{u,v}$ is a multiplier on $L^p_\omega(G)$. From Theorem 2, we obtain for every $\theta \in \widetilde{G^p_\omega}$,

$$(M_{u,v}f)\theta^{-1} \in L^2(G),\ \forall f \in C_c(G) \tag{2.2}$$

and there exists $\Phi_{\theta,u,v} \in L^\infty(\widehat{G})$ such that

$$\mathcal{F}((M_{u,v}f)\theta^{-1})(\chi) = \Phi_{\theta,u,v}(\chi)\mathcal{F}(f\theta^{-1})(\chi),\ a.e. \tag{2.3}$$

Let $\mathcal{O}$ be a countable orthonormal basis of H and let F be the set of finite linear combinations of elements of $\mathcal{O}$. We have

$$|\Phi_{\theta,u,v}(\chi)| \le C_\omega \|M_{u,v}\|,\ \forall \chi \in \widehat{G}\backslash N_{u,v},$$

where $N_{u,v}$ is a set of measure zero. Without loss of generality, we can modify $\Phi_{\theta,u,v}$ on $N = \cup_{(u,v)\in F\times F} N_{u,v}$ in order to obtain

$$|\Phi_{\theta,u,v}(\chi)| \le C_\omega \|M_{u,v}\| \le C_\omega \|M\|\|u\|\|v\|,\ \forall u,\ v \in F, a.e.$$

For fixed $\chi \in \widehat{G}\backslash N$

$$F\times F \ni (u,v) \longrightarrow \Phi_{\theta,u,v}(\chi) \in \mathbb{C}$$

is a sesquilinear and continuous form on $F \times F$ and since F is dense in H, we conclude that there exists an unique map

$$H \times H \ni (u,v) \longrightarrow \widetilde{\Phi}_{\theta,u,v}(\chi) \in \mathbb{C}$$

such that

$$\widetilde{\Phi}_{\theta,u,v}(\chi) = \Phi_{\theta,u,v}(\chi),\ \forall u, v \in F.$$

Consequently, there exists an unique map

$$\Phi_\theta : \widehat{G} \longrightarrow \mathcal{L}(H)$$

such that

$$\langle \Phi_\theta(\chi)[u], v\rangle = \widetilde{\Phi}_{\theta,u,v}(\chi),\ \forall u,\ v \in H.$$

It is clear that

$$\|\Phi_\theta(\chi)\| = \sup_{\|u\|=1,\|v\|=1} |\langle \Phi_\theta(\chi)[u], v\rangle| \le C_\omega \|M\|,\ a.e.$$

Fix $\theta \in \widetilde{G^p_\omega}$ and $f \in C_c(G)$. For every $\chi \in \widehat{G}$, we have $\widehat{f\theta^{-1}}(\chi)u \in H$. Next for almost every $\chi \in \widehat{G}$, we obtain

$$\langle \Phi_\theta(\chi)[\widehat{f\theta^{-1}}(\chi)u], v\rangle = \langle \Phi_\theta(\chi)[u], v\rangle \widehat{f\theta^{-1}}(\chi)$$

$$= \Phi_{\theta,u,v}(\chi)\widehat{f\theta^{-1}}(\chi) = \mathcal{F}((M_{u,v}f)\theta^{-1})(\chi)$$
$$= \mathcal{F}(\langle M[fu], v\rangle\theta^{-1})(\chi).$$

Consequently,

$$\mathcal{F}^{-1}(\langle\Phi_\theta(.)[\widehat{f\theta^{-1}}(.)u], v\rangle)(x) = \langle M[fu](x), v\rangle\theta^{-1}(x), \tag{2.4}$$

for almost every $x \in G$. Now, consider the function Ψ_θ on $\widehat{G}$ defined for almost every $\chi \in \widehat{G}$ by the formula

$$\Psi_\theta(\chi) = \Phi_\theta(\chi)[\widehat{f\theta^{-1}}(\chi)u]$$

and observe that $\Psi_\theta \in L^2(\widehat{G}, H)$. Indeed, we have

$$\int_{\widehat{G}} \|\Phi_\theta(\chi)[\widehat{f\theta^{-1}}(\chi)u]\|^2 d\chi$$
$$\leq \int_{\widehat{G}} \|\Phi_\theta(\chi)\|^2 \|\widehat{f\theta^{-1}}(\chi)u\|^2 d\chi$$
$$\leq C_\omega^2 \|M\|^2 \int_{\widehat{G}} |\widehat{f\theta^{-1}}(\chi)|^2 \|u\|^2 d\chi < +\infty.$$

This makes possible to apply Lemma 1, and we get

$$\mathcal{F}^{-1}(\langle\Phi_\theta(.)[\widehat{f\theta^{-1}}u(.)], v\rangle)(x) = \langle\mathcal{F}^{-1}(\Phi_\theta(.)[\widehat{f\theta^{-1}}(.)u])(x), v\rangle,$$

for almost every $x \in G$. It follows from (2.4) that we have

$$M[fu](x)\theta^{-1}(x) = \mathcal{F}^{-1}(\Phi_\theta(.)[\widehat{f\theta^{-1}}(.)u])(x),$$

for almost every $x \in G$ and this yields

$$M[fu]\theta^{-1} \in L^2(G, H).$$

Moreover, we obtain

$$\mathcal{F}(M[fu]\theta^{-1})(\chi) = \Phi_\theta(\chi)[\widehat{f\theta^{-1}}(\chi)u],$$

for almost every $\chi \in \widehat{G}$. $\square$

3. The case $G = \mathbb{R}$. In [4] we have established a more complete version of Theorem 2 concerning multipliers on weighted spaces on $\mathbb{R}$. Let w be a weight on $\mathbb{R}$ and denote by S_ω the operator $S_{1,\omega}$ on $L^2_\omega(\mathbb{R}, H)$. Define

$$I_\omega = [-\ln \rho(S_\omega^{-1}),\ \ln \rho(S_\omega)]$$

and

$$\Omega_\omega = \{z \in \mathbb{C},\ \operatorname{Im} z \in I_\omega\}.$$

For $f \in L^2_\omega(\mathbb{R}, H)$ denote by $(f)_a$ the function

$$(f)_a(x) = f(x)e^{ax},\ \forall a \in I_\omega.$$

We have the following theorem.

Theorem 4 ([4]). *Let ω be a weight on $\mathbb{R}$ and let M be a multiplier on $L^2_\omega(\mathbb{R})$.*
i) We have $(Mf)_a \in L^2(\mathbb{R})$, $\forall f \in C_c^\infty(\mathbb{R})$ and there exists $h_a \in L^\infty(\mathbb{R})$ such that

$$\widehat{(Mf)_a}(x) = h_a(x)\widehat{(f)_a}(x),\ \ \forall a \in I_\omega,\ \forall f \in C_c^\infty(\mathbb{R}),\ a.e.$$

and

$$\|h_a\|_\infty \le C_\omega \|M\|.$$

ii) If $\overset{\circ}{\Omega}_\omega \neq \emptyset$, then there exists $h \in \mathcal{H}^\infty(\overset{\circ}{\Omega}_\omega)$, such that for every $f \in C_c^\infty(\mathbb{R})$,

$$\widehat{Mf}(z) = h(z)\hat{f}(z),\ \forall z \in \overset{\circ}{\Omega}_\omega,$$

where

$$\widehat{Mf}(x + ia) = \widehat{(Mf)_a}(x),\ \forall x + ia \in \overset{\circ}{\Omega}_\omega.$$

Using the same methods as those exposed in Section 2 combined with Theorem 4, we obtain the following interesting version of Theorem 3 in the particular case $G = \mathbb{R}$.

Theorem 5. *Let ω be a weight on $\mathbb{R}$. Let M be a multiplier on $L^2_\omega(\mathbb{R}, H)$. Then*
i) We have $(Mf)_a \in L^2(\mathbb{R}, H)$, $\forall f \in C_c(\mathbb{R}) \otimes H$ and there exists $\Phi_a \in L^\infty(\mathbb{R}, \mathcal{L}(H))$ such that

$$\widehat{(Mf)_a}(x) = \Phi_a(x)[\widehat{(f)_a}(x)],\ \forall a \in I_\omega,\ \forall f \in C_c(\mathbb{R}) \otimes H,\ a.e.$$

and

$$\operatorname{ess}\sup_{x\in\mathbb{R}} \|\Phi_a(x)\| \leq C_\omega \|M\|.$$

ii) If $\overset{\circ}{\Omega}_\omega \neq \emptyset$, *then there exists*

$$\Phi : \overset{\circ}{\Omega}_\omega \longrightarrow \mathcal{L}(H)$$

such that for every $f \in C_c(\mathbb{R}) \otimes H$,

$$\widehat{Mf}(z) = \Phi(z)[\hat{f}(z)],\ \forall z \in \overset{\circ}{\Omega}_\omega,$$

where

$$\widehat{Mf}(x+ia) = \widehat{(Mf)_a}(x),\ \forall x+ia \in \overset{\circ}{\Omega}_\omega.$$

For every $u, v \in H$ *the function*

$$z \longrightarrow \langle \Phi(z)[u], v\rangle$$

is in $\mathcal{H}^\infty(\overset{\circ}{\Omega}_\omega)$.

Since the proof of Theorem 5 is very similar to that of Theorem 3, we omit the details. Notice that following the results of [4] and [6], if $G = \mathbb{R}$, the set $\widetilde{G^p_\omega}$ given by (1.2), that we use in Theorem 3 is isomorphic to the strip Ω_ω and the set $\widetilde{G^{p+}_\omega}$ is isomorphic to the segment I_ω. Applying Theorem 5, we get the following proposition.

Proposition 2. *Let* ω *be a weight on* $\mathbb{R}$. *We have*

$$spec(S_\omega) = \Big\{ z \in \mathbb{C}, \frac{1}{\rho(S_\omega^{-1})} \leq |z| \leq \rho(S_\omega) \Big\}.$$

P r o o f. Let $\alpha \notin spec(S_\omega)$. Then $M = (S_\omega - \alpha I)^{-1}$ is a multiplier. Applying Theorem 5, we get that for every $a \in I_\omega$, there exists $\Phi_a \in L^\infty(\mathbb{R}, H)$ such that

$$\widehat{(Mf)_a}(x) = \Phi_a(x)[\widehat{(f)_a}(x)],\ \forall a \in I_\omega,\ \forall f \in C_c(\mathbb{R}) \otimes H,\ a.e.$$

and

$$\operatorname{ess}\sup_{x\in\mathbb{R}} \|\Phi_a(x)\| \leq C_\omega \|M\|.$$

Replacing f by $(S_\omega - \alpha I)^{-1}g$ in the above formula, we obtain

$$\widehat{(g)_a}(x) = \Phi_a(x)\Big[\mathcal{F}\Big(((S_\omega - \alpha I)g)_a\Big)(x)\Big],\ \forall g \in C_c(\mathbb{R}) \otimes H,\ \forall a \in I_\omega,\ a.e.$$

We have

$$\mathcal{F}\Big(((S_\omega - \alpha I)g)_a\Big)(x) = \int_G (g(t-1) - \alpha g(t))e^{at}e^{-itx}dt$$

$$= \widehat{(g)_a}(x)(e^{-ix}e^a - \alpha),\ \forall g \in C_c(\mathbb{R}) \otimes H,\ \forall a \in I_\omega,\ a.e.$$

Consequently, we get

$$\Phi_a(x)[\widehat{(g)_a}(x)] = \frac{1}{e^{-ix}e^a - \alpha}\widehat{(g)_a}(x),\ a.e.$$

and hence

$$\|\Phi_a(x)\| \geq \frac{1}{e^{-ix}e^a - \alpha},\ a.e$$

This shows that $e^a \neq |\alpha|$, for every $a \in I_\omega$ and from the definition of I_ω it follows that

$$\alpha \notin \Big\{z \in \mathbb{C}, \frac{1}{\rho(S_\omega^{-1})} \leq |z| \leq \rho(S_\omega)\Big\}.$$

We deduce that

$$\Big\{z \in \mathbb{C}, \frac{1}{\rho(S_\omega^{-1})} \leq |z| \leq \rho(S_\omega)\Big\} \subset spec(S_\omega)$$

and this completes the proof. □

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