# Serdica
## Mathematical Journal
# Сердика
## Математическо списание

Serdica Math. J. **32** (2006), 259–268

**Serdica**
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

# FINITE GROUPS AS THE UNION OF PROPER SUBGROUPS

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*Communicated by V. Drensky*

Abstract. As is known, if a finite solvable group $G$ is an $n$-sum group then $n-1$ is a prime power. It is an interesting problem in group theory to study for which numbers $n$ with $n-1>1$ and not a prime power there exists a finite $n$-sum group. In this paper we mainly study finite nonsolvable $n$-sum groups and show that 15 is the first such number. More precisely, we prove that there exist no finite 11-sum or 13-sum groups and there is indeed a finite 15-sum group. Results in [7] and [15] are thus extended and further generalizations are possible.

It is a basic fact in group theory that a finite group is the set-theoretic union of proper subgroups if and only if the group is not cyclic. For a finite group $G$ the least positive integer $n$ such that $G$ is the union of $n$ proper subgroups is defined to be the covering number of $G$. Denote by $\alpha(G)$ the covering number for any finite group $G$ and we call $G$ an $\alpha(G)$-sum group. If $G$ is cyclic we define

2000 *Mathematics Subject Classification:* 20D60,20E15.
*Key words:* Finite Group, simple group, covering number.
*Supported by Cheung Kong Scholar's Programme, National 973 project and RFDP.

$\alpha(G) = \infty$. Covering numbers of finite groups have been well investigated in the past forty years, for references the reader is refered to [7], [10] and [15]. Haber and Rosenfield proved in 1959 that there are no finite 2-sum groups and a finite group $G$ is a 3-sum group if and only if the Klein 4-group is a homomorphic image of $G$ [10]. Along this line Cohn proves in [7] that a finite group $G$ is a 4-sum group if and only if $Z_3 \times Z_3$ or $S_3$ is a homomorphic image of $G$ and $G$ is a 5-sum group if and only if $A_4$ is a homomorphic image of $G$. Tomkinson proves in [15] that there are no finite 7-sum groups and for finite solvable $n$-sum groups one has $n = 1 + p^m$ where $p$ is a prime number, thus the two conjectures by Cohn [7] are confirmed. There are other ways to study how groups can be expressed as a union of proper subgroups, and similar arguments also apply to the study on unions of ideals in ring theory, see [3], [11] and [16] for references.

It is an important problem when a rational integer $n$ is a covering number for some finite group $G$. For any prime power $p^m$, it is easy to verify that the group $Z_p^m : Z_{p^m-1}$ has $1+p^m$ as its covering number where $Z_p^m$ is the elementary abelian $p$-group of order $p^m$ and $Z_{p^m-1}$ is the cyclic group of order $p^m - 1$. Thus one needs only to determine whether or not there exists a finite $n$-sum group for any rational integer $n$ with $n - 1$ not a prime power. For example, Tomkinson asked in [15] if there are 11, 13 or 15-sum groups. Of course it is very helpful and challenging to determine the covering numbers of finite simple groups. For convenience we call an integer $n$ a solvable covering number if there exists a finite solvable $n$-sum group. Note that finite simple groups may have solvable covering numbers, for example, $\alpha(A_5) = 10 = 1 + 3^2$. However, for a non-solvable covering number $n$ a finite $n$-sum group is not solvable. In this paper we continue the study on finite $n$-sum groups, in particular we prove that 15 is the least non-solvable covering number, thus we solve the problem posed by Tomkinson. We also make effort to determine the covering numbers of finite nonabelian simple groups. Our machinery can be possibly applied for further generalizations.

All groups in this paper are finite, notation and terminology are standard. Information on maximal subgroups of finite simple groups can be found in the Atlas of Finite Groups [6], sometimes we use the information implicitly.

For the convenience of the reader we quote without proof some basic results on $n$-sum groups.

**Lemma 1** [7]. *Let $G$ be a finite group and $N$ a normal subgroup of $G$. Then $\alpha(G) \leq \alpha(G/N)$.*

**Lemma 2** [7]. *If a finite group $G = \cup_{j=1}^{n} H_j$ such that $|G : H_j| \leq |G : H_{j+1}|$ for $j \leq n-1$ then $|G| \leq \Sigma_{j\geq 2}|H_j|$ and $|G : H_2| \leq n-1$.*

**Lemma 3** [10]. *If $G$ is an $n$-sum group and $G = \cup_{j=1}^{n} H_j$ then for any $1 \le k \le n$ we have $\cap_{j \ne k} H_j = \cap_{j=1}^{n} H_j$.*

**Lemma 4** [15]. *Let $N$ be a normal subgroup of a finite group $G$. Let $U_1, U_2, \dots, U_h$ be proper subgroups of $G$ containing $N$ and $V_1, V_2, \dots, V_k$ be proper subgroups such that $NV_j = G$ with $|G : V_j| = I_j$ and $I_1 \le I_2 \le \dots \le I_k$. If $G = U_1 \cup \dots \cup U_h \cup V_1 \cup \dots \cup V_k$ then $I_1 \le k$. Furthermore, if $I_1 = k$ then $I_1 = I_2 = \dots = I_k = k$ and $V_i \cap V_j \le U_1 \cup U_2 \cup \dots \cup U_h$.*

**Lemma 5** [13]. *For the symmetric group $S_n$, $\alpha(S_n) = 2^{n-1}$ when $n$ is odd unless $n = 9$; $\alpha(S_n) \le 2^{n-2}$ when $n$ is even; $\alpha(A_n) \ge 2^{n-2}$ for $n \ne 7, 9$, with equality if and only if $n \equiv 2$ modulo 4.*

**Lemma 6** [5]. $\mathrm{PSL}(2,7)$ *is a 15-sum group.*

**Lemma 7** [5]. *Let $G$ be either $\mathrm{GL}_2(q)$, $\mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$, or $\mathrm{PGL}_2(q)$. Then $\alpha(G) = q(q+1)/2$ for $q$ even or $q(q+1)/2 + 1$ for $q$ odd where $q \ne 5, 7, 9$.*

**Lemma 8.** *Let $K$ be one of the following simple groups: $\mathrm{PSL}(2,11)$, $M_{11}$, $M_{12}$, $A_j$, $7 \le j \le 12$. If $G$ is a subgroup of $\mathrm{Aut}(K)$ containing $K$ then $\alpha(G) \ge 15$.*

Proof. Suppose that $G = \cup_{j=1}^{\alpha} H_j$ where $\alpha = \alpha(G)$ and $H_j$'s are maximal subgroups of $G$. First we consider the case where $K \ne H_j$ for any $j$. For this case we have $\alpha(K) \le \alpha(G)$ since $K = \cup(K \cap H_j)$, so we need only to prove that $\alpha(K) \ge 15$.

If $K = A_j$ with $7 \le j \le 12$ then by Lemma 5 we have $\alpha(K) \ge 2^6 > 15$

If $K = \mathrm{PSL}(2,11)$ then by Lemma 7 $\alpha(K) \ge 67 > 15$. If $K = M_{11}$ then by [13] $\alpha(M_{11}) = 23 > 15$.

If $K = M_{12}$ then there need at least 12 maximal subgroups of $M_{12}$ to cover all cyclic Sylow 11-subgroups. Let $E$ be a cyclic subgroup of $M_{12}$ of order 10, then $E$ is self-centralizing in $M_{12}$ and has $|M_{12}|/10 = 2^5 \cdot 3^3 \cdot 11$ conjugates. Note that every maximal subgroup of $M_{12}$ containing an element of order 11 does not contain elements of order 10. By checking the maximal subgroups $M$ of $M_{12}$ containing an element of order 10 we see that $M$ is not isomorphic to $M_{11}$ and contains at most $2^4 \cdot 3^2 \cdot 11$ conjugates of $E$. Hence we need at least 6 $(= (2^5 \cdot 3^3 \cdot 11)/(2^4 \cdot 3^2 \cdot 11))$ more maximal subgroups to cover elements of order 10 and thus $\alpha(M_{12}) \ge 12 + 6 = 18 > 15$.

Now we suppose that $K$ is one of $H_j$'s, say $j = 1$. Then $K$ is a proper subgroup of $G$ and $G = \mathrm{Aut}(K)$. Since $\mathrm{Out}(M_{11}) = 1$, $K$ is not isomorphic to

$M_{11}$. If $K = \mathrm{PSL}(2,11)$ we see the lemma is true by Lemma 7. If $K = M_{12}$ then all maximal subgroups of $G$ other than $K$ have index at least 24, thus by Lemma 2 we have that $|G| \leq \Sigma_{j>1}|H_j| \leq \alpha(G)|G|/24$, whence $\alpha(G) \geq 24 > 15$. If $K = A_j$ with $j = 7, 11$ then by Lemma 5 $\alpha(G) \geq 2^6 > 15$. If $K = A_9$, let $x = (1\,2\,3\,4\,5\,6\,7)(8\,9)$. Now $x$ is not in $K$ and the maximal subgroups of $G$ containing a conjugate of $x$ are isomorphic to $S_7 \times Z_2$. Thus $\alpha(G) \geq 1 + 9!/(2((9-2)!)) = 1 + 9(9-1)/2 \geq 37 > 15$. Finally we consider the case where $K = A_j$ with $j = 8, 10, 12$. Then $x := (1\,2\,\ldots\,j)$ is not in $K$ and the maximal subgroups of $G$ containing a conjugate of $x$ have index in $G$ at least $15 (j = 8), 45 (j = 10)$ or $66$ $(j = 12)$ respectively, since $x$ does not fix any figures $i \leq j$. Evidently $\langle x \rangle$ is self-centralizing, so any maximal subgroup $M$ of $G$ contains at most $|M|/|\langle x \rangle|$ conjugates of $x$. Since $|M| \leq |G|/15$ and $G$ has $|G|/|\langle x \rangle|$ conjugates of $x$, we need at least 15 maximal subgroups of $G$ other than $K$ to cover conjugates of $x$, so $\alpha(G) \geq 15$. □

**Lemma 9** .*Let $H$ be a noncyclic solvable subgroup of* $\mathrm{GL}(3,2)$ *or* $\mathrm{GL}(2,3)$. *Then $\alpha(H) \leq 8$.*

Proof. If $H \leq \mathrm{GL}(3,2)$ then $H$ is isomorphic to a subgroup of $S_4$ or $D_{21}$. So every chief factor of $H$ is of order at most 7. This is also true for $H \leq \mathrm{GL}(2,3)$. Thus $\alpha(H) \leq 8$ by [15]. □

**Lemma 10.** *Let $G$ be a finite group and $H$ a normal subgroup of $G$ which is the direct product of nonsolvable minimal normal subgroups $N_j$'s. Then any normal subgroup $M$ of $G$ contained in $H$ is also a direct product of some of these $N_j$'s.*

Proof. First note that each $N_j$ is a direct product of isomorphic nonabelian simple groups. We need only to prove that for any $y \in M$ if $y = x_1 x_2 \ldots x_n$ with $x_i \in N_i (i \leq n)$ then $N_j \leq M$ whenever $x_j \neq 1$. Suppose that $x_j \neq 1$ then there is an element $x \in N_j$ such that $[x, x_j] \neq 1$. Now $[x, x_j] = [x, y] \in M, M \cap N_j \neq 1$, so $N_j \leq M$. We are done. □

**Theorem 11.** *There are no* 11- *or* 13-*sum groups.*

Proof. Suppose toward a contradiction that there are 11- or 13-groups and let $G = \cup_{j=1}^{n} H_j$ be an $n$-group of minimal possible order with $H_j$'s maximal in $G$ and $n = 11$ or 13. Then the core $\mathrm{Core}_G(\cap H_j) = 1$ by Lemma 1. It follows that the Frattini subgroup $\Phi(G) = 1$. Set $I_j = |G : H_j|, j = 1, 2, \ldots, n$. By Lemma 2 there are at least two of $I_j$'s at most $n - 1$.

*Step 1.* If $I_j \leq n-1$ then either $H_j$ is normal in $G$ or $\overline{G} := G/\mathrm{Core}_G(H_j)$ is not solvable.

Proof. If this is not true let $H_j$ be not normal in $G$ with $I_j \leq n-1$ and $\overline{G}$ solvable. Since $H_j$ is maximal in $G$, the Fitting subgroup $F(\overline{G})$ is a minimal normal subgroup of $\overline{G}$ and $\overline{G} = \overline{H_j}F(\overline{G})$ with $\overline{H_j} \cap F(\overline{G}) = 1$. Thus $F(\overline{G})$ is of order at most $I_j \leq 13-1 = 12$. Since $H_j$ is not normal in $G$, $F(\overline{G})$ is not in the center of $\overline{G}$. Set $p^m = |F(\overline{G})|$ where $p$ is a prime, then $p^m \leq n-2$ since $n = 11$ or 13. If $\overline{H_j}$ is cyclic then $\alpha(G) \leq \alpha(\overline{G}) \leq p^m + 1 \leq n-1$ by [15], a contradiction. Thus $\overline{H_j}$ is not cyclic. It follows that $p^m = 8$ or 9 and $\overline{H_j} \leq \mathrm{GL}(3,2)$ or $\mathrm{GL}(2,3)$. By Lemma 9 we have $\alpha(G) \leq \alpha(\overline{H_j}) = 9 < n$, again a contradiction. We are done.

*Step 2.* $F(G) = 1$.

Proof. If $F(G)$ is not trivial let $N$ be a nontrivial solvable minimal normal subgroup of $G$. By Lemma 3 for any $1 \leq s \neq t \leq n$ we have $\cap_{j\neq s} H_j = \cap_{j \neq t} H_j = \cap H_j$. Since $\mathrm{Core}_G(\cap H_j) = 1$ there are at least two $H_j$'s such that $NH_j = G$ and $N \cap H_j = 1$. Let $I_t$ be the least of these $I_j$'s with $NH_j = G$, then $|N| = I_t$ and by Lemma 4 we have $I_j \leq n-1$. Note that $|N|$ is a prime power and $n = 11$ or 13, so one has $|N| \leq n-2$. If $H_t$ is normal in $G$ then $N$ is in the center of $G$, so $|N|$ is a prime and $\alpha(G) = |N|+1 \leq n-1$ by [7], which is a contradiction. Hence $H_t$ is not normal in $G$. Since $H_t$ is maximal in $G$, $F(G/\mathrm{Core}_G(H_t)) = N\,\mathrm{Core}_G(H_t)/\mathrm{Core}_G(H_t)$. By Step 1 $G/\mathrm{Core}_G(H_t)$ is not solvable, thus $|N| = 8$ and $G/\mathrm{Core}_G(H_t) \cong Z_2^3 : \mathrm{PSL}(3,2)$ because $|N| \leq n-2 \leq 11$ and all proper subgroups of $\mathrm{PSL}(3,2)$ are solvable. If for some $x \in N, H_t^x \neq H_j$ for any $j$ then $H_t^x = \cup(H_j \cap H_t^x)$. Hence $n \geq \alpha(H_t^x) = \alpha(G/N)$, which is contradictory to the minimality of $G$. Thus $H_t^x = H_j$ for some $j$.

We claim that $\mathrm{Core}_G(H_t) \neq 1$. Suppose otherwise that $\mathrm{Core}_G(H_t) = 1$. Let $y$ be an involution of $H_t$, then there is an involution $u$ in $N$ such that $o(yu) = 4$ and $(yu)^2 \in N$. If an $H_s$ contains a conjugate of $yu$ then $N \cap H_s \neq 1$, so $N \leq H_s$ (otherwise $NH_s = G$ and thus $N \cap H_s = 1$). It follows that $H_s/N$ is a maximal subgroup of $\mathrm{PSL}(3,2)$ containing an involution, whence $H_s/N \cong S_4$ and $I_s = 7$ by [6]. So we need at least 7 $H_j$'s containing $N$ to cover all conjugates of $yu$, which implies that $13 \geq n \geq 8+7 = 15$, impossible. Thus we have $\mathrm{Core}_G(H_t) \neq 1$, as claimed. Let $T$ be the generalized Fitting subgroup $F^*(\mathrm{Core}_G(H_t))$, then $T \neq 1$. Note that $T \leq H_t^x$ for any $x \in N$, so the number $f$ of $H_j$'s not containing $T$ is at most $n-8 \leq 5$. For convenience we may assume that $TH_1 = TH_2 = \cdots = TH_f = G$ and $I_1 \leq I_2 \leq \cdots \leq I_f$. By Lemma 4 we have $I_1 \leq f$. Since $f \leq 5$ and $|N| = 8$, $N \leq H_1$(otherwise $NH_j = G$ and $I_j = |N| = 8$, a contradiction). If $I_1 = f$ then by Lemma 4, $I_j = f(j = 1, 2, \ldots, f)$ and $H_1 \cap H_2 \leq H_t$. As above we have $N \leq H_j (1 \leq j \leq f)$. Thus $N \leq H_1 \cap H_2 \leq H_t$, which is contradictory to the choice of $H_t$. Therefore $I_1 < f \leq 5$. It follows that $G/\mathrm{Core}_G(H_1)$ is solvable.

By Step 1 $H_1$ is normal in $G$ and thus $I_1 = 2$ or 3. Noticing that $\Phi(G) = 1$, we see that $F^*(G)$ is the direct product of minimal normal subgroups of $G$. Since $\mathrm{Core}_G(H_t)$ is normal in $G$, $T \leq F^*(G)$. So there is a minimal normal subgroup $S$ of $G$ such that $SH_1 = TH_1 = G$. It follows immediately that $S$ is solvable and $S \cap H_1 = 1$, so $G = S \times H_1$. Now $S$ is in the center of $G$ and by [7] we have $\alpha(G) \leq 3 + 1 = 4$, contradicting the assumption on $G$.

*Step 3.* $F^*(G) = N_1 \times N_2 \times \cdots \times N_m$ where $N_j$'s are minimal normal subgroups of $G$ and isomorphic to $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(3,2)$, $\mathrm{PSL}(2,11)$, $M_{11}$, $M_{12}$ or $A_i, 5 \leq i \leq 12$. And $G/F^*(G)$ is cyclic of order dividing 6.

Proof. Let $N$ be an arbitrary minimal normal subgroup of $G$. By Step 2, $N$ is a direct product of isomorphic nonabelian simple groups. By Lemma 3 there is an integer $t$ such that $NH_t = G$ and $I_t \leq n - 1 \leq 12$. Since $N \cap \mathrm{Core}_G(H_t) = 1$, $N$ is isomorphic to a normal subgroup of $G/\mathrm{Core}_G(H_t)$. Note that $G/\mathrm{Core}_G(H_t)$ is a primitive permutation group of degree $I_t$. By [9] we know that $N$ is simple and isomorphic to the simple groups listed above. Thus the outmorphism group $\mathrm{Out}(N)$ is an elementary abelian 2- or 3-group [6]. Since $\Phi(G) = F(G) = 1, F^*(G)$ is the direct product of minimal normal subgroups $N_j$'s of $G$. Now $G/F^*(G)$ acts by conjugation on $N_j$'s and it follows that $G/F^*(G)$ is an abelian $\{2,3\}$-group. If $G/F^*(G)$ is not cyclic then $\alpha(G/F^*(G)) \leq 4$ by [15], which is a contradiction.

*Step 4.* For each $N_j$ there exists an $H_i$ such that $N_j \leq H_i$.

Proof. If there is an $N_j$ such that $N_jH_i = G$ for any $i$, then $m = 1, F^*(G) = N_1$ and $F^*(G) = \cup_i(F^*(G) \cap H_i)$. So $\alpha(F^*(G)) \leq 13$. By Lemmas 5, 6, 7 and 8, $F^*(G) \cong A_5$. Thus $G \cong S_5$, which is a contradiction since $\alpha(S_5) = 16$ by [7].

*Step 5.* $m \leq 2$.

Proof. Suppose $m \geq 3$. Set $J_r = \{i : N_rH_i = G\}$ for $1 \leq r \leq m$. For convenience and without loss of generality we may assume that $s := |J_1| = \max\{|J_r| : 1 \leq r \leq m\}$ and $N_1H_i = G$ for $1 \leq i \leq s$ with $I_1 \leq I_2 \leq \cdots \leq I_s$. Thus $N_1 \leq H_j$ for $j > s$. Set $M = N_2 \times N_3 \times \cdots \times N_m$, then $M \leq \cap_{j \leq s} H_j$.

We claim that $s < (n+1)/2$. Suppose othertwise that $s \geq (n+1)/2$. Re-label $H_i$'s for $i > s$ such that $N_2H_{s+1} = N_2H_{s+2} = G$ since there are at least two $H_i$'s not containing $N_2$. Thus $N_3 \leq H_{s+1} \cap H_{s+2}$. Now $N_3$ is contained in $s+2$ of $H_i$'s. Note that $s+2 \geq 9$ for $n = 13$ and 8 for $n = 11$, so $n-s-2 \leq 4$. By [15] for $G$ and the normal subgroup $N_3$ we have $I_t \leq n-s-2 \leq 4$ with $N_3H_t = G$

for some $t > s+2$. Hence $H_t$ is normal in $G$ by Step 1, which is impossible since $H_t$ is maximal in $G$ and $N_3$ is nonabelian simple. Thus the claim holds true. By [15] for $G$ and the normal subgroup $N_1$ we have $I_1 \leq s \leq (n-1)/2$. If $I_1 = s$ then $I_1 = I_2 = \cdots = I_s$ with $M \leq H_i$ for $i > s$. Thus $M$ is contained in every $H_i$, which is contradictory to the assumption. So $I_1 < s$. Since $N_1H_1 = G$, $H_1$ is not normal in $G$. By Step 1, $G\,\mathrm{Core}_G(H_1)$ is not solvable and $I_1 \geq 5$. From $5 \leq I_1 < s \leq (n-1)/2$ we see that $n = 13$ , $s = 6$ and $I_1 = 5$. Re-labeling $H_i$'s for $i > 6$ we may assume that $N_2H_7 = N_2H_8 = \cdots = N_2H_e = G$ and $N_2 \leq H_i$ for $i > e$. Note that $e - 6 \geq 2$ and $N_3 \leq H_i$ for $i \leq e$. By the definition of $s$ we have $e - s \leq s$, so $8 \leq e \leq 12$. By [15] for $G$ and $N_3$, $5 \leq H_i \leq 13 - e$ for some $i > e$, thus $e = 8$ and $H_i = 5$ for all $i > 8$ and $H_9 \cap H_{10} \leq H_1 \cap H_2 \cap \cdots \cap H_8$, which is a contradiction since $N_1 \leq H_i$ for $i > 6$ and $N_1$ is not contained in $H_1$.

*Step 6.* Last contradiction.

P r o o f. We first prove that $m = 1$, so $F^*(G) = N_1$ is simple. Suppose otherwise that $m = 2$. Since each $H_i$ contains either $N_1$ or $N_2$ we may assume that $N_1 \leq H_1 \cap H_2 \cap \cdots \cap H_s$ with $N_1H_i = G$ for $i > s$ and $s \geq (n+1)/2 \geq 6$. Thus $N_2 \leq H_i$ and $|G : H_i| \geq 5$ for $i > s$ (see the proof in Step 5). By [15] for $G$ and the normal subgroup $N_1$ and since $N_2$ is not contained in $H_j$ for some $j \leq s$, $5 \leq |G : H_t| < n - s$ for some $t > s$. It follows that $n = 13, s = 7$ and $|G : H_t| = 5$, so $N_1 \cong A_5$. By the definition of $s$ we know that there is at most one $H_i (i \leq 7)$ containing $N_2$. Now we consider cases.

*Case I.* $N_2 \leq H_i (i \leq 7)$, say $i = 1$. Again by [15] for $G$ and $N_2$ we have $5 \leq |G : H_f| < 6$ for some $1 < f \leq 7$, thus $|G : H_f| = 5$ and $G/\mathrm{Core}_G(H_f) \leq S_5$ with $N_2 \cong A_5$. Now $F^*(G) \cong A_5 \times A_5$ and $|G/F^*(G)|$ is at most 2. Hence we have the following three possibilities: $G \cong A_5 \times A_5, A_5 \times S_5$ or $(N_1 \times N_2) : \langle x \rangle$ where $x$ is of order 2 and $N_1\langle x \rangle \cong N_2\langle x \rangle \cong S_5$. Since $\alpha(A_5) = 10$ we see that $G = (N_1 \times N_2) : \langle x \rangle$. Let $x_i \in N_i$ such that $xx_i = x_ix$ with $o(xx_i) = 6$ for $i = 1, 2$. Evidently $xx_1x_2$ is of order 6 and is not contained in $F^*(G)$ and $\langle xx_1x_2 \rangle$ is self-centralizing in $G$. So $xx_1x_2 \in H_j$ for some $j > 1$. Since $H_j$ contains $N_i (i = 1$ or $2)$, $H_j/N_i$ is isomorphic to a maximal subgroup of $S_5$ containing an element of order 6. Thus, by [6] $H_j/N_i \cong Z_2 \times S_3$ and $H_j$ contains exactly $|H_j|/6$ $= (5!/2)(12/6) = 120$ conjugates of $xx_1x_2$. Since $xx_1x_2$ has $(5!/2)(5!/6) = 1200$ conjugates in $G$, we need at least 10 of these $H_i$'s $(i > 1)$ to cover all conjugates of $xx_1x_2$. Let $y_i \in N_i$ be an involution such that $xy_i$ is of order 4 $(i = 1, 2)$. Then $\langle xy_1y_2 \rangle$ is self-centralizing of order 4 in $G$ and is not contained in $F^*(G)$, so there are $(5!/2)(5!/4) = 1800$ conjugates of $xy_1y_2$ in $G$. Let $xy_1y_2$ be contained in $H_k$ $(k > 1)$, then $H_k/N_j \cong S_4$ or $Z_5 : Z_4$ where $N_j \leq H_k$. It follows immediately that $H_k$ does not contain a conjugate of $xx_1x_2$. Since $H_k$ contains at most $(5!/2)(4!/4) = 360$ conjugates of $xy_1y_2$ we need at least $5 = 1800/360$ of $H_i$'s to cover all conjugates of $xy_1y_2$. Therefore $13 \geq \alpha(G) \geq 1 + 10 + 5 = 16$ which is absurd.

*Case II.* $N_2H_i = G$ for $i \leq 7$. By [15] for $G$ and $N_2$ we have $5 \leq |G : H_f| < 7$ for some $1 \leq f \leq 7$, thus $|G : H_f| = 5$ or 6 and $G/\mathrm{Core}_G(H_f) \leq S_6$ with $N_2 \cong A_5$ or $A_6$. Now $F^*(G) \cong A_5 \times A_5$ or $A_5 \times A_6$ and $G/F^*(G)$ is of order at most 2. Hence we have the following possibilities: $G \cong A_5 \times A_5, A_5 \times S_5, A_5 \times A_6, A_5 \times S_6$ or $(N_1 \times N_2) : \langle x\rangle$ where $x$ is of order 2 and $N_1\langle x\rangle \cong S_5$ and $N_2\langle x\rangle \cong S_5, S_6$ or $A_6 \times Z_2$. Since $\alpha(A_5) = 10$, $A_5$ cannot be a direct factor of $G$. Thus $G = (N_1 \times N_2) : \langle x\rangle$. If $N_1 \cong N_2 \cong A_5$ then as proved in Case I we know that we need at least $10 + 5$ maximal subgroups to cover all elements of orders 4 and 6 outside $F^*(G)$, so $N_2 \cong A_6$. By [6] $G$ contains exactly $(4!)(6!/5) = 24 \times 144 = 3456$ elements $ab$ of order 5 with $a \in N_1 \setminus \{1\}$ and $b \in N_2 \setminus \{1\}$, and each maximal subgroup of $G$ contains at most $144 \times 4 = 576$ such elements. Thus we need at least $3456/576 = 6$ maximal subgroups of $G$ to cover all such elements of order 5. Since $N_2 \cong A_6$ there is an element $v \in N_2$ such that $\langle v\rangle$ is of order 3 corresponding to a product of two 3-cycles in $A_6$ and such that $xv = vx$. Let $u \in N_1$ be an element of order 3 such that $xu = ux$. Then $\langle xuv\rangle$ is of order 6 and self-centralizing in $G$. Now $G$ contains $|G|/6$ conjugates of $xuv$ and the maximal subgroups $M$ of $G$ containing a conjugate of $xuv$ are isomorphic to $N_iL\langle x\rangle$ where $N_i \leq M$ with $i = 1$ or 2 and $L \cong Z_3^2 : Z_4$ or $S_3 (L \leq N_j, j \leq 2, j \neq i)$. Hence $M$ is of order at most $|G|/10$ and does not contain 5-elements of the type $ab$ with $a \in N_1 \setminus \{1\}$ and $b \in N_2 \setminus \{1\}$. We now have $\alpha(G) \geq 10 + 6 > 13$, a contradiction. Therefore $m = 1$, as claimed.

Note that $F^*(G) \cong \mathrm{PSL}(3,2)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,11)$, $M_{11}$, $M_{12}$ or $A_j, 5 \leq j \leq 12$, $G/F^*(G)$ is of order at most 3. By Lemma 8 and [7] we know that $F^*(G)$ is a proper subgroup of $G$ and $F^*(G) \cong \mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$ or $A_6$ (note that $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$). By Lemma 5 $G$ is not isomorphic to $S_6$. Now by [6] $G$ contains an element $x$ of order $m'$ outside $F^*(G)$ such that all maximal subgroups of $G$ containing an element of order $m'$ are conjugate to $N_G(\langle x\rangle)$. The information about $(G, m', N_G(\langle x\rangle), |G|/|N_G(\langle x\rangle)|)$ are as follows [6]:

$$(\mathrm{PSL}(2,7) : Z_2,\ 8,\ D_{16},\ 21),\ (\mathrm{PSL}(2,8) : Z_3,\ 9,\ Z_9 : Z_6,\ 28),$$

$$(A_6.2_2,\ 10,\ D_{20},\ 36), (A_6.2_3,\ 9,\ D_{16},\ 45).$$

Thus $\alpha(G) \geq |G|/|N_G(\langle x\rangle)| \geq 21 > 13$, which is contradictory to the assumption on $G$. We are done. □

## REFERENCES


[1] A. R. Ashrafi. On the $n$-sum groups $n = 6, 7$. *Southeast Asian Bull. Math.* **22**, *2* (1998), 111–114.

[2] M. Bruckheimer, A. C. Bryan, A. Muir. Groups which are the union of three subgroups. *Amer. Math. Monthly* **77** (1970), 52–57.

[3] K. P. S. Bhaskara Rao, J. D. Reid. Abelian groups that are unions of proper subgroups. *Bull. Austral. Math. Soc.* **45**, *1* (1992), 1–7.

[4] R. A. Bryce, V. Fedri, L. Serena. Covering groups with subgroups. *Bull. Austral. Math. Soc.* **55**, *3* (1997), 469–476.

[5] R. A. Bryce, V. Fedri, L. Serena. Subgroup coverings of some linear groups. *Bull. Austral. Math. Soc.* **60**, *2* (1999), 227–238.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson. Atlas of Finite groups. Clarendon Press, Oxford, 1985.

[7] J. H. E. Cohn. On $n$-sum groups. *Math. Scand.* **75**, *1* (1994), 44–58.

[8] R. R. Crawford, K. D. Wallace. On the number of subgroups of index two—an application of Goursat's theorem for groups. *Math. Mag.* **48** (1975), 172-174.

[9] J. D. Dixon, B. Mortimer. Permutation groups. Graduate Texts in Mathematics, vol. **163**. New York, Springer-Verlag, 1996.

[10] S. Haber, A. Rosenfeld. Groups as unions of proper subgroups. *Amer. Math. Monthly* **66** (1959), 491–494.

[11] C. Gottlieb. On finite unions of ideals and cosets. *Comm. Algebra* **22**, *8* (1994), 3087–3097.

[12] A. Mann. Finite groups whose $n$-maximal subgroups are subnormal. *Trans. Amer. Math. Soc.* **132** (1968), 395–409.

[13] A. Maroti. Covering the symmetric groups with proper subgroups. *J. Combin. Theory Ser. A* **110**, *1* (2005), 97–111.

[14] M. Suzuki. Group Theory I. Translated from the Japanese by the author. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 247. Springer-Verlag, Berlin-New York, 1982.

[15] M. J. Tomkinson. Groups as the union of proper subgroups. *Math. Scand.* **81**, *2* (1997), 191–198.

[16] M. J. Tomkinson. Groups covered by finitely many cosets or subgroups. *Comm. Algebra* **15**, *4* (1987), 845–859.

[17] J. P. Zhang. Finite groups with many conjugate elements. *J. Algebra* **170**, *2* (1994), 608–624.

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*Received June 2, 2006*