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LARGE AND MODERATE DEVIATIONS PRINCIPLES FOR RECURSIVE KERNEL ESTIMATOR OF A MULTIVARIATE DENSITY AND ITS PARTIAL DERIVATIVES

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Abstract. In this paper we prove large and moderate deviations principles for the recursive kernel estimator of a probability density function and its partial derivatives. Unlike the density estimator, the derivatives estimators exhibit a quadratic behaviour not only for the moderate deviations scale but also for the large deviations one. We provide results both for the pointwise and the uniform deviations.

1. Introduction. Let $X_1, \ldots, X_n$ be a sequence of independent and identically distributed $\mathbb{R}^d$-valued random vectors with bounded probability density f. Let (h_n) be a positive sequence such that $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} nh_n^d = \infty$; the recursive kernel estimator of f is defined as

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$$f_n(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_i^d}K\left(\frac{x - X_i}{h_i}\right) \tag{1}$$

where the kernel K is a continuous function such that $\lim_{\|x\|\to+\infty} K(x) = 0$ and $\int_{\mathbb{R}^d} K(x)dx = 1$. The estimate (1) is a recursive version of the well-known Rosenblatt kernel estimate (see Rosenblatt [13] and Parzen [11]); it was first discussed by Wolverton and Wagner [18], Yamato [20], and Davies [3]. The estimator (1) is easily updated each time an additionnal observation becomes available without resorting to past data, through the recursive relationship

$$f_n(x) = \frac{n-1}{n}f_{n-1}(x) + \frac{1}{nh_n^d}K\left(\frac{x - X_n}{h_n}\right).$$

The weak and strong consistency of the recursive estimator of the density was studied by many authors; let us cite, among many others, Devroye [5], Menon et al. [9] and Wertz [17]. The law of the iterated logarithm of the recursive density estimator was established by Wegman and Davies [16] and Roussas [14]. For other works on recursive density estimation, the reader is referred to the papers of Wegman [15], Ahmad and Lin [1], and Carroll [2].

Recently, large and moderate deviations results have been proved for the Rosenblatt density estimator and its derivatives. The large deviations principle has been studied by Louani [8] and Worms [19]. Gao [7] and Mokkadem et al. [10] extend these results and provide moderate deviations principles. The large and moderate deviations of the derivatives of the Rosenblatt density estimator are given in Mokkadem et al [10]. The purpose of this paper is to establish large and moderate deviations principles for the recursive density estimator f_n and its derivatives.

Let us recall that a $\mathbb{R}^m$-valued sequence $(Z_n)_{n\geq 1}$ satisfies a large deviations principle (LDP) with speed (ν_n) and good rate function I if:

(a) (ν_n) is a positive sequence such that $\lim_{n\to\infty} \nu_n = \infty$;

(b) $I : \mathbb{R}^m \to [0, \infty]$ has compact level sets;

(c) for every borel set $B \subset \mathbb{R}^m$,

$$
\begin{aligned}
-\inf_{x\in\overset{\circ}{B}} I(x) &\leq \liminf_{n\to\infty} \nu_n^{-1}\log\mathbb{P}\left[Z_n\in B\right] \\
&\leq \limsup_{n\to\infty} \nu_n^{-1}\log\mathbb{P}\left[Z_n\in B\right] \leq -\inf_{x\in\overline{B}} I(x),
\end{aligned}
$$

where $\overset{\circ}{B}$ and $\overline{B}$ denote the interior and the closure of B respectively. Moreover, let (v_n) be a nonrandom sequence that goes to infinity; if (v_nZ_n) satisfies a LDP, then (Z_n) is said to satisfy a moderate deviations principle (MDP).

For any d-uplet $[\alpha]=\left(\alpha_1,\ldots,\alpha_d\right)\in\mathbb{N}^d$, set $|\alpha|=\alpha_1+\cdots+\alpha_d$, let

$$
\partial^{[\alpha]}f(x)=\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}}(x)
$$

denote the $[\alpha]$-th partial derivative of f (if $|\alpha|=0$, then $\partial^{[\alpha]}f\equiv f$) and, for any $j\in\mathbb{N}$, let $D^{(j)}f$ denote the j-th differential of f. The recursive kernel estimator of the $[\alpha]$-th partial derivative of f is defined as

$$
\partial^{[\alpha]}f_n(x)=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_i^{d+|\alpha|}}\partial^{[\alpha]}K\left(\frac{x-X_i}{h_i}\right),
$$

where the kernel K is chosen such that $\partial^{[\alpha]}K\not\equiv 0$ and the bandwidth such that $\lim_{n\to\infty} nh_n^{d+2|\alpha|}=\infty$.

Our first aim is to establish pointwise LDP for the recursive kernel density estimator f_n. It turns out that expliciting the rate function in this case is more complex than either for the Rosenblatt kernel estimator, or for the derivatives estimators. That is the reason why, in this particular framework, we only consider bandwidths defined as $(h_n)\equiv(cn^{-a})$ with $c>0$ and $a\in]0,1/d[$. We then prove that the sequence $(f_n(x)-f(x))$ satisfies a LDP with speed $\left(nh_n^d\right)$ and rate function

$$
I_{a,x}: t\mapsto f(x)I_a\left(\frac{t}{f(x)}+1\right)
$$

where $I_a(t)$ is the Fenchel-Legendre transform of the function ψ_a defined as follows:

$$
\psi_a(u)=\int_{[0,1]\times\mathbb{R}^d} s^{-ad}\left(e^{s^{ad}uK(z)}-1\right)dsdz.
$$

Our second aim is to provide pointwise LDP for the derivative estimators $\partial^{[\alpha]} f_n$ (with $|\alpha| \geq 1$). In this case, we consider more general bandwidths defined as $h_n = h(n)$ for all n, where h is a regularly varying function with exponent $(-a)$, $a \in]0, 1/(d+2|\alpha|)[$. We prove that the sequence

$$\left(\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right)$$

satisfies a LDP of speed $\left(nh_n^{d+2|\alpha|}\right)$ and quadratic rate function $J_{a,[\alpha],x} : \mathbb{R} \to \mathbb{R}$ defined by

$$(2) \qquad \begin{cases} \text{if} \quad f(x) \neq 0, & J_{a,[\alpha],x} : t \mapsto \frac{t^2(1+a(d+2|\alpha|))}{2f(x)\int_{\mathbb{R}^d}\left[\partial^{[\alpha]}K(z)\right]^2 dz} \\ \text{if} \quad f(x) = 0, & J_{a,[\alpha],x}(0) = 0 \quad \text{and} \quad J_{a,[\alpha],x}(t) = \infty \quad \text{for} \quad t \neq 0. \end{cases}$$

Our third aim is to prove pointwise MDP for the density estimator and for its derivatives. For any d-uplet $[\alpha]$ such that $|\alpha| \geq 0$, any positive sequence (v_n) satisfying

$$\lim_{n\to\infty} v_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} = 0,$$

and general bandwidths (h_n), we prove that the random sequence

$$v_n\left(\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right)$$

satisfies a LDP of speed $\left(nh_n^{d+2|\alpha|}/v_n^2\right)$ and rate function $J_{a,[\alpha],x}(\cdot)$ defined by Equation (2).

Let us point out that the rate function $J_{a,[\alpha],x}$ is larger (by a factor $1+a(d+2|\alpha|)$) than the rate function obtained for the Rosenblatt estimator in Mokkadem et al [10]; this means that the recursive estimators $\partial^{[\alpha]} f_n(x)$, $|\alpha| \geq 0$, are more concentrated around $\partial^{[\alpha]} f(x)$ than the Rosenblatt estimators.

Finally, we give a uniform version of the previous results. More precisely, let U be a subset of $\mathbb{R}^d$; we establish large and moderate deviations principles for the sequence $\left(\sup_{x\in U}\left|\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right|\right)$ in the case either U is bounded or all the moments of f are finite.

2. Assumptions and results.

2.1. Pointwise LDP for the density estimator. The assumptions required on the kernel K and the bandwidth (h_n) are the following.

(H1) $K : \mathbb{R}^d \to \mathbb{R}$ is a bounded and integrable function , $\int_{\mathbb{R}^d} K(z)dz = 1$ and $\lim_{\|z\|\to\infty} K(z) = 0$.

(H2) $h_n = cn^{-a}$ with $0 < a < 1/d$ and $c > 0$.

Before stating our results, we need to introduce the rate function for the LDP of the density estimator. Let $\psi_a : \mathbb{R} \to \mathbb{R}$ and $I_a : \mathbb{R} \to \mathbb{R}$ be the functions defined as:

$$\psi_a(u) = \int_{[0,1]\times\mathbb{R}^d} s^{-ad}\left(e^{s^{ad}uK(z)} - 1\right) dsdz \quad \text{and} \quad I_a(t) = \sup_{u\in\mathbb{R}}\{ut - \psi(u)\}$$

(where $s \in [0, 1]$, $z \in \mathbb{R}^d$) and set

$$S_+ = \left\{x \in \mathbb{R}^d; K(x) > 0\right\} \quad \text{and} \quad S_- = \left\{x \in \mathbb{R}^d; K(x) < 0\right\}.$$

The following proposition gives the properties of the functions ψ_a and I_a; in particular, the behaviour of the rate function I_a, which differs depending on whether K is non-negative or not, is explicited.

Proposition 1. *Let λ be the Lebesgue measure on $\mathbb{R}^d$ and let Assumption* (H1) *holds.*

(i) *ψ_a is strictly convex, twice continuously differentiable on $\mathbb{R}$, and I_a is a good rate function on $\mathbb{R}$.*

(ii) *If $\lambda(S_-) = 0$, then $I_a(t) = +\infty$ when $t < 0$, $I_a(0) = \lambda(S_+)$, I_a is strictly convex on $\mathbb{R}$ and continuous on $]0, +\infty[$, and for any $t > 0$*

$$I_a(t) = t\left(\psi_a'\right)^{-1}(t) - \psi_a\left(\left(\psi_a'\right)^{-1}(t)\right). \tag{3}$$

(iii) *If $\lambda(S_-) > 0$, then I_a is finite and strictly convex on $\mathbb{R}$ and* (3) *holds for any $t \in \mathbb{R}$.*

(iv) *In both cases, the strict minimum of I_a is achieved by $I_a(1) = 0$.*

Remark 1. The following relations are straightforward, and will be used in the sequel:

$$I_a(t) = \begin{cases} \sup_{u>0}\{ut - \psi_a(u)\} & \text{if } t > 1 \\ \sup_{u<0}\{ut - \psi_a(u)\} & \text{if } t < 1. \end{cases} \tag{4}$$

We can now state the LDP for the density estimator.

Theorem 1 (Pointwise LDP for the density estimator)**.** *Let Hypotheses* (H1)–(H2) *hold and assume that f is continuous at x. Then, the sequence $(f_n(x) - f(x))$ satisfies a LDP with speed (nh_n^d) and rate function defined as follows:*

$$\begin{cases} \text{if } \ f(x) \neq 0, & I_{a,x} : t \mapsto f(x) I_a \left(1 + \frac{t}{f(x)}\right) \\ \text{if } \ f(x) = 0, & I_{a,x}(0) = 0 \ \text{ and } \ I_{a,x}(t) = +\infty \ \text{ for } \ t \neq 0. \end{cases}$$

2.2. Pointwise LDP for the derivatives estimators. Let $[\alpha]$ be a d-uplet such that $|\alpha| \geq 1$. To establish pointwise LDP for $\partial^{[\alpha]} f_n$, we need the following assumptions.

(H3) $h_n = h(n)$ where the function h is locally bounded and varies regularly with exponent $(-a)$, $0 < a < 1/(d + 2|\alpha|)$.

(H4) i) K is $|\alpha|$-times differentiable on $\mathbb{R}^d$ and $\lim_{\|x\| \to \infty} \|D^{(j)} K(x)\| = 0$ for any $j \in \{0, \dots, |\alpha| - 1\}$.
ii) $\partial^{[\alpha]} K : \mathbb{R}^d \to \mathbb{R}$ is a bounded and integrable function and $\int_{\mathbb{R}^d} \left[\partial^{[\alpha]} K(x)\right]^2 dx \neq 0$.

(H5) f is $|\alpha|$-times differentiable on $\mathbb{R}^d$ and its j-th differentials $D^{(j)} f$ are bounded on $\mathbb{R}^d$ for any $j \in \{0, \dots, |\alpha| - 1\}$.

Remark 2. A positive (not necessarily monotone) function L defined on $]0, \infty[$ is slowly varying if $\lim_{t \to \infty} L(tx)/L(t) = 1$; a function G is said to vary regularly with exponent ρ, $\rho \in \mathbb{R}$, if and only if there exists a slowly varying function L such that $G(x) = x^\rho L(x)$ (see, for example, Feller [6] page 275). Typical examples of regularly varying functions with exponent ρ are x^ρ, $x^\rho \log x$, $x^\rho \log\log x$, $x^\rho \log x / \log\log x$, and so on. An important consequence of (H3) which will be used in the sequel is:

$$\text{if} \quad \beta a < 1, \quad \text{then} \quad \lim_{n \to \infty} \frac{1}{n h_n^\beta} \sum_{i=1}^{n} h_i^\beta = \frac{1}{1 - a\beta}. \tag{5}$$

Theorem 2 (Pointwise LDP for the derivatives estimators)**.** *Let $|\alpha| \geq 1$ and assume that* (H1), (H3)–(H5) *hold and that $\partial^{[\alpha]} f$ is continuous at x. Then,* *the sequence* $\left(\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right)$ *satisfies a LDP with speed* $(nh_n^{d+2|\alpha|})$ *and rate function* $J_{a,[\alpha],x}$ *defined by* (2).

2.3. Pointwise MDP for the density estimator and its derivatives. Let (v_n) be a positive sequence; we assume that

(H6) $\lim_{n\to\infty} v_n = \infty$ and $\lim_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} = 0.$

(H7) i) There exists an integer $q \geq 2$ such that $\forall s \in \{1, \ldots, q-1\}$, $\forall j \in \{1, \ldots, d\}$, $\int_{\mathbb{R}^d} y_j^s K(y) dy_j = 0$, and $\int_{\mathbb{R}^d} \left|y_j^q K(y)\right| dy < \infty$.
ii) $\lim_{n\to\infty} \frac{v_n}{n} \sum_{i=1}^{n} h_i^q = 0.$
iii) $\partial^{[\alpha]} f$ is q-times differentiable on $\mathbb{R}^d$ and $M_q = \sup_{x\in\mathbb{R}^d} \|D^q \partial^{[\alpha]} f(x)\| < +\infty$.

Remark 3. When $h_n = O(n^{-a})$, with $0 < a < 1/(d+2|\alpha|)$, (H6) and (H7) ii) hold for instance for $(v_n) \equiv (n^b)$ for any $b \in]0, \min\{aq; (1 - a(d+2|\alpha|))/2\}[$.

The following theorem gives the MDP for the density estimator and its derivatives.

Theorem 3 (Pointwise MDP)**.** *When* $|\alpha| = 0$*, let Assumptions* (H1), (H3), (H6) *and* (H7) *hold; when* $|\alpha| \geq 1$*, let* (H1), (H3)–(H7) *hold. If* $\partial^{[\alpha]} f$ *is q-times differentiable at* x*, then the sequence* $\left(v_n \left(\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right)\right)$ *satisfies a LDP with speed* $\left(nh_n^{d+2|\alpha|}/v_n^2\right)$ *and good rate function* $J_{a,[\alpha],x}$ *defined in* (2).

2.4. Uniform LDP and MDP for the density estimator and its derivatives. To establish uniform large deviations principles for the density estimator and its derivatives, we need the following additionnal assumptions:

(H8) i) There exists $\xi > 0$ such that $\int_{\mathbb{R}^d} \|x\|^\xi f(x) dx < \infty$.
ii) f is uniformly continuous.

(H9) i) $\partial^{[\alpha]} K$ is Hölder continuous.
ii) There exists $\gamma > 0$ such that $z \mapsto \|z\|^\gamma \partial^{[\alpha]} K(z)$ is a bounded function.

(H10) $\lim_{n\to\infty} \frac{v_n^2 \log(1/h_n)}{nh_n^{d+2|\alpha|}} = 0$ and $\lim_{n\to\infty} \frac{v_n^2 \log v_n}{nh_n^{d+2|\alpha|}} = 0.$

(H11) i) There exists $\zeta > 0$ such that $\int_{\mathbb{R}^d} \|z\|^{\zeta} |K(z)| \, dz < \infty$.
ii) There exists $\eta > 0$ such that $z \mapsto \|z\|^{\eta} \partial^{[\alpha]} f(z)$ is a bounded function.

Remark 4. When $h_n = O(n^{-a})$ with $a \in]0, 1/(d+2|\alpha|)[$, (H10) holds for instance with $(v_n) \equiv (n^b)$ for any $b \in]0, (1 - a(d+2|\alpha|))/2[$.

Set $U \subseteq \mathbb{R}^d$; in order to state in a compact form the uniform large and moderate deviations principles for the density estimator and its derivatives on U, we consider the large deviations case as the special case $(v_n) \equiv 1$ and we set:

$$
\begin{aligned}
g_U(\delta) &= \begin{cases} \|f\|_{U,\infty} I_a\left(1 + \frac{\delta}{\|f\|_{U,\infty}}\right) & \text{if } |\alpha| = 0 \text{ and } (v_n) \equiv 1 \\ \dfrac{\delta^2 (1 + a(d+2|\alpha|))}{2\|f\|_{U,\infty} \int_{\mathbb{R}^d} \left[\partial^{[\alpha]} K\right]^2 (z) dz} & \text{otherwise,} \end{cases} \\
\tilde{g}_U(\delta) &= \min\{g_U(\delta), g_U(-\delta)\},
\end{aligned}
$$

where $\|f\|_{U,\infty} = \sup_{x \in U} |f(x)|$.

Remark 5. The functions $g_U(\cdot)$ and $\tilde{g}_U(\cdot)$ are non-negative, continuous, increasing on $]0, +\infty[$ and decreasing on $]-\infty, 0[$, with a unique global minimum in 0 ($\tilde{g}_U(0) = g_U(0) = 0$). They are thus good rate functions (and $g_U(\cdot)$ is strictly convex).

Theorem 4 below states uniform LDP and MDP for $\left(\partial^{[\alpha]} f_n - \partial^{[\alpha]} f\right)$ on U in the case U is bounded, and Theorem 5 in the case U is unbounded.

Theorem 4 (Uniform deviations on a bounded set)**.** *In the case* $|\alpha| = 0$*, let* (H1), (H2), (H7), (H9) i)*, and* (H10) *hold. In the case* $|\alpha| \geq 1$*, let* (H3)–(H5), (H7), (H9) i) *and* (H10) *hold. Moreover, assume either that* $(v_n) \equiv 1$ *or that* (v_n) *satisfies* (H6)*. Then for any bounded subset* U *of* $\mathbb{R}^d$ *and for all* $\delta > 0$*,*

$$
\lim_{n \to \infty} \frac{v_n^2}{n h_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x \in U} v_n \left|\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right| \geq \delta\right] = -\tilde{g}_U(\delta). \tag{6}
$$

Theorem 5 (Uniform deviations on an unbounded set)**.** *Let Assumptions* (H1), (H7)–(H11) *hold. Moreover,*

- *in the case* $|\alpha| = 0$ *and* $(v_n) \equiv 1$*, let (H2) holds;*
- *in the case* $|\alpha| \geq 1$ *and* $(v_n) \equiv 1$*, or* $|\alpha| \geq 0$ *and* (v_n) *satisfies* (H6)*, let* (H3)–(H5) *hold.*

Then for any subset U of $\mathbb{R}^d$ and for all $\delta > 0$,

$$-\tilde{g}_U(\delta) \le \liminf_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right| \ge \delta\right]$$
$$\le \limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right| \ge \delta\right] \le -\frac{\xi}{\xi+d}\tilde{g}_U(\delta).$$

The following corollary is a straightforward consequence of Theorem 5.

Corollary 1. *Under the assumptions of Theorem* 5*, if $\int_{\mathbb{R}^d} \|x\|^\xi f(x)dx < \infty$ $\forall \xi \in \mathbb{R}$, then for any subset U of $\mathbb{R}^d$,*

$$\lim_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x)\right| \ge \delta\right] = -\tilde{g}_U(\delta). \tag{7}$$

Comment. Theorem 4 and Corollary 1 are LDP for the sequence $(\sup_{x\in U}|f_n(x) - f(x)|)$. As a matter of fact, since the sequence $(\sup_{x\in U}|f_n(x) - f(x)|)$ is positive and since $\tilde{g}_U$ is continuous on $[0, +\infty[$, increasing and goes to infinity as $\delta \to \infty$, the application of Lemma 5 in Worms [19] allows to deduce from (6) or (7) that $(\sup_{x\in U}|f_n(x) - f(x)|)$ satisfies a LDP with speed $\left(nh_n^d\right)$ and good rate function $\tilde{g}_U$ on $\mathbb{R}_+$.

3. Proofs.

Let $(\Psi_n^{[\alpha]})$ and $(B_n^{[\alpha]})$ be the sequences defined as

$$\Psi_n^{[\alpha]}(x) = \frac{1}{n}\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}}\left(\partial^{[\alpha]}K\left(\frac{x-X_i}{h_i}\right) - \mathbb{E}\left[\partial^{[\alpha]}K\left(\frac{x-X_i}{h_i}\right)\right]\right),$$
$$B_n^{[\alpha]}(x) = \mathbb{E}\left[\partial^{[\alpha]} f_n(x)\right] - \partial^{[\alpha]} f(x).$$

We have:

$$\partial^{[\alpha]} f_n(x) - \partial^{[\alpha]} f(x) = \Psi_n^{[\alpha]}(x) + B_n^{[\alpha]}(x). \tag{8}$$

Theorems 1, 2 and 3 are consequences of (8) and the following propositions.

Proposition 2 (Pointwise LDP and MDP for $(\Psi_n^{[\alpha]})$)**.**

1. *Under the assumptions* (H1) *and* (H2)*, the sequence $(f_n(x) - \mathbb{E}(f_n(x)))$ satisfies a LDP with speed $\left(nh_n^d\right)$ and rate function $I_{a,x}$.*

2. *Let* $|\alpha| \geq 1$ *and assume that* (H3), (H4) *hold, then the sequence* $\left(\Psi_n^{[\alpha]}(x)\right)$ *satisfies a LDP with speed* $\left(nh_n^{d+2|\alpha|}\right)$ *and rate function* $J_{a,[\alpha],x}$.

3. *When* $|\alpha| = 0$, *let Assumptions* (H1), (H2) *and* (H6) *hold and when* $|\alpha| \geq 1$, *let* (H3), (H4) *and* (H6) *hold, then the sequence* $\left(v_n \Psi_n^{[\alpha]}(x)\right)$ *satisfies a LDP with speed* $\left(nh_n^{d+2|\alpha|}/v_n^2\right)$ *and rate function* $J_{a,[\alpha],x}$.

Proposition 3 (Uniform LDP and MDP for $(\Psi_n^{[\alpha]})$)**.**

1. *In the case* $|\alpha| = 0$, *let* (H1), (H2), (H9) i) *and* (H10) *hold. In the case* $|\alpha| \geq 1$, *let* (H3)–(H5), (H9) i) *and* (H10) *hold. Moreover, assume either that* $(v_n) \equiv 1$ *or that* (v_n) *satisfies* (H6)*; then for any bounded subset* U *of* $\mathbb{R}^d$ *and for all* $\delta > 0$,

$$\lim_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\partial^{[\alpha]}\Psi_n(x)\right| \geq \delta\right] = -\tilde{g}_U(\delta).$$

2. *Let Assumptions* (H1), (H8)–(H11) *hold. Moreover,*

   - *in the case* $|\alpha| = 0$ *and* $(v_n) \equiv 1$, *let* (H2) *holds,*
   - *in the case either* $|\alpha| \geq 1$ *and* $(v_n) \equiv 1$, *or* $|\alpha| \geq 0$ *and* (v_n) *satisfies* (H6), *let* (H3)–(H5) *hold.*

   Then for any subset U *of* $\mathbb{R}^d$ *and for all* $\delta > 0$,

$$\begin{aligned} -\tilde{g}_U(\delta) &\leq \liminf_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\partial^{[\alpha]}\Psi_n(x)\right| \geq \delta\right] \\ &\leq \limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\partial^{[\alpha]}\Psi_n(x)\right| \geq \delta\right] \leq -\frac{\xi}{\xi+d}\tilde{g}_U(\delta). \end{aligned}$$

Proposition 4 (Pointwise and uniform convergence rate of $B_n^{[\alpha]}$)**.** *Let Assumptions* (H1), (H3)–(H5) *and* (H7) i) *hold.*

1) *If* $\partial^{[\alpha]} f$ *is q-times differentiable at* x, *then*

$$\mathbb{E}\left(\partial^{[\alpha]} f_n(x)\right) - \partial^{[\alpha]} f(x) = O\left(\frac{\sum_{i=1}^n h_i^q}{n}\right).$$

2) If (H7) iii) *holds, then:*

$$\lim_{n\to\infty} \frac{n}{\sum_{i=1}^{n} h_i^q} \sup_{x\in\mathbb{R}^d} \left|\mathbb{E}\left(\partial^{[\alpha]} f_n(x)\right) - \partial^{[\alpha]} f(x)\right| \quad \leq \quad \frac{M_q}{q!} \int_{\mathbb{R}^d} \|z\|^q \left|K(z)\right| dz.$$

Set $x \in \mathbb{R}^d$; since the assumptions of Theorems 1 and 2 guarantee that $\lim_{n\to\infty} B_n^{[\alpha]}(x) = 0$, Theorem 1 (respectively Theorem 2) is a straightforward consequence of the application of Part 1 (respectively of Part 2) of Proposition 2. Moreover, under the assumptions of Theorem 3, we have, by application of Part 1 of Proposition 4, $\lim_{n\to\infty} v_n B_n^{[\alpha]}(x) = 0$; Theorem 3 thus straightfully follows from the application of Part 3 of Proposition 2. Finally, Theorems 4 and 5 are obtained by applying Proposition 3 together with the second part of Proposition 4.

We now state a preliminary lemma, which will be used in the proof of Propositions 2 and 3. For any $u \in \mathbb{R}$, set

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{E}\left[\exp\left(u \frac{nh_n^{d+2|\alpha|}}{v_n} \Psi_n^{[\alpha]}(x)\right)\right], \\
\Lambda_x^L(u) &= f(x)\left(\psi_a(u) - u\right), \\
\Lambda_x^M(u) &= \frac{u^2}{2(1 + a\,(d + 2|\alpha|))} f(x) \int_{\mathbb{R}^d} \left[\partial^{[\alpha]} K(z)\right]^2 dz.
\end{aligned}$$

Lemma 1 (Convergence of $\Lambda_{n,x}$)**.**

- *In the case* $|\alpha| = 0$ *and* $(v_n) \equiv 1$*, let* (H1) *and* (H2) *hold;*
- *In the case either* $|\alpha| \geq 1$ *and* $(v_n) \equiv 1$*, or* $|\alpha| \geq 0$ *and* (v_n) *satisfies* (H6)*, let* (H1), (H3) *and* (H4) *hold.*

1. (Pointwise convergence)
If f is continuous at x, then for all $u \in \mathbb{R}$,

$$\lim_{n\to\infty} \Lambda_{n,x}(u) = \Lambda_x(u) \tag{9}$$

where

$$\Lambda_x(u) = \begin{cases} \Lambda_x^L(u) & \textit{when } \; v_n \equiv 1 \;\; \textit{and} \;\; |\alpha| = 0 \\ \Lambda_x^M(u) & \textit{when either } \; v_n \to \infty \;\; \textit{and} \;\; |\alpha| \geq 0 \\ & \qquad\qquad \textit{or } \; v_n \equiv 1 \;\; \textit{and} \;\; |\alpha| \geq 1. \end{cases}$$

2. *(Uniform convergence)*
   If f is uniformly continuous, then the convergence (9) *holds uniformly in $x \in U$.*

Our proofs are now organized as follows: Lemma 1 is proved in Section 3.1, Proposition 2 in Section 3.2, Proposition 3 in Section 3.3 and Proposition 4 in Section 3.4. Section 3.5 is devoted to the proof of Proposition 1 on the rate function I_a.

3.1. Proof of Lemma 1. Set $u \in \mathbb{R}$, $u_n = u/v_n$, $Y_i = \partial^{[\alpha]} K\left(\frac{x-X_i}{h_i}\right)$ and $a_n = nh_n^{d+2|\alpha|}$. We have:

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{v_n^2}{a_n}\log\mathbb{E}\left[\exp\left(u_n a_n \Psi_n^{[\alpha]}(x)\right)\right] \\
&= \frac{v_n^2}{a_n}\log\mathbb{E}\left[\exp\left(\frac{u_n a_n}{n}\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}}\left(Y_i - \mathbb{E}(Y_i)\right)\right)\right] \\
&= \frac{v_n^2}{a_n}\sum_{i=1}^n \log\mathbb{E}\left[\exp\left(\frac{u_n a_n}{nh_i^{d+|\alpha|}}Y_i\right)\right] - \frac{uv_n}{n}\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}}\mathbb{E}(Y_i).
\end{aligned}$$

By Taylor expansion, there exists $c_{i,n}$ between 1 and $\mathbb{E}\left[\exp\left(\frac{u_n a_n}{nh_i^{d+|\alpha|}}Y_i\right)\right]$ such that

$$\begin{aligned}
&\log\left(\mathbb{E}\left[\exp\left(\frac{u_n a_n}{nh_i^{d+|\alpha|}}Y_i\right)\right]\right) \\
&\quad= \mathbb{E}\left[\exp\left(\frac{u_n a_n}{nh_i^{d+|\alpha|}}Y_i\right) - 1\right] - \frac{1}{2c_{i,n}^2}\left(\mathbb{E}\left[\exp\left(\frac{u_n a_n}{nh_i^{d+|\alpha|}}Y_i\right) - 1\right]\right)^2
\end{aligned}$$

and $\Lambda_{n,x}$ can be rewritten as

$$\begin{aligned}
&\Lambda_{n,x}(u) \\
&\quad= \frac{v_n^2}{a_n}\sum_{i=1}^n \mathbb{E}\left[\exp\left(\frac{u_n a_n}{nh_i^{d+|\alpha|}}Y_i\right) - 1\right] \\
&\qquad -\frac{v_n^2}{2a_n}\sum_{i=1}^n \frac{1}{c_{i,n}^2}\left(\mathbb{E}\left[\exp\left(\frac{u_n a_n}{nh_i^{d+|\alpha|}}Y_i\right)\right] - 1\right)^2 \\
&\qquad -\frac{uv_n}{n}\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}}\mathbb{E}(Y_i). \qquad (10)
\end{aligned}$$

For proving Lemma 1, we consider two cases:

3.1.1. First case: either $v_n \to \infty$ or $|\alpha| \geq 1$. A Taylor's expansion implies the existence of $c'_{i,n}$ between 0 and $\frac{u_n a_n}{n h_i^{d+|\alpha|}} Y_i$ such that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{u_n a_n}{n h_i^{d+|\alpha|}} Y_i\right) - 1\right] \\
&= \frac{u_n a_n}{n h_i^{d+|\alpha|}} \mathbb{E}(Y_i) + \frac{1}{2}\left(\frac{u_n a_n}{n h_i^{d+|\alpha|}}\right)^2 \mathbb{E}(Y_i^2) + \frac{1}{6}\left(\frac{u_n a_n}{n h_i^{d+|\alpha|}}\right)^3 \mathbb{E}\left(Y_i^3 e^{c'_{i,n}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{u^2 a_n}{2n^2} \sum_{i=1}^{n} \frac{1}{h_i^{2d+2|\alpha|}} \mathbb{E}\left[Y_i^2\right] + R_{n,x}^{(1)}(u) \\
&= f(x) \frac{u^2 a_n}{2n^2} \sum_{i=1}^{n} \frac{1}{h_i^{d+2|\alpha|}} \int_{\mathbb{R}^d} \left(\partial^{[\alpha]} K(z)\right)^2 dz + R_{n,x}^{(1)}(u) + R_{n,x}^{(2)}(u)
\end{aligned}$$

with

$$\begin{aligned}
R_{n,x}^{(1)}(u) &= \frac{1}{6} \frac{u^3 a_n^2}{n^3 v_n} \sum_{i=1}^{n} h_i^{-3d-3|\alpha|} \mathbb{E}\left(Y_i^3 e^{c'_{i,n}}\right) \\
&\quad - \frac{v_n^2}{2a_n} \sum_{i=1}^{n} \frac{1}{c_{i,n}^2} \left(\mathbb{E}\left[\exp\left(\frac{u_n a_n}{n h_i^{d+|\alpha|}} Y_i\right) - 1\right]\right)^2 \\
R_{n,x}^{(2)}(u) &= \frac{u^2 a_n}{2n^2} \sum_{i=1}^{n} \frac{1}{h_i^{d+2|\alpha|}} \int_{\mathbb{R}^d} \left(\partial^{[\alpha]} K(z)\right)^2 \left[f(x - h_i z) - f(x)\right] dz.
\end{aligned}$$

Since $|Y_i| \leq \|\partial^{[\alpha]} K\|_\infty$, we have

$$\left|\frac{u_n a_n}{n h_i^{d+|\alpha|}} Y_i\right| \leq |u_n| \|\partial^{[\alpha]} K\|_\infty,$$

so that

$$c'_{i,n} \leq |u_n| \|\partial^{[\alpha]} K\|_\infty, \tag{11}$$

and

$$\frac{1}{c_{i,n}^2} \leq \exp\left(2|u_n| \|\partial^{[\alpha]} K\|_\infty\right). \tag{12}$$

Noting that $\mathbb{E}|Y_i|^3 \leq h_i^d \|f\|_\infty \int_{\mathbb{R}^d} \left|\partial^{[\alpha]} K(z)\right|^3 dz$. Hence, using (5) and (11), there exists a positive constant c_1 such that, for n large enough,

$$\begin{aligned} & \left| \frac{u^3 a_n^2}{n^3 v_n} \sum_{i=1}^n h_i^{-3d-3|\alpha|} \mathbb{E}\left(Y_i^3 e^{c'_{i,n}}\right) \right| \\ (13) \qquad & \leq \quad c_1 \frac{\|f\|_\infty |u|^3 e^{|u_n| \|\partial^{[\alpha]} K\|_\infty} h_n^{|\alpha|}}{(1 + a(2d + 3|\alpha|))\, v_n} \int_{\mathbb{R}^d} \left|\partial^{[\alpha]} K(z)\right|^3 dz \end{aligned}$$

which goes to 0 as $n \to \infty$ since either $v_n \to \infty$ or $|\alpha| \geq 1$.
In the same way, in view of (5) and (12), there exists a positive constant c_2 such that, for n large enough,

$$\begin{aligned} & \left| \frac{v_n^2}{2a_n} \sum_{i=1}^n \frac{1}{c_{i,n}^2} \left(\mathbb{E}\left[\exp\left(\frac{u_n a_n}{n h_i^{d+|\alpha|}} Y_i \right) - 1 \right] \right)^2 \right| \\ (14) \qquad & \leq \quad c_2 u^2 \|f\|_\infty^2 e^{4|u_n| \|\partial^{[\alpha]} K\|_\infty} \left(\int_{\mathbb{R}^d} \left|\partial^{[\alpha]} K(z)\right| dz \right)^2 h_n^d \end{aligned}$$

which goes to 0 as $n \to \infty$. The combination of (13) and (14) ensures that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} |R_{n,x}^{(1)}(u)| = 0.$$

Now, since f is continuous, we have $\lim_{i\to\infty} |f(x - h_i z) - f(x)| = 0$, and thus, by the dominated convergence theorem, (H4)ii) implies that

$$\lim_{i\to\infty} \int_{\mathbb{R}^d} \left(\partial^{[\alpha]} K(z)\right)^2 |f(x - h_i z) - f(x)|\, dz = 0.$$

Since, in view of (5), the sequence $\left(\frac{u^2 a_n}{2n^2} \sum_{i=1}^n \frac{1}{h_i^{d+2|\alpha|}} \right)$ is bounded, it follows that $\lim_{n\to\infty} |R_{n,x}^{(2)}(u)| = 0$. The pointwise convergence (9) then follows.
In the case f is uniformly continuous, we have $\lim_{i\to\infty} \sup_{x\in\mathbb{R}^d} |f(x - h_i z) - f(x)| = 0$ and thus, using the same arguments as previously, we obtain $\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} |R_{n,x}^{(2)}(u)| = 0$.
We then deduce that $\lim_{n\to\infty} \sup_{x\in U} |\Lambda_{n,x}(u) - \Lambda_x(u)| = 0$ which concludes the proof of Lemma 1 in this case.

3.1.2. Second case: $|\boldsymbol{\alpha}| = 0$ and $(\boldsymbol{v_n}) \equiv 1$. Using assumption (H2) and in view of (10), we have

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{1}{a_n}\sum_{i=1}^{n}\mathbb{E}\left[e^{\frac{ua_n}{nh_i^d}Y_i}-1\right]-\frac{1}{2a_n}\sum_{i=1}^{n}\frac{1}{c_{i,n}^2}\left(\mathbb{E}\left[e^{\frac{ua_n}{nh_i^d}Y_i}-1\right]\right)^2\\
&\quad -\frac{u}{n}\sum_{i=1}^{n}h_i^{-d}\mathbb{E}(Y_i)\\
&= \frac{1}{nh_n^d}\sum_{i=1}^{n}h_i^d\int_{\mathbb{R}^d}\left[e^{u\left(\frac{i}{n}\right)^{ad}K(z)}-1\right]f(x)dz-u\int_{\mathbb{R}^d}K(z)f(x)dz\\
&\quad -R_{n,x}^{(3)}(u)+R_{n,x}^{(4)}(u)
\end{aligned}$$

with

$$\begin{aligned}
R_{n,x}^{(3)}(u) &= \frac{1}{2nh_n^d}\sum_{i=1}^{n}\frac{1}{c_{i,n}^2}\left(\mathbb{E}\left[e^{\frac{uh_n^d}{h_i^d}Y_i}-1\right]\right)^2\\
R_{n,x}^{(4)}(u) &= \frac{1}{nh_n^d}\sum_{i=1}^{n}h_i^d\int_{\mathbb{R}^d}\left(e^{\frac{uh_n^d}{h_i^d}K(z)}-1\right)\left[f(x-h_iz)-f(x)\right]dz\\
&\quad -\frac{u}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^d}K(z)\left[f(x-h_iz)-f(x)\right]dz.
\end{aligned}$$

Using the bound (14), we have $\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}|R_{n,x}^{(3)}(u)|=0$.
Since $|e^t-1|\leq |t|e^{|t|}$, we have

$$\begin{aligned}
\left|R_{n,x}^{(4)}(u)\right| &\leq \frac{1}{nh_n^d}\sum_{i=1}^{n}h_i^d\int_{\mathbb{R}^d}\left|\left(e^{\frac{uh_n^d}{h_i^d}K(z)}-1\right)\left[f(x-h_iz)-f(x)\right]\right|dz\\
&\quad +\frac{|u|}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^d}|K(z)|\,|f(x-h_iz)-f(x)|\\
&\leq |u|e^{|u|\|K\|_\infty}\frac{1}{nh_n^d}\sum_{i=1}^{n}h_i^d\int_{\mathbb{R}^d}|K(z)|\,|f(x-h_iz)-f(x)|\,dz\\
&\quad +\frac{|u|}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^d}|K(z)|\,|f(x-h_iz)-f(x)|\,.
\end{aligned}$$

In the case f is continuous, since in view of (5) the sequence $\left(\frac{1}{nh_n^d}\sum_{i=1}^{n}h_i^d\right)$ is bounded, the dominated convergence theorem ensures that $\lim_{n\to\infty} R_{n,x}^{(4)}(u) = 0$. In the case f is uniformly continuous, set $\varepsilon > 0$ and let $M > 0$ such that $2\|f\|_\infty \int_{\|z\|\geq M} |K(z)|\, dz < \varepsilon/2$. We need to prove that for n sufficiently large

$$\sup_{x\in\mathbb{R}^d} \int_{\|z\|\leq M} |K(z)|\, |f(x-h_i z) - f(x)|\, dz \quad \leq \quad \varepsilon/2$$

which is a straightforward consequence of the uniform continuity of f. It follows from analysis considerations that

$$\lim_{n\to\infty} \Lambda_{n,x}(u) \quad = \quad f(x)\int_{\mathbb{R}^d}\left[\int_0^1 s^{-ad}\left(e^{s^{ad}uK(z)} - 1 - us^{ad}K(z)\right) ds\right] dz,$$

and thus Lemma 1 is proved. "'□

3.2. Proof of Proposition 2. To prove Proposition 2, we apply Proposition 1, Lemma 1 and the following result (see Puhalskii [12]).

Lemma 2. *Let (Z_n) be a sequence of real random variables, (ν_n) a positive sequence satisfying $\lim_{n\to\infty}\nu_n = +\infty$, and suppose that there exists some convex non-negative function Γ defined (i.e. finite) on $\mathbb{R}$ such that*

$$\forall u \in \mathbb{R}, \quad \lim_{n\to\infty} \frac{1}{\nu_n}\log \mathbb{E}\left[\exp(u\nu_n Z_n)\right] = \Gamma(u).$$

If the Legendre transform $\tilde{\Gamma}$ of Γ is a strictly convex function, then the sequence (Z_n) satisfies a LDP of speed (ν_n) and good rate function $\tilde{\Gamma}$.

In our framework, when $|\alpha| = 0$ and $v_n \equiv 1$, we take $Z_n = f_n(x) - \mathbb{E}(f_n(x))$, $\nu_n = nh_n^d$ with $h_n = cn^{-a}$ where $0 < a < 1/d$ and $\Gamma = \Lambda_x^L$. In this case, the Legendre transform of $\Gamma = \Lambda_x^L$ is the rate function $I_{a,x} : t \mapsto f(x)I_a\left(\frac{t}{f(x)} + 1\right)$ which is strictly convex by Proposition 1. Otherwise, we take $Z_n = v_n\left(\partial^{[\alpha]}f_n(x) - \mathbb{E}\left(\partial^{[\alpha]}f_n(x)\right)\right)$, $\nu_n = nh_n^{d+2|\alpha|}/v_n^2$ and $\Gamma = \Lambda_x^M$; $\tilde{\Gamma}$ is then the quadratic rate function $J_{a,[\alpha],x}$ defined in (2) and thus Proposition 2 follows. □

3.3. Proof of Proposition 3. In order to prove Proposition 3, we first establish some lemmas.

Lemma 3. *Let $\phi_a : \mathbb{R}+ \to \mathbb{R}$ be the function defined for $\delta > 0$ as*

$$\phi_a(\delta) \quad = \quad \begin{cases} (\psi_a')^{-1}\left(\frac{\delta}{\|f\|_{U,\infty}} + 1\right) & \textit{if} \quad (v_n) \equiv 1 \quad \textit{and} \quad |\alpha| = 0, \\ \frac{\delta(1+a(d+2|\alpha|))}{\|f\|_{U,\infty}\int_{\mathbb{R}^d}\left[\partial^{[\alpha]}K\right]^2(z)dz} & \textit{otherwise.} \end{cases}$$

1. $\sup_{u\in\mathbb{R}}\{u\delta - \sup_{x\in U}\Lambda_x(u)\}$ *equals* $g_U(\delta)$ *and is achieved for* $u = \phi_a(\delta) > 0$.

2. $\sup_{u\in\mathbb{R}}\{-u\delta - \sup_{x\in U}\Lambda_x(u)\}$ *equals* $g_U(-\delta)$ *and is achieved for* $u = \phi_a(-\delta) < 0$.

Proof of Lemma 3. We just prove the first part, the proof of the second one being similar. First, let us consider the case $(v_n) \equiv 1$ and $|\alpha| = 0$. Since $e^t \geq 1 + t$ $(\forall t)$, we have $\psi(u) \geq u$ and therefore,

$$\begin{aligned} u\delta - \sup_{x\in U}\Lambda_x(u) &= u\delta - \|f\|_{U,\infty}(\psi_a(u) - u) \\ &= \|f\|_{U,\infty}\left[u\left(\frac{\delta}{\|f\|_{U,\infty}} + 1\right) - \psi_a(u)\right] \end{aligned}$$

The function $u \mapsto u\delta - \sup_{x\in U}\Lambda_x(u)$ has second derivative $-\|f\|_{U,\infty}\psi_a''(u) < 0$ and thus it has a unique maximum achieved for

$$u_0 = (\psi_a')^{-1}\left(\frac{\delta}{\|f\|_{U,\infty}} + 1\right).$$

Now, since ψ_a' is increasing and since $\psi_a'(0) = 1$, we deduce that $u_0 > 0$.
In the case $\lim_{n\to\infty} v_n = \infty$, Lemma 3 is established in the same way by noting that

$$\begin{aligned} u\delta - \sup_{x\in U}\Lambda_x(u) &= u\delta - \sup_{x\in U}\Lambda_x^M(u) \\ &= u\delta - \frac{u^2}{2\left(1 + a\left(d + 2|\alpha|\right)\right)}\|f\|_{U,\infty}\int_{\mathbb{R}^d}\left[\partial^{[\alpha]}K(z)\right]^2 dz. \quad \square \end{aligned}$$

Lemma 4.

- *In the case* $|\alpha| = 0$ *and* $(v_n) \equiv 1$, *let* (H1) *and* (H2) *hold;*
- *In the case either* $|\alpha| \geq 1$ *and* $(v_n) \equiv 1$, *or* $|\alpha| \geq 0$ *and* (v_n) *satisfies* (H6), *let* (H1), (H3) *and* (H4) *hold.*

Then for any $\delta > 0$,

$$\begin{aligned} \lim_{n\to\infty}\frac{v_n^2}{nh_n^{d+2|\alpha|}}\log\sup_{x\in U}\mathbb{P}\left[v_n\left(\partial^{[\alpha]}f_n(x) - \mathbb{E}\left(\partial^{[\alpha]}f_n(x)\right)\right) \geq \delta\right] &= -g_U(\delta) \\ \lim_{n\to\infty}\frac{v_n^2}{nh_n^{d+2|\alpha|}}\log\sup_{x\in U}\mathbb{P}\left[v_n\left(\partial^{[\alpha]}f_n(x) - \mathbb{E}\left(\partial^{[\alpha]}f_n(x)\right)\right) \leq -\delta\right] &= -g_U(-\delta) \\ \lim_{n\to\infty}\frac{v_n^2}{nh_n^{d+2|\alpha|}}\log\sup_{x\in U}\mathbb{P}\left[v_n\left|\partial^{[\alpha]}f_n(x) - \mathbb{E}\left(\partial^{[\alpha]}f_n(x)\right)\right| \geq \delta\right] &= -\tilde{g}_U(\delta). \end{aligned}$$

Proof of Lemma 4. Set $b_n = nh_n^{d+2|\alpha|}/v_n^2$, $S_n(x) = v_n\Psi_n^{[\alpha]}(x)$, and $\delta > 0$. In the sequel, $\Lambda_x(u)$ is defined as in (9).
We first note that, for any $u > 0$,

$$\begin{aligned}
\mathbb{P}\left[S_n(x) \geq \delta\right] &= \mathbb{P}\left[e^{b_n u S_n(x)} \geq e^{b_n u\delta}\right] \\
&\leq e^{-b_n u\delta}\mathbb{E}\left[e^{b_n u S_n(x)}\right] \\
&\leq e^{-b_n u\delta}e^{b_n\Lambda_{n,x}(u)} \\
&\leq e^{-b_n(u\delta-\Lambda_x(u))}e^{b_n(\Lambda_{n,x}(u)-\Lambda_x(u))}.
\end{aligned}$$

Therefore, for every $u > 0$,

$$\sup_{x\in U}\mathbb{P}\left[S_n(x) \geq \delta\right] \leq e^{-b_n\left(u\delta-\sup_{x\in U}\Lambda_x(u)\right)}e^{b_n\sup_{x\in U}|\Lambda_{n,x}(u)-\Lambda_x(u)|}. \tag{15}$$

Similarly, we prove that, for every $u < 0$,

$$\sup_{x\in U}\mathbb{P}\left[S_n(x) \leq -\delta\right] \leq e^{-b_n\left(-u\delta-\sup_{x\in U}\Lambda_x(u)\right)}e^{b_n\sup_{x\in U}|\Lambda_{n,x}(u)-\Lambda_x(u)|}. \tag{16}$$

The application of Lemma 3 to (15) and (16) yields

$$\begin{aligned}
\sup_{x\in U}\mathbb{P}\left[S_n(x) \geq \delta\right] &\leq e^{-b_n g_U(\delta)}e^{b_n\sup_{x\in U}|\Lambda_{n,x}(\phi_a(\delta))-\Lambda_x(\phi_a(\delta))|} \\
\sup_{x\in U}\mathbb{P}\left[S_n(x) \leq -\delta\right] &\leq e^{-b_n g_U(-\delta)}e^{b_n\sup_{x\in U}|\Lambda_{n,x}(\phi_a(-\delta))-\Lambda_x(\phi_a(-\delta))|}
\end{aligned}$$

and the second part of Lemma 1 provides

$$\begin{aligned}
\lim_{n\to\infty}\sup_{x\in U}|\Lambda_{n,x}(\phi_a(\delta)) - \Lambda_x(\phi_a(\delta))| &= 0 \\
\lim_{n\to\infty}\sup_{x\in U}|\Lambda_{n,x}(\phi_a(-\delta)) - \Lambda_x(\phi_a(-\delta))| &= 0.
\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{b_n}\log\sup_{x\in U}\mathbb{P}\left[S_n(x) \geq \delta\right] &\leq -g_U(\delta) \\
\limsup_{n\to\infty}\frac{1}{b_n}\log\sup_{x\in U}\mathbb{P}\left[S_n(x) \leq -\delta\right] &\leq -g_U(-\delta)
\end{aligned}$$

and thus, setting $\tilde{g}_U(\delta) = \min\{g_U(\delta), g_U(-\delta)\}$,

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \sup_{x\in U} \mathbb{P}\left[|S_n(x)| \geq \delta\right] \quad \leq \quad -\tilde{g}_U(\delta).$$

In order to conclude the proof of Lemma 4, let us note that there exists $x_0 \in \overline{U}$ such that $f(x_0) = \|f\|_{U,\infty}$. The application of Proposition 2 at the point x_0 thus yields

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left[S_n(x_0) \geq \delta\right] &= -g_U(\delta) \\
\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left[S_n(x_0) \leq -\delta\right] &= -g_U(-\delta) \\
\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left[|S_n(x_0)| \geq \delta\right] &= -\tilde{g}_U(\delta).
\end{aligned}$$

The latter relation being due to the straightforward bounds

$$\begin{aligned}
\max\{\mathbb{P}\left[S_n(x_0) \geq \delta\right], \mathbb{P}\left[S_n(x_0) \leq -\delta\right]\} &\leq \mathbb{P}\left[|S_n(x_0)| \geq \delta\right] \\
&\leq 2\max\{\mathbb{P}\left[S_n(x_0) \geq \delta\right], \mathbb{P}\left[S_n(x_0) \leq -\delta\right]\}.
\end{aligned}$$

□

Lemma 5. *Let Assumptions* (H1), (H3), (H4) i), (H9) i) *and* (H10) *hold and assume that either* $(v_n) \equiv 1$ *or* (H6) *holds.*

1. *If U is a bounded set, then, for any $\delta > 0$, we have*

$$\lim_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right] \quad \leq \quad -\tilde{g}_U(\delta).$$

2. *If U is an unbounded set, then, for any $b > 0$ and $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U, \|x\|\leq w_n} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right] \quad \leq \quad db - \tilde{g}_U(\delta)$$

where $w_n = \exp\left(bnh_n^{d+2|\alpha|}/v_n^2\right)$.

P r o o f o f L e m m a 5. Set $\rho \in]0, \delta[$, let β denote the Hölder order of $\partial^{[\alpha]}K$, and $\|\partial^{[\alpha]}K\|_H$ its corresponding Hölder norm. Set $w_n = \exp\left(bnh_n^{d+2|\alpha|}/v_n^2\right)$ and

$$R_n \quad = \quad \left(\frac{\rho n}{2\|\partial^{[\alpha]}K\|_H v_n \sum_{j=1}^n h_j^{-(d+\beta+|\alpha|)}}\right)^{\frac{1}{\beta}}.$$

We begin with the proof of the second part of Lemma 5. There exist $N'(n)$ points of $\mathbb{R}^d$, $y_1^{(n)}, y_2^{(n)}, \ldots, y_{N'(n)}^{(n)}$ such that the ball $\{x \in \mathbb{R}^d; \|x\| \leq w_n\}$ can be covered by the $N'(n)$ balls $B_i^{(n)} = \{x \in \mathbb{R}^d; \|x - y_i^{(n)}\| \leq R_n\}$ and such that $N'(n) \leq 2\left(\frac{2w_n}{R_n}\right)^d$. Considering only the $N(n)$ balls that intersect $\{x \in U; \|x\| \leq w_n\}$, we can write

$$\{x \in U; \|x\| \leq w_n\} \subset \cup_{i=1}^{N(n)} B_i^{(n)}.$$

For each $i \in \{1, \ldots, N(n)\}$, set $x_i^{(n)} \in B_i^{(n)} \cap U$. We then have:

$$\begin{aligned}
\mathbb{P}\left[\sup_{x \in U, \|x\| \leq w_n} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right] &\leq \sum_{i=1}^{N(n)} \mathbb{P}\left[\sup_{x \in B_i^{(n)}} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right] \\
&\leq N(n) \max_{1 \leq i \leq N(n)} \mathbb{P}\left[\sup_{x \in B_i^{(n)}} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right].
\end{aligned}$$

Now, for any $i \in \{1, \ldots, N(n)\}$ and any $x \in B_i^{(n)}$,

$$\begin{aligned}
v_n \left|\Psi_n^{[\alpha]}(x)\right| &\leq v_n \left|\Psi_n^{[\alpha]}(x_i^{(n)})\right| \\
&\quad + \frac{v_n}{n} \sum_{j=1}^{n} \frac{1}{h_j^{d+|\alpha|}} \left|\partial^{[\alpha]} K\left(\frac{x - X_j}{h_j}\right) - \partial^{[\alpha]} K\left(\frac{x_i^{(n)} - X_j}{h_j}\right)\right| \\
&\quad + \frac{v_n}{n} \sum_{j=1}^{n} \frac{1}{h_j^{d+|\alpha|}} \mathbb{E}\left|\partial^{[\alpha]} K\left(\frac{x - X_j}{h_j}\right) - \partial^{[\alpha]} K\left(\frac{x_i^{(n)} - X_j}{h_j}\right)\right| \\
&\leq v_n \left|\Psi_n^{[\alpha]}(x_i^{(n)})\right| + \frac{2v_n}{n} \|\partial^{[\alpha]} K\|_H \sum_{j=1}^{n} \frac{1}{h_j^{d+|\alpha|}} \left(\frac{\|x - x_i^{(n)}\|}{h_j}\right)^\beta \\
&\leq v_n \left|\Psi_n^{[\alpha]}(x_i^{(n)})\right| + \frac{2v_n}{n} \|\partial^{[\alpha]} K\|_H \sum_{j=1}^{n} h_j^{-(d+\beta+|\alpha|)} R_n^\beta \\
&\leq v_n \left|\Psi_n^{[\alpha]}(x_i^{(n)})\right| + \rho.
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
\mathbb{P}\left[\sup_{x \in U, \|x\| \leq w_n} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right] &\leq N(n) \max_{1 \leq i \leq N(n)} \mathbb{P}\left[v_n \left|\Psi_n^{[\alpha]}(x_i^{(n)})\right| \geq \delta - \rho\right] \\
&\leq N(n) \sup_{x \in U} \mathbb{P}\left[v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta - \rho\right].
\end{aligned}$$

Let us at first assume that

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log N(n) \leq db. \tag{17}$$

The application of Lemma 4 then yields

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U,\|x\|\leq w_n} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right] \\
&\leq \limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log N(n) - \tilde{g}_U(\delta-\rho) \\
&\leq db - \tilde{g}_U(\delta-\rho).
\end{aligned}$$

Since this inequality holds for any $\rho \in]0,\delta[$, Part 2 of Lemma 5 thus follows from the continuity of $\tilde{g}_U$.
Let us now establish Relation (17). By definition of $N(n)$ and w_n, we have $\log N(n) \leq \log N'(n) \leq dbnh_n^{d+2|\alpha|}/v_n^2 + (d+1)\log 2 - d\log R_n$, with

$$\begin{aligned}
&\frac{v_n^2}{nh_n^{d+2|\alpha|}} \log R_n \\
&= \frac{v_n^2}{\beta nh_n^{d+2|\alpha|}} \left[\log\rho + \log n - \log\left(2\|\partial^{[\alpha]}K\|_H\right) - \log v_n - \log\left(\sum_{j=1}^n h_j^{-(d+\beta+|\alpha|)}\right)\right],
\end{aligned}$$

which goes to zero in view of (H10) and (5). Thus, (17) is proved, and the proof of part 2 of Lemma 5 is completed.

Let us now consider part 1 of Lemma 5. This part is proved by following the same steps as for part 2, except that the number $N(n)$ of balls covering U is at most the integer part of $(\Delta/R_n)^d$, where Δ denotes the diameter of $\overline{U}$. Relation (17) then becomes

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log R_n \leq 0$$

and Lemma 5 is proved. $\square$

Lemma 6. *Let Assumptions* (H1), (H3), (H4) i) *and* (H11) ii) *hold. Assume that either* $(v_n) \equiv 1$ *or* (H6), (H10) *and* (H11) i) *hold. Moreover assume* *that $\partial^{[\alpha]}f$ is continuous. For any $b>0$ if we set $w_n=\exp\left(bnh_n^{d+2|\alpha|}/v_n^2\right)$ then, for any $\rho>0$, we have, for n large enough,*

$$\sup_{x\in U,\|x\|\geq w_n}\frac{v_n}{n}\sum_{i=1}^n\frac{1}{h_i^{d+|\alpha|}}\left|\mathbb{E}\left[\partial^{[\alpha]}K\left(\frac{x-X_i}{h_i}\right)\right]\right|\leq\rho.$$

Proof of Lemma 6. We have

$$(18)\quad \frac{v_n}{n}\sum_{i=1}^n\frac{1}{h_i^{d+|\alpha|}}\mathbb{E}\left[\partial^{[\alpha]}K\left(\frac{x-X_i}{h_i}\right)\right] = \frac{v_n}{n}\sum_{i=1}^n\int_{\mathbb{R}^d}K(z)\partial^{[\alpha]}f(x-h_iz)dz.$$

Set $\rho>0$. In the case $(v_n)\equiv 1$, set M such that $\|\partial^{[\alpha]}f\|_\infty\int_{\|z\|>M}|K(z)|\,dz\leq\rho/2$; we have

$$\begin{aligned}
&\frac{v_n}{n}\sum_{i=1}^n\frac{1}{h_i^{d+|\alpha|}}\left|\mathbb{E}\left[\partial^{[\alpha]}K\left(\frac{x-X_i}{h_i}\right)\right]\right|\\
&\leq \frac{\rho}{2}+\partial^{[\alpha]}f(x)\int_{\|z\|\leq M}|K(z)|\,dz\\
&\quad+\frac{1}{n}\sum_{i=1}^n\int_{\|z\|\leq M}|K(z)|\left|\partial^{[\alpha]}f(x-h_iz)-\partial^{[\alpha]}f(x)\right|dz.
\end{aligned}$$

Lemma 6 then follows from the fact that $\partial^{[\alpha]}f$ fulfills (H11) ii). As a matter of fact, this condition implies that $\lim_{\|x\|\to\infty,x\in\overline{U}}\partial^{[\alpha]}f(x)=0$ and that the third term in the right-hand-side of the previous inequality goes to 0 as $n\to\infty$ (by the dominated convergence).
Let us now assume that $\lim_{n\to\infty}v_n=\infty$; relation (18) can be rewritten as

$$\begin{aligned}
\frac{v_n}{n}\sum_{i=1}^n\frac{1}{h_i^{d+|\alpha|}}\mathbb{E}\left[\partial^{[\alpha]}K\left(\frac{x-X_i}{h_i}\right)\right] &= \frac{v_n}{n}\sum_{i=1}^n\int_{\|z\|\leq w_n/2}K(z)\partial^{[\alpha]}f(x-h_iz)dz\\
&\quad+\frac{v_n}{n}\sum_{i=1}^n\int_{\|z\|>w_n/2}K(z)\partial^{[\alpha]}f(x-h_iz)dz.
\end{aligned}$$

Set $\rho>0$; on the one hand, we have

$$\begin{aligned}
\|x\|\geq w_n \quad\text{and}\quad \|z\|\leq w_n/2 &\Rightarrow \|x-h_iz\|\geq w_n\left(1-h_i/2\right)\\
&\Rightarrow \|x-h_iz\|\geq w_n/2\quad\text{for } n \text{ large enough.}
\end{aligned}$$

Set $M_f = \sup_{x\in\mathbb{R}^d} \|x\|^\eta \partial^{[\alpha]} f(x)$. Assumption (H11) ii) implies that, for n sufficiently large,

$$
\begin{aligned}
\sup_{\|x\|\geq w_n} \frac{v_n}{n}\sum_{i=1}^n \int_{\|z\|\leq w_n/2} & \left|K(z)\partial^{[\alpha]} f(x-h_i z)\right| dz \\
& \leq \sup_{\|x\|\geq w_n} \frac{v_n}{n}\sum_{i=1}^n \int_{\|z\|\leq w_n/2} |K(z)|\, M_f \|x-h_i z\|^{-\eta} dz \\
& \leq 2^\eta \frac{v_n}{w_n^\eta} M_f \int_{\mathbb{R}^d} |K(z)|\, dz \\
\leq & \quad \frac{\rho}{2}.
\end{aligned}
$$

On the other hand, we note that, in view of assumptions (H10) and (H11) i),

$$
\begin{aligned}
& \sup_{\|x\|\geq w_n} \frac{v_n}{n}\sum_{i=1}^n \int_{\|z\|> w_n/2} \left|K(z)\partial^{[\alpha]} f(x-h_i z)\right| dz \\
\leq\ & 2^\zeta \frac{v_n}{w_n^\zeta} \|\partial^{[\alpha]} f\|_\infty \int_{\|z\|>w_n/2} \|z\|^\zeta\, |K(z)|\, dz \\
\leq\ & \frac{\rho}{2}
\end{aligned}
$$

(for n large enough). As a matter of fact, we have by assumptions (H6) and (H10), $\forall \xi > 0$

$$
\frac{v_n}{w_n^\xi} = \exp\left\{-\xi \log w_n \left(1 - \frac{\log v_n}{\xi \log w_n}\right)\right\} \overset{n\to\infty}{\longrightarrow} 0.
$$

This conclude the proof of Lemma 6. $\square$

Since $\partial^{[\alpha]} K$ is a bounded function that vanishes at infinity, we have $\lim_{\|x\|\to\infty} |\Psi_n^{[\alpha]}(x)| = 0$ for every given $n \geq 1$. Moreover, since $\partial^{[\alpha]} K$ is assumed to be continuous, $\Psi_n^{[\alpha]}$ is continuous, and this ensures the existence of a random variable s_n such that

$$
\left|\Psi_n^{[\alpha]}(s_n)\right| \quad = \quad \sup_{x\in U} \left|\Psi_n^{[\alpha]}(x)\right|.
$$

Lemma 7. *Let Assumptions* (H1), (H3), (H4) i), (H8) i), (H9) ii) *and* (H10) *hold. Suppose either* $(v_n) \equiv 1$ *or* (H6) *and* (H11) *hold. For any* $b > 0$*, set*

$w_n = \exp\left(bnh_n^{d+2|\alpha|}/v_n^2\right)$; *then, for any* $\delta > 0$, *we have*

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\|s_n\| \geq w_n \quad \textit{and} \quad v_n \left|\Psi_n^{[\alpha]}(s_n)\right| \geq \delta\right] \leq -b\xi. \tag{19}$$

Proof of Lemma 7. We first note that $s_n \in \overline{U}$ and therefore

$$\begin{aligned}
&\|s_n\| \geq w_n \quad \text{and} \quad v_n \left|\Psi_n^{[\alpha]}(s_n)\right| \geq \delta \\
&\quad \Rightarrow \quad \|s_n\| \geq w_n \quad \text{and} \quad \frac{v_n}{n}\left|\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}} \partial^{[\alpha]} K\left(\frac{s_n - X_i}{h_i}\right)\right| \\
&\qquad\qquad + \frac{v_n}{n}\mathbb{E}\left|\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}} \partial^{[\alpha]} K\left(\frac{s_n - X_i}{h_i}\right)\right| \geq \delta \\
&\quad \Rightarrow \quad \|s_n\| \geq w_n \quad \text{and} \quad \frac{v_n}{n}\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}} \left|\partial^{[\alpha]} K\left(\frac{s_n - X_i}{h_i}\right)\right| \geq \delta \\
&\qquad\qquad - \sup_{\|x\|\geq w_n, x\in\overline{U}} \frac{v_n}{n}\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}} \mathbb{E}\left|\partial^{[\alpha]} K\left(\frac{x - X_i}{h_i}\right)\right|
\end{aligned}$$

Set $\rho \in]0, \delta[$; the application of Lemma 6 ensures that, for n large enough,

$$\begin{aligned}
&\|s_n\| \geq w_n \quad \text{and} \quad v_n \left|\Psi_n^{[\alpha]}(s_n)\right| \geq \delta \\
&\quad \Rightarrow \quad \|s_n\| \geq w_n \quad \text{and} \quad \frac{v_n}{n}\sum_{i=1}^n \frac{1}{h_i^{d+|\alpha|}} \left|\partial^{[\alpha]} K\left(\frac{s_n - X_i}{h_i}\right)\right| \geq \delta - \rho.
\end{aligned}$$

Set $\kappa = \sup_{x\in\mathbb{R}^d} \|x\|^\gamma |\partial^{[\alpha]} K(x)|$ (see Assumption (H9) ii)). We obtain, for n sufficiently large,

$$
\begin{aligned}
&\|s_n\| \geq w_n \quad \text{and} \quad v_n \left|\Psi_n^{[\alpha]}(s_n)\right| \geq \delta \\
&\quad \Rightarrow \quad \|s_n\| \geq w_n \quad \text{and} \quad \exists i \in \{1, \dots, n\} \quad \text{such that} \\
&\qquad\qquad\qquad\qquad \frac{v_n}{h_i^{d+|\alpha|}} \left|\partial^{[\alpha]} K\left(\frac{s_n - X_i}{h_i}\right)\right| \geq \delta - \rho \\
&\quad \Rightarrow \quad \|s_n\| \geq w_n \quad \text{and} \quad \exists i \in \{1, \dots, n\} \quad \text{such that} \\
&\qquad\qquad\qquad\qquad \kappa h_i^{\gamma} \geq \frac{h_i^{d+|\alpha|}}{v_n} \|s_n - X_i\|^{\gamma} (\delta - \rho) \\
&\quad \Rightarrow \quad \|s_n\| \geq w_n \quad \text{and} \quad \exists i \in \{1, \dots, n\} \quad \text{such that} \\
&\qquad\qquad\qquad\qquad \big|\|s_n\| - \|X_i\|\big| \leq \left[\frac{\kappa v_n h_i^{\gamma-d-|\alpha|}}{\delta - \rho}\right]^{\frac{1}{\gamma}} \\
&\quad \Rightarrow \quad \|s_n\| \geq w_n \quad \text{and} \quad \exists i \in \{1, \dots, n\} \quad \text{such that} \\
&\qquad\qquad\qquad\qquad \|X_i\| \geq \|s_n\| - \left[\frac{\kappa v_n h_i^{\gamma-d-|\alpha|}}{\delta - \rho}\right]^{\frac{1}{\gamma}} \\
&\quad \Rightarrow \quad \exists i \in \{1, \dots, n\} \quad \text{such that} \\
&\qquad\qquad\qquad\qquad \|X_i\| \geq w_n \left(1 - u_{n,i}\right) \quad \text{with}
\end{aligned}
$$

$$
u_{n,i} = w_n^{-1} v_n^{\frac{1}{\gamma}} h_i^{\frac{\gamma-d-|\alpha|}{\gamma}} \left(\frac{\kappa}{\delta - \rho}\right)^{\frac{1}{\gamma}}.
$$

Assume for the moment that

$$
\lim_{n \to \infty} u_{n,i} = 0. \tag{20}
$$

It then follows that $1 - u_{n,i} > 0$ for n sufficiently large; therefore we can deduce that (see Assumption (H8) i)):

$$
\begin{aligned}
\mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad v_n \left|\Psi_n^{[\alpha]}(s_n)\right| \geq \delta\right] &\leq \sum_{i=1}^{n} \mathbb{P}\left[\|X_i\|^{\xi} \geq w_n^{\xi} \left(1 - u_{n,i}\right)^{\xi}\right] \\
&\leq \sum_{i=1}^{n} \mathbb{E}\left(\|X_i\|^{\xi}\right) w_n^{-\xi} \left(1 - u_{n,i}\right)^{-\xi} \\
&\leq n\mathbb{E}\left(\|X_1\|^{\xi}\right) w_n^{-\xi} \max_{1 \leq i \leq n} \left(1 - u_{n,i}\right)^{-\xi}.
\end{aligned}
$$

Consequently,

$$\frac{v_n^2}{nh_n^{d+2|\alpha|}}\log\mathbb{P}\left[\|s_n\|\geq w_n \quad \text{and} \quad v_n\left|\Psi_n^{[\alpha]}(s_n)\right|\geq\delta\right]$$
$$\leq \frac{v_n^2}{nh_n^{d+2|\alpha|}}\left[\log n+\log\mathbb{E}\left(\|X_1\|^{\xi}\right)-b\xi\frac{nh_n^{d+2|\alpha|}}{v_n^2}-\xi\log\max_{1\leq i\leq n}(1-u_{n,i})\right],$$

and, thanks to assumption (H10), it follows that

$$\limsup_{n\to\infty}\frac{v_n^2}{nh_n^{d+2|\alpha|}}\log\mathbb{P}\left[\|s_n\|\geq w_n \quad \text{and} \quad v_n\left|\Psi_n^{[\alpha]}(s_n)\right|\geq\delta\right] \leq -b\xi.$$

Let us now prove relation (20). We expand

$$u_{n,i}=\exp\left(-b\frac{nh_n^{d+2|\alpha|}}{v_n^2}\left[1-\frac{1}{b\gamma}\frac{v_n^2\log v_n}{nh_n^{d+2|\alpha|}}-\frac{\gamma-d-|\alpha|}{b\gamma}\frac{v_n^2\log(h_i)}{nh_n^{d+2|\alpha|}}\right]\right)\left(\frac{\kappa}{\delta-\rho}\right)^{\frac{1}{\gamma}}$$

and assumptions (H6) and (H10) ensure that $\lim_{n\to\infty}u_{n,i}=0$ and thus Lemma 7 is proved. □

P r o o f o f P r o p o s i t i o n 3. Let us at first note that the lower bound

$$\liminf_{n\to\infty}\frac{v_n^2}{nh_n^{d+2|\alpha|}}\log\mathbb{P}\left[\sup_{x\in U}v_n\left|\Psi_n^{[\alpha]}(x)\right|\geq\delta\right]\geq-\tilde{g}_U(\delta) \tag{21}$$

follows from the application of Proposition 2 at a point $x_0\in\overline{U}$ such that $f(x_0)=\|f\|_{U,\infty}$.

In the case U is bounded, Proposition 3 is thus a straightforward consequence of (21) and of the first part of Lemma 5. Let us now consider the case U is unbounded.

Set $\delta>0$ and, for any $b>0$ set $w_n=\exp\left(bnh_n^{d+2|\alpha|}/v_n^2\right)$. Since, by definition of s_n,

$$\mathbb{P}\left[\sup_{x\in U}v_n\left|\Psi_n^{[\alpha]}(x)\right|\geq\delta\right]$$
$$\leq \mathbb{P}\left[\sup_{x\in U,\|x\|\leq w_n}v_n\left|\Psi_n^{[\alpha]}(x)\right|\geq\delta\right]+\mathbb{P}\left[\|s_n\|\geq w_n \quad \text{and} \quad v_n\left|\Psi_n^{[\alpha]}(s_n)\right|\geq\delta\right]$$

it follows from Lemmas 5 and 7 that

$$\limsup_{n\to\infty}\frac{v_n^2}{nh_n^{d+2|\alpha|}}\log\mathbb{P}\left[\sup_{x\in U}v_n\left|\Psi_n^{[\alpha]}(x)\right|\geq\delta\right] \leq \max\{-b\xi;db-\tilde{g}_U(\delta)\}$$

and consequently

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right] \quad \leq \quad \inf_{b>0} \max\{-b\xi; db - \tilde{g}_U(\delta)\}.$$

Since the infimum in the right-hand-side of the previous bound is achieved for $b = \tilde{g}_U(\delta)/(\xi + d)$ and equals $-\xi\tilde{g}_U(\delta)/(\xi + d)$, we obtain the upper bound

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{d+2|\alpha|}} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n^{[\alpha]}(x)\right| \geq \delta\right] \quad \leq \quad -\frac{\xi}{\xi + d}\tilde{g}_U(\delta)$$

which concludes the proof of Proposition 3. $\square$

Proof of Proposition 4. Let us set $g = \partial^{[\alpha]} f$, $D^j g$ $\left(j \in \{1, \ldots, q\}\right)$ the j-th differential of g, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ and $y^{(j)} = (y, \ldots, y) \in (\mathbb{R}^d)^j$. With these notations,

$$D^j g(x)(y^{(j)}) = \sum_{\alpha_1+\cdots+\alpha_d=j} \frac{\partial^j g}{\partial y_1^{\alpha_1} \ldots \partial y_d^{\alpha_d}}(x) y_1^{\alpha_1} \ldots y_d^{\alpha_d}.$$

By successive integrations by parts (and using the fact that the partial derivatives of K vanish at infinity, see Assumption (H4)i)), we have

$$\begin{aligned}
\mathbb{E}\left[\partial^{[\alpha]} f_n(x)\right] &= \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h_i^{d+|\alpha|}} \mathbb{E}\left[\partial^{[\alpha]} K\left(\frac{x - X_i}{h_i}\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h_i^{d+|\alpha|}} \int_{\mathbb{R}^d} \partial^{[\alpha]} K\left(\frac{x - y}{h_i}\right) f(y)dy \\
&= \frac{1}{n}\sum_{i=1}^{n} \frac{1}{h_i^{d}} \int_{\mathbb{R}^d} K\left(\frac{x - y}{h_i}\right) g(y)dy \\
&= \frac{1}{n}\sum_{i=1}^{n} \int_{\mathbb{R}^d} K(y) g(x - h_i y)dy.
\end{aligned}$$

Hence, using assumption (H7)i) and the fact that $\partial^{[\alpha]} f$ is q-times differentiable, it comes

$$\begin{aligned}
&\mathbb{E}\Big[\partial^{[\alpha]} f_n(x)\Big] - \partial^{[\alpha]} f(x) \\
&= \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbb{R}^d} K(y)\big[g(x-h_iy)-g(x)\big]dy \\
&= \frac{1}{n}\sum_{i=1}^{n} h_i^q \int_{\mathbb{R}^d} K(y)\left[\frac{g(x-h_iy)-g(x)-\sum_{j=1}^{q-1}\frac{(-1)^j}{j!}h_i^jD^jg(x)(y^{(j)})}{h_i^q}\right]dy.
\end{aligned}$$

Let us set

$$\begin{aligned}
U_i(x) &= \int_{\mathbb{R}^d} K(y)\left[\frac{g(x-h_iy)-g(x)-\sum_{j=1}^{q-1}\frac{(-1)^j}{j!}h_i^jD^jg(x)(y^{(j)})}{h_i^q}\right]dy \quad \text{and} \\
U_\infty(x) &= \frac{(-1)^q}{q!}\int_{\mathbb{R}^d} D^qg(x)(y^{(q)})K(y)dy.
\end{aligned}$$

We clearly have

$$\lim_{i\to\infty} U_i(x) = U_\infty(x) \tag{22}$$

and therefore, $\forall\varepsilon>0$, $\exists i_0\in\mathbb{R}$ such that $\forall i\geq i_0$, $|U_i(x)-U_\infty(x)|\leq\varepsilon$.

- If $\sum_i h_i^q=\infty$, then

$$\begin{aligned}
&\left|\frac{n}{\sum_{i=1}^n h_i^q}\left[\mathbb{E}\left(\partial^{[\alpha]} f_n(x)\right)-\partial^{[\alpha]} f(x)\right]-U_\infty(x)\right| \\
&= \left|\frac{\sum_{i=1}^n h_i^qU_i(x)}{\sum_{i=1}^n h_i^q}-U_\infty(x)\right| \\
&\leq \frac{\sum_{i=1}^{i_0-1} h_i^q\left|U_i(x)-U_\infty(x)\right|+\sum_{i=i_0}^n h_i^q\left|U_i(x)-U_\infty(x)\right|}{\sum_{i=1}^n h_i^q} \\
&\leq 2\varepsilon.
\end{aligned}$$

- If $\sum_i h_i^q<\infty$, we can write

$$\frac{n}{\sum_{i=1}^n h_i^q}\left[\mathbb{E}\left(\partial^{[\alpha]} f_n(x)\right)-\partial^{[\alpha]} f(x)\right] = \frac{\sum_{i=1}^n h_i^qU_i(x)}{\sum_{i=1}^n h_i^q}.$$

In view of (22), for x fixed and for all $i \in \mathbb{N}$, the sequence $(U_i(x))_i$ is bounded and thus Part 1 of Proposition 4 is completed. Let us now prove Part 2. Since $(U_i(x))_i$ is bounded by $\sup_{x\in\mathbb{R}^d} \|D^q g(x)\| = M_q$ (see Assumption (H7) iii)), Part 2 follows. $\square$

3.5. Proof of Proposition 1.

- Since $\left|e^t - 1\right| \le |t|\, e^{|t|}\ \forall t \in \mathbb{R}$, and thanks to the boundedness and integrability of K, we have

$$\int_{[0,1]\times\mathbb{R}^d} s^{-ad} \left| e^{s^{ad}uK(z)} - 1 \right| dsdz \quad \le \quad |u|e^{|u|\|K\|_\infty} \int_{[0,1]\times\mathbb{R}^d} |K(z)|\, dsdz < \infty$$

which ensures the existence of ψ_a. It is straightforward to check that ψ_a is twice differentiable, with

$$\begin{aligned} \psi_a'(u) &= \int_{[0,1]\times\mathbb{R}^d} K(z) e^{s^{ad}uK(z)} dsdz, \\ \psi_a''(u) &= \int_{[0,1]\times\mathbb{R}^d} s^{ad} \left(K(z)\right)^2 e^{s^{ad}uK(z)} dsdz. \end{aligned}$$

Since $\psi_a''(u) > 0\ \forall u \in \mathbb{R}$, ψ_a' is increasing on $\mathbb{R}$, and ψ_a is strictly convex on $\mathbb{R}$. It follows that its Cramer transform I_a is a good rate function on $\mathbb{R}$ (see Dembo and Zeitouni [4]) and (i) of Proposition 1 is proved.

- Let us now assume that $\lambda(S_-) = 0$. We then have

$$\lim_{u\to-\infty} \psi_a'(u) = 0 \quad \text{and} \quad \lim_{u\to+\infty} \psi_a'(u) = +\infty,$$

so that the range of ψ_a' is $]0, +\infty[$. Moreover $\lim_{u\to-\infty} \psi_a(u) = -\lambda(S_+)/(1-ad)$ (which can be $-\infty$). This implies in particular that $I_a(0) = \lambda(S_+)/(1-ad)$. Now, when $t < 0$, $\lim_{u\to-\infty} (ut - \psi_a(u)) = +\infty$, and $I_a(t) = +\infty$. Since ψ_a' is increasing with range $]0, +\infty[$, when $t > 0$, $\sup_u(ut - \psi_a(u))$ is reached for $u_0(t)$ such that $\psi_a'(u_0(t)) = t$, i.e. for $u_0(t) = (\psi_a')^{-1}(t)$; this prove (3). (Note that, since $\psi_a''(t) > 0$, the function $t \mapsto u_0(t)$ is differentiable on $]0, +\infty[$). Now, differentiating (3), we have

$$\begin{aligned} I_a'(t) &= u_0(t) + tu_0'(t) - \psi_a'(u_0(t))u_0'(t) \\ &= (\psi_a')^{-1}(t) + tu_0'(t) - tu_0'(t) \\ &= (\psi_a')^{-1}(t). \end{aligned}$$

Since $(\psi_a')^{-1}$ is an increasing function on $]0,+\infty[$, it follows that I_a is strictly convex on $]0,+\infty[$ (and differentiable). Thus (ii) is proved.
Now, since $\lambda(S_-)=0$, $\psi_a'(0)=\int_{[0,1]\times\mathbb{R}^d} K(z)dsdz=1$; we have

$$I_a'(t)=0 \quad\Leftrightarrow\quad (\psi_a')^{-1}(t)=0 \quad\Leftrightarrow\quad \psi_a'(0)=t \quad\Leftrightarrow\quad t=1.$$

Then $I_a'(1)=0$, and $I_a(1)=0$ is the unique global minimum of I_a on $]0,+\infty[$. This proves (iv) when $\lambda(S_-)=0$.

- Assume that $\lambda(S_-)>0$. In this case, ψ_a' can be rewritten as

$$\begin{aligned}\psi_a'(u) &= \int_{[0,1]\times\left(\mathbb{R}^d\cap S_+\right)} K(z)e^{s^{ad}uK(z)}dsdz \\ &\quad + \int_{[0,1]\times\left(\mathbb{R}^d\cap S_-\right)} K(z)e^{s^{ad}uK(z)}dsdz\end{aligned}$$

and we have

$$\lim_{u\to-\infty}\psi_a'(u)=-\infty \quad\text{and}\quad \lim_{u\to+\infty}\psi_a'(u)=+\infty$$

so that the range of ψ_a' is $\mathbb{R}$ in this case. The proof of (iii) and the case $\lambda(S_-)>0$ of (iv) follows the same lines as previously, except that, in the present case, $(\psi_a')^{-1}$ is defined on $\mathbb{R}$, and not only on $]0,+\infty[$. $\square$

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