Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **33** (2007), 267–286

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

LINEAR COLLIGATIONS AND DYNAMIC SYSTEM CORRESPONDING TO OPERATORS IN THE BANACH SPACE

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Communicated by E. I. Horozov

Abstract. New concepts of linear colligations and dynamic systems, corresponding to the linear operators, acting in the Banach spaces, are introduced. The main properties of the transfer function and its relation to the dual transfer function are established.

Introduction. The theory of operator colligations and open operator system are the main tool to study linear non-self adjoint operators [2]. However, generalizing of these constructions in general form for Banach spaces is still unsolved problem. A new approach to study linear operators in Banach spaces and linear dynamic systems, defined in Banach space, are presented in the article.

2000 *Mathematics Subject Classification:* Primary 47A48, 93B28, 47A65; Secondary 34C94.
Key words: Banach space, Dynamic system, Colligations, Coupling.

1. Linear colligations in Banach spaces and associated dynamic systems.

1.1. Let A be a linear bounded operator acting in the Banach space B[1]. Consider Banach spaces E and F, and operators

$$\sigma_E : E \to E, \quad K : E \to F, \quad \sigma_F : F \to F, \quad \varphi : E \to B, \quad \psi : B \to F.$$

Define a linear colligation as the assembly

$$\alpha = (\sigma_E, E, \varphi, B, A, B, \psi, F, \sigma_F, K), \tag{1.1}$$

where B is the inner space, E, F are the outer spaces of the colligation α, A is the main operator, φ, ψ are the canal operators, σ_E, σ_F are the metric operators, and K is the deforming operator.

We will associate the linear dynamic system $\mathcal{F}_\alpha = \{R_\alpha, S_\alpha\}$, describing evolution from 0 to T $(0 < T < \infty)$,

$$R_\alpha : \begin{cases} i\dfrac{d}{dt}x(t) + Ax(t) = \varphi\sigma_E u(t), \\ x(0) = x_0, \qquad (0 \le t \le T), \end{cases} \tag{1.2}$$

where $x(t) \in B$ is an internal state of the system $\mathcal{F}_\alpha$, $u(t) \in E$ is the input of the system $\mathcal{F}_\alpha$ and $x_0 \in B$ is the initial condition;

$$S_\alpha : v(t) = Ku(t) - i\psi x(t), \tag{1.3}$$

where the vector function $v(t) \in F$ is the output of the system $\mathcal{F}_\alpha$, and also $x(t)$ is the solution of (1.2). In other words the mapping R is such that

$$R_\alpha(u(t), x_0) = x(t), \tag{1.4}$$

and the transfer mapping S_α is such that

$$S_\alpha(u(t), x_0) = (v(t), x_T), \tag{1.5}$$

where $x_T = x(T)$ is the solution of $x(t)$ (1.2) at the point $t = T$.

If $B = H$ and E, F are Hilbert spaces, then the linear system $\mathcal{F} = \{R, S\}$, as well as the colligation α, may be extended owing to the external space only (i.e. owing to increase in the number of canal bonds) to the open system (for which the energy of conservation law is valid) associated with a local colligation [2].

Applied A into local colligation [2],

$$\Delta_1 = (A, H, \varphi_1, E_1, \sigma_1),$$

where $A - A^* = i\varphi_1^* \sigma_1 \varphi_1$. Form the following colligation

$$\hat{\Delta} = (A, H, \hat{\varphi}, \hat{E}, \hat{\sigma}),$$

where $\hat{E} = E \oplus E \oplus F \oplus F \oplus E_1$, and $\hat{\sigma}$ has the form

$$\hat{\sigma} = \begin{bmatrix} 0 & \sigma_E & 0 & 0 & 0 \\ \sigma_E^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_F & 0 \\ 0 & 0 & \sigma_F^* & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1 \end{bmatrix},$$

also $\hat{\varphi} f = (\varphi^* f, 0, \psi f, 0, \varphi_1 f) \in \hat{E}$; $f \in H$.

Elementary calculations shows that $\hat{\triangle}$ is a local colligation

$$A - A^* = i\hat{\varphi}^* \hat{\sigma} \hat{\varphi}.$$

Associating $\hat{\triangle}$ with an open system $\hat{\mathcal{F}}_\triangle = \{\hat{R}_\triangle, \hat{S}_\triangle\}$, where

$$\hat{R}_\Delta : \begin{cases} i\dfrac{d}{dt}h(t) + Ah(t) = \varphi\sigma_E\xi_2^- + \psi^*\sigma_F\xi_4^- + \varphi_1^*\sigma_1\xi_5^-, \\ h(0) = h_0, \end{cases}$$

then the transfer mapping of $\hat{S}_\Delta$ has the form

$$\hat{S}_\Delta : \begin{cases} \xi_1^+ = \xi_1^- - i\varphi^* h(t), \\ \xi_2^+ = \xi_2^-, \\ \xi_3^+ = \xi_3^- - i\psi h(t), \\ \xi_4^+ = \xi_4^-, \\ \xi_5^+ = \xi_5^- - i\varphi_1^* h(t), \end{cases}$$

where $\xi^\pm(t) = (\xi_1^\pm, \xi_2^\pm, \xi_3^\pm, \xi_4^\pm, \xi_5^\pm)$ are input and output signals of the systems $\hat{\mathcal{F}}_\Delta$ from the space $\hat{E}$.

Applying signals, presented as

$$\xi^-(t) = (0, \xi_2^-(t), K\xi_2^-(t), 0, 0),$$

to the input of the system $\hat{\mathcal{F}}_\Delta$, and projecting output signals onto the third component

$$\xi^+(t) = (0, 0, \xi_3^+(t), 0, 0)$$

we obtain the system $\mathcal{F}\{R, S\}$ (1.2), (1.3).

Proposition 1.1. *An open system $\mathcal{F}_\alpha$ (1.2), (1.3), corresponding to the linear colligation α, has extension (owing to increase in the number external spaces) to open system $\hat{\mathcal{F}}$, associated with a local colligation $\hat{\Delta}$ where $B = H$; E, F are Hilbert spaces.*

Thus, the problem of studying common linear systems may be reduced to the analysis of the open systems, associated with local colligations. However, for some problems it is more convenient to work with the initial system rather than with its extension. For Banach spaces B, E, and F such extension, in principle, is presumably impossible.

1.2. Let us consider linear colligation α and $\tilde{\alpha}$ where,

$$\tilde{\alpha} = (\sigma_F, F, \tilde{\varphi}, \tilde{B}, \tilde{A}, \tilde{B}, \tilde{\psi}, E, \sigma_F, \tilde{K}). \tag{1.6}$$

We associate with a linear colligation $\tilde{\alpha}$ a linear dynamic system $\tilde{\mathcal{F}}_{\tilde{\alpha}} = \{\tilde{R}_{\tilde{\alpha}}, \tilde{S}_{\tilde{\alpha}}\}$, describing evaluation in the backward, relative to the system $\mathcal{F}_\alpha$ (1.2), (1.3), direction from T $(0 < T < \infty)$ to 0,

$$\tilde{R}_{\tilde{\alpha}} : \begin{cases} i\dfrac{d}{dt}\tilde{x}(t) + \tilde{A}\tilde{x}(t) = \tilde{\varphi}\tilde{u}(t), \\ \tilde{x}(T) = \tilde{x}_T, \ (0 \le t \le T), \end{cases} \tag{1.7}$$

where $\tilde{x}(t) \in \tilde{B}$ is the internal state of the system $\tilde{\mathcal{F}}_{\tilde{\alpha}}$, $\tilde{u}(t) \in F$ is the input of the open system $\tilde{\mathcal{F}}_{\tilde{\alpha}}$, and $\tilde{x}_T \in \tilde{B}$ is the initial condition;

$$\tilde{S}_{\tilde{\alpha}} : \tilde{v}(t) = \tilde{K}\tilde{u}(t) + i\sigma_E\tilde{\psi}\tilde{x}(t), \tag{1.8}$$

where $\tilde{v}(t) \in E$ is the output of $\tilde{F}_{\tilde{\alpha}}$.

It is reasonable that mappings $\tilde{R}_{\tilde{\alpha}}$ and $\tilde{S}_{\tilde{\alpha}}$ consist in

$$\begin{aligned} \tilde{R}_{\tilde{\alpha}}(\tilde{u}(t), \tilde{x}_T) &= \tilde{x}(t), \\ \tilde{S}_{\tilde{\alpha}}(\tilde{u}(t), \tilde{x}_T) &= (\tilde{v}(t), \tilde{x}_0), \end{aligned} \tag{1.9}$$

where $\tilde{x}_0 = \tilde{x}(0)$.

Definition 1.1. *The colligation $\tilde{\alpha}$ is called dual colligation, relative to the α, if there exists a linear bounded operator S from the space B into $\tilde{B}$ such that the following relations holds*

$$
(1.10)\qquad
\begin{aligned}
&1)\quad SA - \tilde{A}S = i\tilde{\varphi}\sigma_F\psi,\\
&2)\quad S\varphi\sigma_E = \tilde{\varphi}\sigma_F K,\\
&3)\quad \sigma_E\tilde{\psi}S = \tilde{K}\sigma_F\psi,\\
&4)\quad \tilde{K}\sigma_F K = \sigma_E.
\end{aligned}
$$

Thus the operator S, ensuring duality of colligation α and $\tilde{\alpha}$, we shall call the dual operator of the colligations α and $\tilde{\alpha}$.

Remark 1.1. If the spaces $B = \tilde{B} = H$ and $E = F$ are Hilbert spaces, then the conditions (1.10) on the assumption that $\tilde{A} = A^*$, $S = I_H$, $\varphi = \tilde{\varphi} = \psi^*$, $\psi = \tilde{\psi}$, $K = \tilde{K} = I_E$ ($\sigma_E = \sigma_F$ are self adjoint operators), reduce to the colligation condition of the local colligation.

Remark 1.2. If the operators K, σ_E, σ_F are invertible, then it is not difficult to show that for colligation α there always exists dual colligation $\tilde{\alpha}$.

Indeed, let S be a linear boundedly invertible operator from the Banach space B into the Banach space $\tilde{B}$.

We define the following operators

$$
\begin{aligned}
\tilde{A} &= SAS^{-1} - iS\varphi\sigma_E K^{-1}\psi S^{-1},\\
\tilde{\varphi} &= S\varphi\sigma_E K^{-1}\sigma^{-1},\\
\tilde{\psi} &= K^{-1}\psi S^{-1},\\
\tilde{K} &= \sigma_E K^{-1}\sigma_F^{-1}.
\end{aligned}
$$

Elementary calculation shows that the above assembly defines the colligation $\tilde{\alpha}$ dual to the colligation α. The duality conditions (1.10) superimpose some bonds on the systems $\mathcal{F}_\alpha$ (1.2), (1.3) and $\tilde{\mathcal{F}}_{\tilde{\alpha}}$ (1.7), (1.8), which we call the dual bonds.

Particularly, if $S = I$, then the colligation

$$\tilde{\alpha} = (\sigma_F, F, \tilde{\varphi} = \varphi\sigma_E K^{-1}\sigma_F^{-1}, B, \tilde{A}, B, \tilde{\psi} = K^{-1}\psi, E, \sigma_E, \tilde{K} = \sigma_E K^{-1}\sigma_F^{-1}),$$

is dual to α, where

$$\tilde{A} = A - i\varphi\sigma_E K^{-1}\psi.$$

Proposition 1.2. *Suppose that the colligation $\tilde{\alpha}$ is dual to the colligation α. Then mappings $\tilde{R}_{\tilde{\alpha}}$ (1.7) and $\tilde{S}_{\tilde{\alpha}}$ (1.8) of the open system $\tilde{\mathcal{F}}_{\tilde{\alpha}}$ are defined by output $v(t)$ and x_T of the open system $\mathcal{F}_{\alpha}$ (1.2), (1.3) as follows*

$$\begin{aligned} \tilde{R}_{\tilde{\alpha}}(v(t), Sx_T) &= \tilde{x}(t) = Sx(t), \\ \tilde{S}_{\tilde{\alpha}}(\sigma_F v(t), Sx_T) &= (\sigma_E u(t), Sx_0), \end{aligned}$$

where S is the dual operator of the colligations α and $\tilde{\alpha}$ in (1.10)*.*

Proof. Let us apply operator S to the equation (1.2). Then, in virtue of the conditions 1) and 2) (1.10), we obtain

$$i\frac{d}{dt}Sx(t) + \tilde{A}Sx(t) = -i\tilde{\varphi}\sigma_F\psi x(t) + \tilde{\varphi}\sigma_F K u(t).$$

Using (1.3), then we have

$$i\frac{d}{dt}Sx(t) + \tilde{A}Sx(t) = \tilde{\varphi}\sigma_F v(t) \quad \text{and} \quad \tilde{x}_T = Sx_T.$$

Applying the operator $\tilde{K}\sigma_F$ to the expression (1.3), then we obtain

$$\sigma_E u(t) = \tilde{K}\sigma_F v(t) + i\tilde{\psi}Sx(t). \quad \square$$

1.3. Define a concept of contraction. Note that for the dual colligation α and $\tilde{\alpha}$ the contracting procedure [2] must be carried out in "the opposite direction" since the respective dual system $\mathcal{F}_{\alpha}$ and $\tilde{\mathcal{F}}_{\tilde{\alpha}}$ describe the evolution in mutually inverse directions.

Consider the linear the colligation α and α' where,

$$\alpha' = (\sigma_{E'}, E', \varphi', B', A', B', \psi', F', \sigma_{F'}, K'), \tag{1.11}$$

with $F = E'$ and $\sigma_{E'} = \sigma_F$.

Definition 1.2. *The linear colligation*

$$\alpha^0 = \alpha' \vee \alpha = (\sigma_{E^0}, E^0, \varphi^0, B^0, A^0, B^0, \psi^0, F^0, \sigma_{F^0}, K^0), \tag{1.12}$$

is called the coupling of the linear colligation α and α', where $B^0 = B'\dot{+}B$ and the main operator A^0 equal to

$$A^0 = \begin{bmatrix} A' & i\varphi'\sigma_{E'}\psi \\ 0 & A \end{bmatrix} : B^0 \to B^0, \tag{1.12'}$$

and

$$\begin{aligned} \varphi^0 &= \varphi \dot{+} \varphi' \sigma_{E'} K \sigma_E^{-1}, \\ \psi^0 &= K'\psi + \psi', \\ K^0 &= K'K, \end{aligned} \tag{1.12''}$$

where $E^0 = E$, $\sigma_{E^0} = \sigma_E$, $F^0 = F'$, $\sigma_F = \sigma_{F'}$.

Naturally, we suppose that σ_E is an invertible operator.

Proposition 1.3. *The linear dynamic system* $\mathcal{F}^0 = \{R^0, S^0\}$, *associated with* $\alpha^0 = \alpha' \vee \alpha$, *is defined by* $\mathcal{F}_\alpha = \{R_\alpha, S_\alpha\}$ *and* $\mathcal{F}_{\alpha'} = \{R_{\alpha'}, S_{\alpha'}\}$, *corresponding to the colligation* α *and* α' *as follows*

$$\begin{aligned} R^0 &= R'_\alpha S_\alpha \dot{+} R_\alpha, \\ S^0 &= S'_\alpha S_\alpha. \end{aligned} \tag{1.13}$$

P r o o f. We write down componentwise the equation for the mapping R^0,

$$i\frac{d}{dt}\begin{bmatrix} x' \\ x \end{bmatrix} + \begin{bmatrix} A'x' + i\varphi'\sigma_{E'}\psi x \\ Ax \end{bmatrix} = \begin{bmatrix} \varphi'\sigma_{E'}Ku \\ \varphi\sigma_E u \end{bmatrix}.$$

Then

$$\begin{aligned} i\frac{d}{dt}x' + A'x' &= \varphi'\sigma_{E'}\left[Ku - i\psi x\right], \\ i\frac{d}{dt}x + Ax &= \varphi\sigma_E u, \end{aligned}$$

which it proves the first relation (1.13). The prove of the second relation directly follows from the equation

$$S^0 = K^0 - i\psi^0 R^0 = S'S. \quad \square$$

Consider the linear colligations $\tilde{\alpha}$ and $\tilde{\alpha}'$, dual with respect to α and α' where

$$\tilde{\alpha}' = (\sigma_{F'}, F', \tilde{\varphi}', \tilde{B}', \tilde{A}', \tilde{B}', \tilde{\psi}', E', \sigma_{E'}, \tilde{K}'). \tag{1.14}$$

Definition 1.3. *The colligation*

$$\tilde{\alpha}^0 = \tilde{\alpha} \vee \tilde{\alpha}' = (\sigma_{F'}, F', \tilde{\varphi}^0, \tilde{B}^0, \tilde{A}^0, \tilde{B}^0, \tilde{\psi}^0, E, \sigma_E, \tilde{K}^0) \tag{1.15}$$

is called $*$-coupling of the colligations $\tilde{\alpha}$ and $\tilde{\alpha}'$ (when $E = F$, $\sigma_{E'} = \sigma_F$), with respect to colligation (1.12)*, where $\tilde{B}^0 = \tilde{B}' \dot{+} \tilde{B}$, and the operators $\tilde{A}^0$, $\tilde{\varphi}^0$, $\tilde{\psi}^0$ are equals to*

$$(1.15') \qquad \begin{aligned} \tilde{A}^0 &= \begin{bmatrix} A & 0 \\ -i\tilde{\varphi}\sigma_{E'}\tilde{\psi}' & \tilde{A} \end{bmatrix}, \\ \tilde{\varphi}^0 &= \tilde{\varphi}' \dot{+} \tilde{\varphi}\tilde{K}', \\ \tilde{\psi}^0 &= \sigma_E^{-1}\tilde{K}\sigma_{E'}\tilde{\psi}' + \tilde{\psi}, \\ \tilde{K}^0 &= \tilde{K}\tilde{K}', \end{aligned}$$

where the operator σ_E^{-1} exists and bounded.

Proposition 1.4. *The linear dynamic system $\tilde{\mathcal{F}}^0 = \{\tilde{R}^0, \tilde{S}^0\}$, associated with $\tilde{\alpha}^0 = \tilde{\alpha} \vee \tilde{\alpha}'$, is defined by the relations*

$$(1.16) \qquad \begin{cases} \tilde{R}' = \tilde{R}' \dot{+} \tilde{R}\tilde{S}', \\ \tilde{S}^0 = \tilde{S}\tilde{S}'. \end{cases}$$

Proof. The proof is similar to that of Proposition 1.3. □

1.4. Let us show that the duality of the colligations α and $\tilde{\alpha}$ in (1.10) is inherited under the coupling.

Theorem 1.1. *Suppose that the coupling $\alpha^0 = \alpha' \vee \alpha$ of the linear colligations α and α' corresponds the $*$-coupling $\tilde{\alpha}^0 = \tilde{\alpha} \vee \tilde{\alpha}'$ of the dual colligations $\tilde{\alpha}$ and $\tilde{\alpha}'$, where the colligation $\tilde{\alpha}$ is dual to α, and $\tilde{\alpha}'$ is dual to the colligation α'. Then the colligations α^0 and $\tilde{\alpha}^0$ are also dual, and the dual operator S^0 is the direct sum of the respective dual operators, i.e. $S^0 = S' \dot{+} S$.*

Proof. Let us verify that there took place the relation 1) (1.10)

$$\begin{aligned} S^0A^0 - \tilde{A}^0S^0 &= \begin{bmatrix} S'A' - \tilde{A}'S' & iS'\varphi'\sigma_{E'}\psi \\ i\tilde{\varphi}\sigma_F\tilde{\psi}'S' & SA - \tilde{A}S \end{bmatrix} \\ &= i\left\{\tilde{\varphi}'\sigma_F'\psi' + \tilde{\varphi}'\sigma_{F'}K'\psi + \tilde{\varphi}\tilde{K}'\sigma_{F'}\psi + \tilde{\varphi}\sigma_F\psi\right\}, \end{aligned}$$

in virtue of 1) – 3) (1.10) for the dual colligations α (α') and $\tilde{\alpha}$ ($\tilde{\alpha}'$). If we use the definitions (1.12″) and (1.15′) for the corresponding colligation we obtain the following expression required

$$i(\tilde{\varphi}' + \tilde{\varphi}\tilde{K}')\sigma_{F'}(K'\psi + \psi') = i\tilde{\varphi}^0\sigma_{F'}\psi^0.$$

The proof of 2) (1.10) is verify directly

$$S^0\varphi^0\sigma_E = S\varphi\sigma_E + S'\varphi'\sigma_{E'}K = \tilde{\varphi}^0\sigma_{F'}K^0,$$

in virtue of respective formulas (1.12), (1.15) and (1.10).

The relations 3) and 4) (1.10) are proved similarly. □

1.5. Let us apply the signals of the form $u(t) = e^{i\lambda t}u_0$, $(\lambda \in \mathbb{C})$ to the input of the system (1.2). Then it is reasonable to expect that for the "complex frequency λ" there exists internal state $x(t) = e^{i\lambda t}x_0$, and output $v(t) = e^{i\lambda t}v_0$. Hence

$$\begin{cases} x_0 = (A - \lambda I)^{-1}\varphi\sigma_E u_0, \\ v_0 = S_\alpha(\lambda)u_0, \end{cases} \tag{1.17}$$

where the transfer function $S_\alpha(\lambda)$ of the linear colligation α is

$$S_\alpha(\lambda) = K - i\psi(A - \lambda I)^{-1}\varphi\sigma_E. \tag{1.18}$$

Similarly, if the plain wave is applied to the input of the system $\tilde{\mathcal{F}}_{\tilde{\alpha}}$ (1.7), (1.8) we get

$$\begin{cases} \tilde{x}_0 = (\tilde{A} - \lambda I)^{-1}\tilde{\varphi}\tilde{u}_0, \\ \tilde{v}_0 = S_{\tilde{\alpha}}(\lambda)\tilde{u}_0, \end{cases} \tag{1.19}$$

where,

$$S_{\tilde{\alpha}}(\lambda) = \tilde{K} + i\sigma_E\tilde{\psi}(\tilde{A} - \lambda I)^{-1}\tilde{\varphi} \tag{1.20}$$

is the transfer function of the colligation $\tilde{\alpha}$ dual with respect to α.

From the Proposition 1.2 it follows the next

Proposition 1.5. *If the colligation $\tilde{\alpha}$ is dual with respect to α, then for the transfer functions $S_\alpha(\lambda)$ (1.18) and $S_{\tilde{\alpha}}(\lambda)$ (1.20) the relation*

$$S_{\tilde{\alpha}}(\lambda)\sigma_F S_\alpha(\lambda) = \sigma_E, \tag{1.21}$$

is valid.

Thus, the duality between colligations $\tilde{\alpha}$ and α means that the transfer function $S_\alpha(\lambda)$ has the left invertible $\sigma_E^{-1}S_{\tilde{\alpha}}(\lambda)\sigma_F$ if σ_F is invertible.

Definition 1.4. *The subspace*

$$B_y(\alpha) = \mathrm{Span}\{A^n \varphi f : f \in E; \quad n \in \mathbb{Z}_+\}, \tag{1.22}$$

is called the controllability space of the systems $\mathcal{F}_\alpha$ (1.2), (1.3).

To define the abservability space for the system $\mathcal{F}_\alpha$ it is necessary to consider the dual system $\tilde{\mathcal{F}}_{\tilde{\alpha}}$ (1.7), (1.8), describing evolution in the backward direction. It is reasonable to suppose that the colligation $\tilde{\alpha}$ is dual to α.

Let $B_y(\tilde{\alpha})$ be the controllability space of the system $\tilde{\mathcal{F}}_{\tilde{\alpha}}$,

$$B_y(\tilde{\alpha}) = \mathrm{Span}\{\tilde{A}^n \tilde{\varphi}\ f' : f' \in E'; \quad n \in \mathbb{Z}_+\}.$$

Suppose that the dual operator S is boundedly invertible then the observability space of the system $\mathcal{F}_\alpha$ is called the subspace

$$B_H(\alpha) = S^{-1} B_y(\tilde{\alpha}). \tag{1.23}$$

It is obvious that the controllability of system $\tilde{\mathcal{F}}_{\tilde{\alpha}}$ means observability of system $\mathcal{F}_\alpha$ and vice versa, i.e.

$$B_H(\alpha) = S^{-1} B_y(\tilde{\alpha}), B_H(\alpha) = S B_y(\alpha),$$

where the colligation $\tilde{\alpha}$ is dual to the colligation α. The inverse duality for the colligations α and $\tilde{\alpha}$ must also exist (see Subsection 1.6).

The system $\mathcal{F}_\alpha$ (1.2), (1.3) corresponds to the colligation α, is called a controllable (observable) colligation if

$$B = B_y(\alpha) \quad (B = S^{-1} B_y(\tilde{\alpha})).$$

Note that the controllability or observability assumptions means that the duality conditions (1.10) are also dependent.

Lemma 1.1. *Suppose that the colligation* α *is controllable and observable. Furthermore, let the colligation* $\tilde{\alpha}$ *exist, such that for the transfer function* $S_\alpha(\lambda)$ (1.18) *and* $S_{\tilde{\alpha}}(\lambda)$ (1.20), *the relation* (1.21) *is valid. Then if there exist operator* $S : B \to \tilde{B}$ *such that* 1) *and* 4) (1.10) *are valid, then*

$$\left[\sigma_E \tilde{\psi} S - \tilde{K} \sigma_F \psi\right] (A - \lambda I)^{-1} \varphi \sigma_E = \sigma_E \tilde{\psi} (\tilde{A} - \lambda I)^{-1} \left[S \varphi \sigma_E - \tilde{\varphi} \sigma_F K\right]. \tag{1.24}$$

Formula (1.24) follows from (1.21). That is the validity of 2) or 3) (1.10) implies validity of the others.

1.6. The duality of the colligations $\tilde{\alpha}$ and α means that the function $S_\alpha(\lambda)$ (1.18) has left invertible (1.21). However the presence of unilateral invertibility does not guarantee, in general case, the operator invertibility on the whole.

Definition 1.5. *The linear colligation α is called $*$-dual to the colligation $\tilde{\alpha}$, if there exists the linear operator Q, mapping the Banach space $\tilde{B}$ into B, so that*

$$\begin{aligned} &1)\quad AQ - Q\tilde{A} = i\varphi\sigma_E\tilde{\psi},\\ &2)\quad K\tilde{\psi} = \psi Q,\\ &3)\quad Q\tilde{\varphi} = \varphi\tilde{K},\\ &4)\quad K\sigma_E^{-1}\tilde{K} = \sigma_F^{-1}. \end{aligned} \tag{1.25}$$

Remark 1.3. Suppose that the dual operator S, enhancing duality (1.10) of the colligations $\tilde{\alpha}$ and α, has bounded invertible and let there exist σ_E^{-1}, σ_F^{-1}, K^{-1}, $\tilde{K}^{-1}$. Then using 1) (1.10) we conclude that

$$AS^{-1} - S^{-1}A = iS^{-1}\tilde{\varphi}\sigma_F\psi S^{-1}.$$

Using the relations 2) and 3) (1.10), presented as

$$S^{-1}\tilde{\varphi}\sigma_F = \varphi\sigma_E K^{-1}, \psi S^{-1} = \sigma_F^{-1}\tilde{K}^{-1}\sigma_E\tilde{\psi},$$

we obtain

$$AS^{-1} - S^{-1}\tilde{A} = i\varphi\sigma_E\tilde{\psi},$$

coinciding with 1) (1.25) if $Q = S^{-1}$.

It is not difficult to ascertain that the relations 2) – 4) (1.10) lead to the conditions 2) and 3) (1.25) if $Q = S^{-1}$. And from 4) (1.10) it follows that

$$(\sigma_E^{-1}\tilde{K})(\sigma_F K) = I_E,$$

and since the right invertible coincides with the left invertible (for bounded invertible operator), then

$$\sigma_F K\sigma_E^{-1}\tilde{K} = I_F,$$

and we obtain 4) (1.25).

Thus $*$-duality conditions (1.25) follow from conditions (1.10) if S is boundedly invertible $Q = S^{-1}$ and the operators K^{-1}, $\tilde{K}^{-1}$, σ_E^{-1}, σ_F^{-1} exist and are bounded.

Similarly to (1.21) it follows from the conditions of $*$-duality that

$$S_{\alpha}(\lambda)\sigma_E^{-1}S_{\tilde{\alpha}}(\lambda) - \sigma_F^{-1} = 0. \tag{1.26}$$

The relations (1.21) and (1.26) follow from the next theorem.

Theorem 1.2. *If the points λ and μ do not belong to the spectra of main operators of the dual and $*$-dual colligations alpha and $\tilde{\alpha}$, respectively, then the formulas*

$$\begin{aligned} &1) \quad \frac{S_{\tilde{\alpha}}(\mu)\sigma_F S_{\alpha}(\lambda) - \sigma_E}{\mu - \lambda} = i\sigma_E\tilde{\psi}(\tilde{A} - \mu)^{-1}S(A - \lambda I)^{-1}\varphi\sigma_E, \\ &2) \quad \frac{S_{\alpha}(\lambda)\sigma_E^{-1}S_{\tilde{\alpha}}(\mu) - \sigma_F^{-1}}{\mu - \lambda} = i\psi(A - \mu)^{-1}Q(\tilde{A} - \mu I)^{-1}\tilde{\varphi}, \end{aligned} \tag{1.27}$$

are valid.

P r o o f. We prove formula 1). Using duality conditions (1.10), we obtain

$$\begin{aligned} S_{\tilde{\alpha}}(\mu)\sigma_F S_{\alpha}(\lambda) - \sigma_E &= i\sigma_E\ \tilde{\psi}(\tilde{A} - \mu)^{-1}\tilde{\varphi}\sigma_F K - i\tilde{K}\sigma_F\psi(A - \lambda I)^{-1}\varphi\sigma_E \\ &\quad +\sigma_E\ \tilde{\psi}(\tilde{A} - \mu I)^{-1}\tilde{\varphi}\sigma_F\ \psi(A - \lambda I)^{-1}\psi\sigma_E \\ &= -i(\mu - \lambda)\sigma_E\tilde{\psi}(\tilde{A} - \mu I)S(A - \lambda I)^{-1}\varphi\sigma_E. \end{aligned}$$

Formula 2) is proved similarly. □

The next theorem shows that from the relations (1.21), (1.26) it follows that $Q = S^{-1}$.

Theorem 1.3. *Let the colligations $\tilde{\alpha}$ and α be dual and the colligations α and $\tilde{\alpha}$ be $*$-dual. Suppose that the colligation α is controllable and observable and the following relations holds*

$$\begin{aligned} &a) \quad SQ\ |_{\overline{\tilde{\varphi}F}} = I_{\overline{\tilde{\varphi}F}}, \\ &b) \quad QS\ |_{\overline{\varphi E}} = I_{\overline{\varphi E}}. \end{aligned}$$

Then the operators Q and S are dually invertible, i.e. $Q = S^{-1}$.

P r o o f. From relations 1) (1.10) and 1) (1.25) it follows that

$$\begin{aligned} \tilde{R}(\lambda)S - SR(\lambda) &= i\tilde{R}(\lambda)\tilde{\varphi}\sigma_F\psi R(\lambda), \\ Q\tilde{R}(\lambda) - R(\lambda)Q &= iR(\lambda)\varphi\sigma_E\tilde{\psi}\tilde{R}(\lambda), \end{aligned}$$

where

$$R(\lambda) = (A - \lambda I)^{-1}, \tilde{R}(\lambda) = (\tilde{A} - \lambda I)^{-1}.$$

Multiplying the first equality by Q from the right and subtracting from it the second equality multiplied by S from the left we obtain

$$SQ\tilde{R}(\lambda) - \tilde{R}(\lambda)SQ = iSR(\lambda)\varphi\sigma_E\tilde{\psi}\tilde{R}(\lambda) - i\tilde{R}(\lambda)\tilde{\varphi}\sigma_F\psi R(\lambda)Q.$$

Applying operator $\tilde{\varphi}$ from the right and using the transfer functions $S_\alpha(\lambda)$ (1.18) and $S_{\tilde{\alpha}}(\lambda)$ (1.20) we derive the equality

$$\begin{aligned} \left[SQ\tilde{R}(\lambda) - \tilde{R}(\lambda)SQ\right]\tilde{\varphi} &= \tilde{R}(\lambda)\tilde{\varphi}\{\tilde{K}^{-1}(S_{\tilde{\alpha}}(\lambda) - \tilde{K}) + \sigma_F(S_\alpha(\lambda) - K) \\ &\quad \times\sigma_E^{-1}(S_{\tilde{\alpha}}(\lambda) - \tilde{K}) + \sigma_F(S_\alpha(\lambda) - K)\sigma_E^{-1}\tilde{K}\}. \end{aligned}$$

Since the right-hand side of the equality equals to zero, in virtue of (1.26), (1.21), and Theorem 1.3(a) we obtain that

$$SQ\tilde{R}(\lambda)\tilde{\varphi}F = \tilde{R}(\lambda)SQ\tilde{\varphi}F = 0.$$

Observability of colligation α leads to the equality $QS = I_{\tilde{B}}$. The relation $QS = I_B$ is proved similarly. $\square$

Definition 1.6. *The linear system $\mathcal{F}_\alpha$* (1.2), (1.3) *is called S-simple, if for the linear colligation α there exists the dual colligation $\tilde{\alpha}$, where operator S is invertible and the following relation holds*

$$B = B_y(\alpha) \vee B_H(\alpha). \qquad (1.28)$$

Definition 1.7. *The colligaton α is called S-simple, if the system $\mathcal{F}_\alpha$* (1.2), (1.3), *associated with the colligation, is S-simple system.*

Hereinafter, we will suppose that dual operator S belonging to the colligations α and $\tilde{\alpha}$ is boundedly invertible unless otherwise stated.

1.7. It is not difficult to see that any analytical operator-function $S(\lambda)$ in the vicinity of the infinite point or any fixed one from $\mathbb{C}$ may be realized as transfer function for some linear colligation α [2].

Theorem 1.4. *Let the analytical in the vicinity of the infinite point operator-function $S(\lambda)$, mapping Banach space E into F such, that $S(\infty) \not\equiv 0$. Then there exists a linear colligation α with the transfer function $S_\alpha(\lambda)$ (1.18), coinciding with $S(\lambda)$, i.e. $S_\alpha(\lambda) = S(\lambda)$.*

P r o o f. Let us define a contour Γ, that encloses the zero point and lies in the domain of analyticity of the function $S(\lambda)$. Then in virtue of integral Cauchy formula [1, 2] for $S(\lambda) - S_\infty$ we have

$$S(\lambda) - S_\infty = \frac{1}{2\pi i} \int\limits_\Gamma \frac{S(\xi) - S_\infty}{\xi - \lambda} d\xi,$$

where λ lies in the holomorphy domain of $S(\lambda)$, and the integration along Γ is in the appropriate direction.

Define the space of vector-function in F over the curve Γ,

$$B_\Gamma(F) = \left\{ f(\xi) \in F, (\xi \in \Gamma); \int\limits_\Gamma \|f(\xi)\|_F \, d\xi < \infty \right\}.$$

Define the linear bounded operators:

$$\begin{aligned} \varphi f &= (S(\xi) - S_\infty)\sigma_E^{-1} f, (f \in E), \\ \psi f(\xi) &= \frac{1}{2\pi} \int\limits_\Gamma f(\xi) d\xi, (f(\xi) \in B_\Gamma(E)), \\ A f(\xi) &= \xi f(\xi), (f(\xi) \in B_\Gamma(E)). \end{aligned}$$

where σ_E is the bounded, invertible, arbitrary operator in E. From the mentioned above Cauchy formula we deduce that

$$S_\alpha(\lambda) = K - i\psi(A - \lambda I)^{-1} \varphi \sigma_E = S(\lambda),$$

where $K = S_\infty$, and the colligation α consist of the elements A, φ, ψ, $B = B_F(E)$, σ_E that were defined earlier. $\square$

By the transfer function $S(\lambda)$ it is reasonable to raise a question of the unambiguity of such reconstruction of the linear colligation α and of the open linear system $\mathcal{F}$, associated with the colligation.

1.8. Consider two linear colligations α and α' with coinciding outer spaces $E = E'$ and $F = F'$, and the operators $\sigma_{E'} = \sigma_E$, $K = K'$.

Definition 1.8. *The colligations α and α' are called linear-equivalent colligations if there exists the linear bounded operator N from B into B' (such that N^{-1} exists and is bounded) which satisfies the conditions*

$$NA = A'N, N\varphi = \varphi', \psi = \psi' N.$$

It is evident, that the transfer functions $S_\alpha(\lambda)$ and $S_{\alpha'}(\lambda)$ for the linear-equivalent colligations α and α' coincide i.e. $S_\alpha(\lambda) = S_{\alpha'}(\lambda)$.

From pointview of the open systems, the linear equivalency means that the realization of the system $\mathcal{F} = \{R, S\}$ by its transfer mapping S, supposed to be known in the different "complex frequencies λ", includes ambiguity, defined by similarity transformation of the linear operator N.

2. Factorization of the transfer functions.

2.1. In various problems of the systems theory and the theory of operator-functions, the problem, concerning the factorization of the transfer function $S_\alpha(\lambda)$, may have particular interest [2]. This problem is reduced to the decomposition, in terms of the coupling operation (Proposition 1.2), into subsystems of the open system $\mathcal{F}_\alpha$ (1.2), (1.3).

Lemma 2.1. *Suppose that the operator S, mapping the Banach space $B_1 \dotplus B_2$ into $\tilde{B}_1 \dotplus \tilde{B}_2$, in terms of the given expansion is presented as*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} : B_1 \dotplus B_2 \to \tilde{B}_1 \dotplus \tilde{B}_2,$$

such that S_{11} is invertible and S_{11}^{-1} is bounded. Then S permits the factorization

$$S = UV,$$

where operators U and V have the triangular form

$$V = \begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix} : B_1 \dot{+} B_2 \to \hat{B}_1 \dot{+} \hat{B}_2,$$

$$U = \begin{bmatrix} U_{11} & 0 \\ U_{21} & U_{22} \end{bmatrix} : \hat{B}_1 \dot{+} \hat{B}_2 \to \tilde{B}_1 \dot{+} \tilde{B}_2.$$

The assertion of a lemma is satisfied for example, if the elements $V_{K,S}$, $U_{K,S}$ of the operators V, U are

$$\begin{aligned} &V_{11} = S_{11}, \quad V_{12} = S_{12}, \qquad\quad V_{22} = P_2, \\ &U_{11} = \tilde{P}_1, \quad U_{21} = S_{21}S_{11}^{-1}, \quad U_{22} = S_{22} - S_{21}S_{11}^{-1}S_{12}, \end{aligned} \tag{2.1}$$

where P_k and $\tilde{P}_k$ are projectors on B_k and $\tilde{B}_k$ $(k = 1, 2)$, respectively, and $\hat{B}_1 \dot{+} \hat{B}_2 = \tilde{B}_1 \dot{+} \tilde{B}_2$.

Note that the operator V^{-1} always exists,

$$V^{-1} = \begin{bmatrix} S_{11}^{-1} & -S_{11}^{-1}S_{12} \\ 0 & P_2 \end{bmatrix}.$$

Assuming that the operator S is invertible, then for $Q = S^{-1}$ one may easily get that

$$Q_{22} = U_{22}^{-1}.$$

Theorem 2.1 (On the factorization)**.** *Let the dual operator S (1.10), of the colligations α and $\tilde{\alpha}$ such that $Q = S^{-1}$ is bounded and operators $\tilde{K}$, K and σ_E are invertible. Suppose that $B = B_1 \dot{+} B_2$, the subspace B_1 is invariant relative to A, and $\tilde{B}_2$ is invariant relative to $\tilde{A}$, where $\tilde{B} = \tilde{B}_1 \dot{+} \tilde{B}_2$, $\overline{SB_k} = \tilde{B}_k$ $(k = 1, 2)$. Let the operator S_{11}^{-1} exist and be invertible. Then the transfer function $S_\alpha(\lambda)$ of the colligation α is equal to*

$$S_\alpha(\lambda) = S_1(\lambda)S_2(\lambda), \tag{2.2}$$

where

$$\begin{aligned} S_1(\lambda) &= K_1 - i\psi P_1(A_1 - \lambda I)^{-1}S_{11}^{-1}\tilde{P}_1\tilde{\varphi}\sigma_F K_1, \\ S_2(\lambda) &= K_2 - iK_2\tilde{\psi}\tilde{P}_2 Q_{22}^{-1}(A_1 - \lambda I)^{-1}P_2\varphi\sigma_E, \end{aligned} \tag{2.3}$$

and $A_k = P_k A \tilde{P}_k$ *(*P_k *and* $\tilde{P}_k$ *are projectors on* B_k *and* $\tilde{B}_k$*) and* K_1*,* K_2 *are boundedly invertible, such that* $K_1 K_2 = K$*,* $K_2 : E \to \hat{E}$*,* $K_1 : \hat{E} \to F$*, and* $\hat{E}$ *is a Banach space.*

P r o o f. The invariance of the subspace B_1 relative to A means that

$$A = \begin{bmatrix} A_1 & \Gamma \\ 0 & A_2 \end{bmatrix}.$$

To calculate Γ we write the condition 1) (1.10) in the form

$$VAV^{-1} - U^{-1}\tilde{A}U = iU^{-1}\tilde{\varphi}\sigma_F \psi V^{-1}, \tag{2.4}$$

in virtue of Lemma 2.1. Since the operators VAV^{-1} and $U^{-1}\tilde{A}U$ are in the respective triangular form then in virtue of 2) (1.10) we have

$$\hat{A}_{1,2} = i\hat{P}_1 V \varphi \sigma_E K^{-1} \psi V^{-1} P_2,$$

where $\hat{A} = VAV^{-1}$. Thus we have calculate the coupling coefficient $\hat{A}_{1,2}$ for the main operator

$$\hat{A} = VAV^{-1} = \begin{bmatrix} \hat{A}_1 & \hat{A}_{1,2} \\ 0 & \hat{A}_2 \end{bmatrix}$$

of the colligation $\hat{\alpha}$, which is linearly equivalent to the initial colligation α,

$$\hat{\alpha} = (\sigma_E, E, \hat{\varphi}, \hat{B}, \hat{A}, \hat{B}, \hat{\psi}, F, \sigma_F, K), \tag{2.5}$$

where

$$\hat{B} = \overline{VB} = \tilde{B}_1 \dot{+} B_2, \quad \hat{\varphi} = V\varphi, \quad \hat{\psi} = \psi V^{-1}.$$

Let us fix an intermediate Banach space $\hat{E}$ and operators

$$K_2 : E \to \hat{E}, \quad K_1 : \hat{E} \to F, \quad \sigma_{\hat{E}} : \hat{E} \to \hat{E},$$

such that $K_1 K_2 = K$, and K_1, K_2, $\sigma_{\hat{E}}$ are boundedly invertible.

Define the linear colligations

$$\begin{aligned} \alpha_1 &= (\sigma_{\hat{E}}, \hat{E}, \tilde{P}_1 \hat{\varphi} \sigma_E K_2^{-1} \sigma_{\hat{E}}^{-1}, \hat{B}_1, \hat{A}_1, \hat{B}_1, \hat{\psi} \tilde{P}_1, F, \sigma_F, K_1), \\ \alpha_2 &= (\sigma_E, E, P_2 \hat{\varphi}, B_2, \hat{A}_2, B_2, K_1^{-1} \hat{\psi} P_2, \hat{E}, \sigma_{\hat{E}}, K_2). \end{aligned} \tag{2.6}$$

Immediate verification shows in virtue of (1.12) that

$$\hat{\alpha} = \alpha_1 \vee \alpha_2.$$

Therefore in virtue of Proposition 1.3 the transfer function of the colligation $\hat{\alpha}$ is equal to

$$S_{\hat{\alpha}}(\lambda) = S_{\alpha_1}(\lambda) S_{\alpha_2}(\lambda),$$

where $S_{\alpha_k}(\lambda)$ are the transfer functions of the colligations $\alpha_k (k = 1, 2)$.

Since the transfer functions of the linearly equivalent colligations α and $\hat{\alpha}$ coincide then

$$S_{\alpha}(\lambda) = S_1(\lambda) S_2(\lambda).$$

Now we calculate $S_k(\lambda)$ $(k = 1, 2)$,

$$\begin{aligned} S_1(\lambda) &= K_1 - i\hat{\psi}\tilde{P}_1(\hat{A}_1 - \lambda I)^{-1}\tilde{P}_1\hat{\varphi}\sigma_E K_2^{-1} \\ &= K_1 - i\psi P_1(A_1 - \lambda I)^{-1} S_{11}^{-1}\tilde{P}_1 S\varphi\sigma_E K_2^{-1} \\ &= K_1 - i\psi P_1(A_1 - \lambda I)^{-1} S_{11}^{-1}\tilde{P}_1\tilde{\varphi}\sigma_F K_1 \end{aligned}$$

Similarly

$$\begin{aligned} S_2(\lambda) &= K_2 - iK_1^{-1}\tilde{\psi} Q\tilde{P}_2 Q_{22}^{-1}(A_2 - \lambda I)^{-1} P_2\varphi\sigma_E \\ &= K_2 - iK_2\tilde{\psi}\tilde{P}_2 Q_{22}^{-1}(A_2 - \lambda I)^{-1} P_2\varphi\sigma_E. \quad \square \end{aligned}$$

2.2. From the relation (2.5), taking into account the respective duality conditions (1.10), we conclude that coupling coefficient of the operator $U^{-1}\tilde{A}U$ is equal to

$$-\tilde{P}_2 U^{-1}\tilde{A} U P_1 = i\tilde{P}_2 U^{-1}\tilde{\varphi}\tilde{K}^{-1}\sigma_E\tilde{\psi} U P_1.$$

By repeating the above reasoning, we obtain that the transfer function

$$S_{\tilde{\alpha}}(\lambda) = \tilde{K} + i\sigma_E\tilde{\psi}(A_2 - \lambda I)^{-1}\tilde{\varphi},$$

of the dual colligation $\tilde{\alpha}$ is factored as

$$S_{\tilde{\alpha}}(\lambda) = \tilde{S}_2(\lambda)\tilde{S}_1(\lambda), \tag{2.7}$$

where

$$\begin{aligned} \tilde{S}_1(\lambda) &= \tilde{K}_1 + i\tilde{K}_1\sigma_F\psi P_1 S_{11}^{-1}(\tilde{A}_1 - \lambda I)^{-1}\tilde{P}_1\tilde{\varphi}, \\ \tilde{S}_2(\lambda) &= \tilde{K}_2 + i\sigma_E\tilde{\psi}\tilde{P}_2(A_2 - \lambda I)^{-1} Q_{22}^{-1} P_2\varphi K_2, \end{aligned} \tag{2.8}$$

the invertible operators $\tilde{K}_1$, $\tilde{K}_2$ are such that

$$\tilde{K}_2\tilde{K}_1 = \tilde{K}, \quad \tilde{K}_1 : F \to \tilde{E}, \quad \tilde{K}_2 : \tilde{E} \to E,$$

and $\tilde{E}$ is an intermediate factorization space (2.7).

Similarly to the decomposition of the colligation $\hat{\alpha}$ into the coupling α_k $(k = 1, 2)$, some metric invertible operator $\sigma_{\tilde{E}}$ appears in the space $\tilde{E}$.

In general, the factorizations of $S_\alpha(\lambda)$ (2.2) and $S_{\tilde{\alpha}}(\lambda)$ (2.7) are not have to be consistent in the sense that for $S_k(\lambda)$ (2.3) and $\tilde{S}_k(\lambda)$ (2.8) the "duality" relations (1.21) are valid. Since for $S_\alpha(\lambda)$ and $S_{\tilde{\alpha}}(\lambda)$ the relation (1.21) holds, then to prove the similar equalities for $S_k(\lambda)$ and $\tilde{S}_k(\lambda)$ it is necessary to show that

$$\tilde{S}_1(\lambda)\sigma_F S_1(\lambda) = \sigma_{\tilde{E}},$$

with some $\sigma_{\tilde{E}}$. To prove that it is necessary to regard that the intermediate spaces $\tilde{E}$ and $\hat{E}$ coincide $(\tilde{E} = \hat{E})$. Now we calculate

$$\begin{aligned}\tilde{S}_1(\lambda)\sigma_F S_1(\lambda) - \sigma_{\tilde{E}} = {} & \tilde{K}_1\sigma_F K_1 - \sigma_{\tilde{E}} + i\tilde{K}_1\sigma_F\tilde{\psi}P_1\tilde{S}_{11}^{-1}(\tilde{A}_1 - \lambda I)^{-1}\tilde{P}_1\tilde{\varphi}\sigma_F K_1\\ & - i\tilde{K}_1\sigma_F\psi P_1(A_1 - \lambda I)^{-1}S_{11}^{-1}\tilde{P}_1\tilde{\varphi}\sigma_F K_1\\ & - i\tilde{K}_1\sigma_F\tilde{\psi}P_1S_{11}^{-1}(\tilde{A}_1 - \lambda I)^{-1}\tilde{P}_1(SA - \tilde{A}S)P_1\\ & \times (A_1 - \lambda I)^{-1}S_{11}^{-1}\tilde{P}_1\tilde{\varphi}\sigma_F K_1,\end{aligned}$$

which in virtue of (1.10) reduce to

$$\tilde{K}_1\sigma_F K_1 - \sigma_{\tilde{E}}.$$

Thus it is necessary to assume that

$$\sigma_{\tilde{E}} = \tilde{K}_1\sigma_F K_1. \tag{2.9}$$

Theorem 2.2. *Suppose that the suppositions of the Theorems* 2.1 *are true. Then the transfer functions* $S_\alpha(\lambda)$ (1.18) *and* $S_{\tilde{\alpha}}(\lambda)$ (1.20) *of the dual colligations* α *and* $\tilde{\alpha}$ *are decomposed simultaneously into the products* (2.2) *and* (2.7) *and there always exists intermediate factorization space* $\hat{E} = \tilde{E}$ *and metric operator* $\sigma_{\tilde{E}}$ *(defined, for example, by* (2.9)*), so that for the factors* $S_k(\lambda)$ (2.3) *and* $\tilde{S}_k(\lambda)$ (2.8) *the duality relations*

$$\tilde{S}_1(\lambda)\sigma_F S_1(\lambda) = \sigma_{\tilde{E}}, \qquad \tilde{S}_2(\lambda)\sigma_{\tilde{E}} S_1(\lambda) = \sigma_E,$$

are valid.

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Received October 10, 2006