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SOME COEFFICIENT ESTIMATES FOR POLYNOMIALS ON THE UNIT INTERVAL

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ABSTRACT. In this paper we present some inequalities about the moduli of the coefficients of polynomials of the form $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$, where $a_0, \ldots, a_n \in \mathbb{C}$. They can be seen as generalizations, refinements or analogues of the famous inequality of P. L. Chebyshev, according to which $|a_n| \le 2^{n-1}$ if $|\sum_{\nu=0}^{n} a_\nu x^\nu| \le 1$ for $-1 \le x \le 1$.

1. Chebyshev Polynomials T_n. Using *de Moivre's theorem*, i.e. the formula

$$\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n,$$

we readily see that $\cos n\theta$ can be expressed as a polynomial of degree n in $\cos\theta$. The polynomial T_n such that

$$T_n(\cos\theta) \equiv \cos n\theta \tag{1}$$

is called the Chebyshev polynomial of the first kind of degree n.

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Formula (1) implies that $|T_n(x)| \le 1$ for $-1 \le x \le 1$ and that

$$T_n\left(\cos\frac{\nu\pi}{n}\right) = (-1)^\nu \qquad (\nu = 0, 1, \dots, n).$$

From (1), it also follows that the zeros of T_n are

$$\cos\frac{(2\nu-1)\pi}{2n} \qquad (\nu = 1, \dots, n).$$

Clearly, they all lie in the open interval $(-1, 1)$. Writing (1) in the form

$$T_n\left(\frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}\right) \equiv \frac{\mathrm{e}^{\mathrm{i}n\theta} + \mathrm{e}^{-\mathrm{i}n\theta}}{2} \qquad (\theta \in \mathbb{R})$$

we see that

$$\tau(z) := z^n T_n\left(\frac{z + z^{-1}}{2}\right) - \frac{z^{2n} + 1}{2}$$

is a polynomial of degree not exceeding $2n$, which vanishes at each point of the unit circle. Since a polynomial of degree at most $2n$ cannot vanish at more than $2n$ points without being identically zero, we conclude that

$$T_n\left(\frac{z + z^{-1}}{2}\right) \equiv \frac{z^n + z^{-n}}{2} \qquad (z \neq 0). \tag{2}$$

Let $T_n(z) := \sum_{\nu=0}^{n} t_{n,\nu} z^\nu$ be the Maclaurin expansion of T_n. Then, comparing the coefficients of z^n on the two sides of (2) we find that

$$t_{n,n} = 2^{n-1}, \tag{3}$$

i.e. the leading coefficient in the Maclaurin expansion of T_n is 2^{n-1}. It is useful to know that

$$T_n(z) := \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2z)^{n-2k}. \tag{4}$$

2. The polynomial inequality of Chebyshev. The polynomials $\{T_n\}$ are known to have several remarkable extremal properties. They play an important role in the Theory of Approximation and related areas of mathematics. Let $\mathcal{P}_m$ denote the class of all algebraic polynomials of degree at most m. For any function $\Phi \in C[a, b]$, let

$$E_n[\Phi] = \inf_{P \in \mathcal{P}_n} \{ \max_{x \in [a,b]} |\Phi(x) - P(x)| \}.$$

A polynomial $P^* \in \mathcal{P}_n$ is said to be a polynomial of best approximation to Φ if

$$E_n[\Phi] = \max_{a \le x \le b} |\Phi(x) - P^*(x)|.$$

By a classical result of P. L. Chebyshev, a polynomial $P^* \in \mathcal{P}_n$ is a polynomial of best approximation to Φ if there exist $n+2$ points

$$a \le x_0 < \cdots < x_{n+1} \le b$$

such that

$$\pm(-1)^\nu [\Phi(x_\nu) - P^*(x_\nu)] = \max_{a \le x \le b} |\Phi(x) - P^*(x)| \qquad (\nu = 0, \ldots, n+1),$$

i.e. if $\Phi(x) - P^*(x)$ takes on the values $\pm \max_{a \le x \le b} |\Phi(x) - P^*(x)|$ in alternating succession at the points $x_0, \ldots, x_{n+1}$. This alternation theorem shows, in particular, that P^* is unique. For a historical perspective of the alternation theorem, we refer the reader to an interesting article of Butzer and Jongemans [2].

As an example, let us find the best uniform approximation of the function $\Phi(x) := x^n$ on $[-1, 1]$ by polynomials in $\mathcal{P}_{n-1}$. Clearly, the polynomial of best approximation must be real. In view of (3), we can write

$$x^n = \frac{T_n(x)}{2^{n-1}} \; + \; \text{ a polynomial in } \mathcal{P}_{n-1}.$$

Since $T_n(x)$ takes the value $+1$ and -1 at $n+1$ points in $[-1, 1]$ and $n+1$ can be written as $(n-1)+2$, the alternation theorem shows that the desired polynomial of best approximation to x^n, in $\mathcal{P}_{n-1}$, is the polynomial

$$x^n - \frac{1}{2^{n-1}} T_n(x).$$

Equivalently, the monic polynomial $P^*(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of best approximation to the function $\Phi(x) \equiv 0$ on the interval $[-1, 1]$ is the Chebyshev polynomial $2^{1-n} T_n(x)$. The maximum deviation of this polynomial from zero is 2^{1-n}.

The above discussion may be summarized as follows.

Theorem A. *Let $T_n(x) := \sum_{\nu=0}^n t_{n,\nu} x^\nu$ be the Chebyshev polynomial of the first kind of degree n. Then, for any polynomial $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ of degree at most n, not identically zero, we have*

$$\frac{|a_n|}{\max_{-1 \le x \le 1} |f(x)|} \le \frac{t_{n,n}}{\max_{-1 \le x \le 1} |T_n(x)|} = 2^{n-1}. \tag{5}$$

3. Upper bound for $|a_n|$ and the distribution of the zeros of f. In (5), equality holds if and only if $f(z) \equiv e^{i\gamma}\, T_n(z)$, $\gamma \in \mathbb{R}$. It was shown by Schur [16] that if the polynomial $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ vanishes at $c = 1$ or $c = -1$ and $|f(x)| \le 1$ for $-1 \le x \le 1$, then

$$|a_n| \le \left(\cos\frac{\pi}{4n}\right)^{2n} 2^{n-1}, \tag{6}$$

where equality holds if and only if

$$f(z) := \mathrm{e}^{\mathrm{i}\gamma} T_n\left(z\cos^2\frac{\pi}{4n} - c\sin^2\frac{\pi}{4n}\right) \qquad (\gamma \in \mathbb{R}).$$

In fact, the following more general result (see [9, Theorem 1]) holds.

Theorem B. *Let $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n such that $f(c) = 0$ for some c belonging to $[-1, -\cos(\pi/2n)) \cup (\cos(\pi/2n), 1]$. Furthermore, let $|f(x)| \le 1$ at the points*

$$\xi_\nu = \xi_{\nu,c} := \frac{c}{|c|}\frac{(1+|c|)\cos(\nu\pi/n)+c-(\operatorname{sign} c)\cos(\pi/2n)}{2\cos^2(\pi/4n)} \qquad (1 \le \nu \le n).$$

Then

$$|a_n| \le \left(\frac{1+\cos(\pi/2n)}{1+|c|}\right)^n 2^{n-1}. \tag{7}$$

The upper bound for $|a_n|$ given in (7) is attained for the polynomial

$$\mathrm{e}^{\mathrm{i}\gamma} T_n\left(\frac{(1+\cos(\pi/2n))z - c + (\operatorname{sign} c)\cos(\pi/2n)}{1+|c|}\right) \qquad (\gamma \in \mathbb{R}),$$

which satisfies the conditions of Theorem B.

The above mentioned result of Schur has also been generalized [12, Theorem 2] as follows.

Theorem C. *Inequality* (6) *holds for any polynomial $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n with real coefficients which has at most $n-1$ distinct zeros in the open interval $(-1, 1)$.*

It was conjectured by K. Mahler and stated by him in some of his lectures (see [20, p. 15]) that if f is as above, then

$$\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|d\theta\right\}\leq 2^{n-1}. \tag{8}$$

It is well-known [18, p. 284d] that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|\mathrm{e}^{\mathrm{i}\theta}-\zeta\right|d\theta=\log^{+}|\zeta|:=\begin{cases}0 & \text{for } |\zeta|\leq 1\\ \log|\zeta| & \text{for } |\zeta|\geq 1.\end{cases}$$

Hence, if $z_1,\ldots,z_n$ are the zeros of f then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left|f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|d\theta=\log|a_n|+\sum_{\nu=1}^{n}\log^{+}|z_\nu|=\log|a_n|+\log\prod_{|z_\nu|>1}|z_\nu|\,,$$

where the product on the right is to be replaced by 1 if $|z_\nu|\leq 1$ for $1\leq\nu\leq n$. Thus (8) is equivalent to the inequality

$$\left(\prod_{|z_\nu|>1}|z_\nu|\right)|a_n|\leq 2^{n-1}. \tag{9}$$

Turán [20] settled the conjecture of Mahler by proving it in the form (9).

Inequality (9) gives a quantitative improvement on (5) only if f has some zeros outside the unit circle. However, as mentioned above, inequality (5) is strict unless $f(z)\equiv\mathrm{e}^{\mathrm{i}\gamma}T_n(z)$, $\gamma\in\mathbb{R}$, and so certainly if some of the zeros of f lie in $\mathbb{C}\backslash[-\cos(\pi/2n),\cos(\pi/2n)]$. Theorem B deals with the case where f has a zero in $[-1,-\cos(\pi/2n))\cup(\cos(\pi/2n),1]$. What if f has some zeros in $\mathbb{C}\backslash[-1,1]$? An answer to this question is to be found in Corollary 1. We shall deduce it from the following theorem, which is a refined version of a result from [11, Theorem 1].

Theorem 1. *Let* $\psi(\zeta):=\sum_{\mu=0}^{m}\gamma_\mu\zeta^\mu$ *be a polynomial of degree* m *such that* $\gamma_0\,\overline{\gamma}_m>0$ *and* $\left|\psi\left(\mathrm{e}^{2\ell\pi\mathrm{i}/m}\right)\right|\leq 1$ *for* $\ell=0,1,\ldots,m-1$. *Furthermore, let* $\zeta_1,\ldots,\zeta_J$ *be amongst the zeros of* ψ, *where* $1\leq J\leq m$. *Then*

$$\frac{|\gamma_0|}{\prod_{j=1}^{J}|\zeta_j|}+|\gamma_m|\prod_{j=1}^{J}|\zeta_j|\leq 1. \tag{10}$$

The following modified form of a famous inequality of C. Visser [21] plays an important role in the proof of Theorem 1.

Lemma 1. *Let* $\phi(\zeta) := \sum_{\mu=0}^{m} c_\mu \zeta^\mu$ *be a polynomial of degree* m *such that* $c_0\overline{c}_m > 0$. *Furthermore, let* $\left|\phi\left(\mathrm{e}^{2\ell\pi\mathrm{i}/m}\right)\right| \le 1$ *for* $\ell = 0, 1, \ldots, m-1$. *Then*

$$|c_0| + |c_m| \le 1.$$

Proof. Let $\omega := \mathrm{e}^{2\ell\pi\mathrm{i}/m}$, where $\ell \in \{1, \ldots, m-1\}$. Then $\omega^m = 1$ and because $\omega \ne 1$, we have

$$\sum_{\mu=0}^{m-1} \left(\mathrm{e}^{2\ell\pi\mathrm{i}/m}\right)^\mu = \sum_{\mu=0}^{m-1} \omega^\mu = \frac{1-\omega^m}{1-\omega} = 0 \qquad (\ell = 1, \ldots, m-1)\,.$$

Hence

$$\sum_{\ell=0}^{m-1} \left(\mathrm{e}^{2\ell\pi\mathrm{i}/m}\right)^\mu = \sum_{\ell=0}^{m-1} \left(\mathrm{e}^{2\mu\pi\mathrm{i}/m}\right)^\ell = 0 \qquad (\mu = 1, \ldots, m-1),$$

and so

$$\sum_{\ell=0}^{m-1} \phi\left(\mathrm{e}^{2\ell\pi\mathrm{i}/m}\right) = \sum_{\ell=0}^{m-1} c_0 + \sum_{\mu=1}^{m-1} c_\mu \sum_{\ell=0}^{m-1} \left(\mathrm{e}^{2\ell\pi\mathrm{i}/m}\right)^\mu + \sum_{\ell=0}^{m-1} c_m = m(c_0 + c_m).$$

Since $c_0\, \overline{c}_m > 0$, this implies that

$$|c_0| + |c_m| = |c_0 + c_m| \le \frac{1}{m} \sum_{\ell=0}^{m-1} \left|\phi\left(\mathrm{e}^{2\ell\pi\mathrm{i}/m}\right)\right| \le 1. \qquad \square$$

Proof of Theorem 1. Let

$$\phi(\zeta) := \psi(\zeta) \prod_{j=1}^{J} \frac{\overline{\zeta}_j \zeta - 1}{\zeta - \zeta_j}.$$

Then $\phi(\zeta) := \sum_{\mu=0}^{m} c_\mu \zeta^\mu$, where

$$c_0 = \frac{\gamma_0}{\prod_{j=1}^{J} \zeta_j} \quad \text{and} \quad c_m = \gamma_m \prod_{j=1}^{J} \overline{\zeta}_j,$$

and so

$$c_0\,\overline{c}_m = \frac{\gamma_0}{\prod_{j=1}^{J}\zeta_j}\,\overline{\gamma}_m\prod_{j=1}^{J}\zeta_j = \gamma_0\,\overline{\gamma}_m > 0.$$

Furthermore, $\left|\phi\left(\mathrm{e}^{2\ell\pi\mathrm{i}/m}\right)\right| \le 1$ for $\ell = 0, 1, \ldots, m-1$ since

$$\left|\frac{\overline{\zeta}_j\zeta - 1}{\zeta - \zeta_j}\right| = 1 \qquad (|\zeta| = 1).$$

Thus, Lemma 1 applies and so (10) holds. □

We are ready to state and prove our refinement of Chebyshev's inequality, which takes into account each and every zero that does not belong to $[-1, 1]$.

Corollary 1. *Let* $f(z) := a_n\prod_{\nu=1}^{n}(z - z_\nu)$, $a_n \ne 0$, *and suppose that*

$$\left|f\left(\cos\frac{k\pi}{n}\right)\right| \le 1 \qquad (k = 0, 1, \ldots, n). \tag{11}$$

Furthermore, for any $z_\nu \notin [-1, 1]$ *let* R_ν *denote the sum of the semi-axes of the ellipse whose foci lie at* ± 1 *and which passes through* z_ν*; otherwise let* $R_\nu = 1$. *Then*

$$\left(R_1\ldots R_n + \frac{1}{R_1\ldots R_n}\right)|a_n| \le 2^n. \tag{12}$$

Proof. Let

$$\psi(\zeta) := \zeta^n f\left(\frac{\zeta + \zeta^{-1}}{2}\right).$$

Then $\psi(\zeta) := \sum_{\nu=0}^{2n}\gamma_\nu\,\zeta^\nu$, where

$$\gamma_0 = \gamma_{2n} = 2^{-n}a_n. \tag{13}$$

In particular, $\gamma_0\,\overline{\gamma}_{2n} = 2^{-2n}|a_n|^2 > 0$. Furthermore,

$$\left|\psi\left(\exp\left(\frac{2\ell\pi\mathrm{i}}{2n}\right)\right)\right| = \left|f\left(\cos\frac{\ell\pi}{n}\right)\right| \le 1 \qquad (\ell = 0, 1, \ldots, 2n-1)$$

since

$$\cos\frac{\ell\pi}{n} = \cos\frac{(2n-\ell)\pi}{n} \qquad (\ell = n+1, \ldots, 2n-1).$$

Thus, ψ satisfies the conditions of Theorem 1 with $m = 2n$.

For any given $R > 1$, let C_R denote the circle $\zeta = Re^{it}$, $0 \le t \le 2\pi$. Also, let $\mathcal{E}_R$ be the ellipse whose foci lie at ± 1 and whose semi-axes are

$$\frac{R + R^{-1}}{2} \quad \text{and} \quad \frac{R - R^{-1}}{2}.$$

It may be noted that as ζ goes around the circle C_R, the point $z := (\zeta + \zeta^{-1})/2$ describes the ellipse $\mathcal{E}_R$ once counterclockwise, starting at the point $(R + R^{-1})/2$. Hence, corresponding to a zero z_ν of f, not belonging to $[-1, 1]$, there is a number ζ_ν lying outside the closed unit disc such that

$$\frac{1}{2}(\zeta_\nu + \zeta_\nu^{-1}) = z_\nu. \tag{14}$$

In fact, $|\zeta_\nu|$ is equal to R_ν, the sum of the semi-axes of the ellipse whose foci lie at ± 1 and which passes through z_ν.

Note that

$$\frac{1}{2}(\zeta + \zeta^{-1}) = z_\nu$$

has a double root at $\zeta = 1$ if $z_\nu = 1$ and a double root at $\zeta = -1$ if $z_\nu = -1$. For any $z_\nu \in (-1, 1)$ there are two different values of ζ_ν, both of modulus 1, that satisfy (14). So, we take $R_\nu = |\zeta_\nu| = 1$ if z_ν is a zero of f that lies somewhere in $[-1, 1]$.

Amongst the zeros of ψ we can therefore count n zeros $\zeta_1, \dots, \zeta_n$ with $|\zeta_\nu| = R_\nu \ge 1$ for $\nu = 1, \dots, n$. Applying Theorem 1 with $m = 2n$, $J = n$ and taking note of (13), by which $\gamma_0 \bar{\gamma}_{2n} > 0$, we see that (12) holds. $\square$

Remark 1. In [11, Corollary 4], inequality (12) was obtained under the assumption that $|f(x)| \le 1$ for $-1 \le x \le 1$. Some other interesting results appear in [7]. The result, stated above as Corollary 1, first appeared in [9], but the proof presented here is different.

It is easily proved by induction that if $R_1, \dots, R_n$ are as in Corollary 1, then

$$\prod_{\nu=1}^{n} \frac{R_\nu^2 + 1}{2} \le \frac{\prod_{\nu=1}^{n} R_\nu^2 + 1}{2}.$$

This is clearly equivalent to the inequality

$$\prod_{\nu=1}^{n} \frac{R_\nu + R_\nu^{-1}}{2} \le \frac{\prod_{\nu=1}^{n} R_\nu + \prod_{\nu=1}^{n} R_\nu^{-1}}{2}, \tag{15}$$

which may be combined with (12) to obtain the following result.

Corollary 2. *Under the conditions of Corollary* 1, *we have*

$$\left(\prod_{\nu=1}^{n} \frac{R_\nu + R_\nu^{-1}}{2}\right) |a_n| \le 2^{n-1}. \tag{16}$$

Since

$$|z_\nu| = \left|\frac{\zeta_\nu + \zeta_\nu^{-1}}{2}\right| \le \frac{R_\nu + R_\nu^{-1}}{2},$$

we see that (12) is considerably stronger than (9). In addition, the hypothesis in Corollary 1 is quite a bit weaker. Instead of supposing $|f(x)|$ to be bounded by 1 on $[-1,1]$ we only require this restriction on $|f(x)|$ to be satisfied at the extrema of T_n.

The following result is an immediate consequence of Corollary 1.

Corollary 3. *For any* $R > 1$, *let* $\mathcal{E}_R$ *be the ellipse whose foci lie at* ± 1 *and the sum of whose semi-axes is* R. *Furthermore, let* $f(z) := a_n \prod_{\nu=1}^{n}(z - z_\nu)$ *satisfy* (11), *and have at most* $n - \ell$ *zeros inside* $\mathcal{E}_R$. *Then*

$$(R^\ell + R^{-\ell})|a_n| \le 2^n. \tag{17}$$

Corollary 3 says in particular that *if* $f(z) \ne 0$ *inside* $\mathcal{E}_R$, *then*

$$(R^n + R^{-n})|a_n| \le 2^n. \tag{18}$$

This estimate is sharp. In order to see this, let us consider the polynomial

$$f_R(z) := \frac{2}{R^n + R^{-n}}\left(T_n(z) + \mathrm{i}\frac{R^n - R^{-n}}{2}\right).$$

Clearly,

$$|f_R(x)| \le \frac{2}{R^n + R^{-n}}\sqrt{1 + \left(\frac{R^n - R^{-n}}{2}\right)^2} = 1 \qquad (-1 \le x \le 1).$$

Using (2) we find that

$$\frac{R^n + R^{-n}}{2} f_R\left(\frac{\zeta + \zeta^{-1}}{2}\right) = \frac{\zeta^n + \zeta^{-n}}{2} + \mathrm{i}\frac{R^n - R^{-n}}{2},$$

which is easily seen to vanish for

$$\zeta = \zeta_\nu = R\mathrm{e}^{(4\nu-1)\pi\mathrm{i}/2n} \qquad (\nu = 1, \ldots, n).$$

Hence, $f_R(z) = 0$ at each of the n points

$$z = z_\nu = \frac{1}{2}\left\{R\mathrm{e}^{(4\nu-1)\pi\mathrm{i}/2n} + R^{-1}\mathrm{e}^{-(4\nu-1)\pi\mathrm{i}/2n}\right\} \qquad (\nu = 1, \ldots, n).$$

Since f_R is a polynomial of degree n, it has no other zeros. Setting

$$A := \frac{R + R^{-1}}{2} \quad \text{and} \quad B := \frac{R - R^{-1}}{2},$$

the points z_ν can also be written as

$$z_\nu = A\cos\frac{(4\nu-1)\pi}{2n} + \mathrm{i}B\sin\frac{(4\nu-1)\pi}{2n} \qquad (\nu = 1, \ldots, n).$$

Thus, f_R has all its zeros on the ellipse $\mathcal{E}_R$. Since $f_R(z)$ is of the form

$$f_R(z) = \frac{2^n}{R^n + R^{-n}} z^n + \sum_{\nu=0}^{n-1} a_\nu z^\nu,$$

we conclude that the bound for $|a_n|$, given by (18), cannot be improved even if (11) is replaced by the stronger condition: "$|f(x)| \le 1$ for $-1 \le x \le 1$".

4. Relevance of the points $\left\{\cos\frac{k\pi}{n}\right\}$. Corollary 1 says that in (5) we may replace $\max_{-1\le x\le 1}|f(x)|$ by the maximum of $|f(x)|$ at the extrema of the (extremal) polynomial T_n. One might wonder about the reason behind it. The following result (see [3, Theorems 4 and 5]) provides an explanation.

Theorem D. *Let $t_0 < \cdots < t_n$ be an arbitrary set of $n+1$ real numbers. Setting $P(z) := \prod_{\nu=0}^{n}(z - t_\nu)$ let $t_{0,k} < \cdots < t_{n-k,k}$ be the zeros of $P^{(k)}$, the $k-$th derivative of P. In addition, let $y_0, \ldots, y_n$ be any set of $n+1$ non-negative numbers, not all zero, and denote by π_n the unique polynomial of degree at most n such that $\pi_n(t_\nu) = (-1)^{n-\nu} y_\nu$ for $\nu = 0, 1, \ldots, n$. Then, for any real polynomial f of degree at most n, other than $f(z) \equiv \pm\pi_n(z)$, such that $|f(t_\nu)| \le y_\nu$ for $\nu = 0, 1, \ldots, n$, we have*

$$\text{(19)} \qquad |f(z)| < |\pi_n(z)| \qquad \left(\left|z - \frac{t_0 + t_n}{2}\right| \ge \left|\frac{t_n - t_0}{2}\right|,\ z \notin \{t_0, t_n\}\right),$$

and

$$|f^{(k)}(z)| < |\pi_n^{(k)}(z)| \qquad (k = 1, \ldots, n) \tag{20}$$

if z satisfies

$$\left| z - \frac{t_{0,k} + t_{n-k,k}}{2} \right| \geq \left| \frac{t_{n-k,k} - t_{0,k}}{2} \right| \,,\ z \notin \{t_{0,k}, t_{n-k,k}\}.$$

Remark 2. The case $k = n$ of (20) says that if $\pi_n(z) := \sum_{\nu=0}^{n} \pi_{n,\nu} z^\nu$, then $|a_n| < |\pi_{n,n}|$ unless $f(z) \equiv \pm\pi_n(z)$. This is a good deal more than what Theorem A says. In the important case where $t_\nu = \cos(\nu\pi/n)$, inequality (19) goes back to S. Bernstein [1] and to Erdős [4]. One may consult [14, Chapter 12] for related results. The following application of (19) gives a meaningful refinement of (5) provided that n is *odd*.

Theorem 2. *For any odd integer n, let $t_0 = -1 < \cdots < t_n = 1$ be a set of $n + 1$ real numbers such that $t_\nu = -t_{n-\nu}$ for $\nu = 0, 1, \ldots, n$. Also, let $y_0, \ldots, y_n$ be a set of $n + 1$ non-negative numbers, not all zero, satisfying the condition $y_\nu = y_{n-\nu}$ for $\nu = 0, 1, \ldots, n$. Finally, let $\pi_n(z) := \sum_{\nu=0}^{n} \pi_{n,\nu} z^\nu$ be the unique polynomial of degree at most n for which $\pi_n(t_\nu) = (-1)^{n-\nu} y_\nu$ for $\nu = 0, 1, \ldots, n$. Then for any real polynomial $f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree at most n, other than $f(z) \equiv \pm\pi_n(z)$, satisfying $|f(t_\nu)| \leq y_\nu$ for $\nu = 0, 1, \ldots, n$, we have*

$$|a_n| + |a_0| < \pi_{n,n}. \tag{21}$$

Proof. Because of the restrictions on $\{t_\nu\}$ and $\{y_\nu\}$, the polynomial π_n must be odd, and so $\pi_{n,0} = 0$. Without loss of generality we may assume a_n to be positive. Applying (19) with $t_0 := -1$ and $t_n := 1$ we see that $|f(z)| < |\pi_n(z)|$ if $|z| \geq 1$, $z \neq \pm 1$. Hence, the polynomial $\pi_n(z) - f(z)$ has all its zeros in $\mathcal{E} := \{z \in \mathbb{C} : |z| < 1\} \cup \{-1, 1\}$. We claim that the zeros cannot all lie at ± 1. In fact, if they did then $\pi_n(x) - f(x)$ would be either strictly positive or strictly negative on $(-1, 1)$. If it was strictly positive then in particular we would have

$$-y_{n-1} - f(t_{n-1}) = \pi_n(t_{n-1}) - f(t_{n-1}) > 0,$$

which is not possible since $|f(t_{n-1})| \leq y_{n-1}$ by hypothesis. Similarly, if $\pi_n(x) - f(x)$ was strictly negative on $(-1, 1)$ then for $x = t_{n-2}$ we would obtain

$f(t_{n-2}) > y_{n-2}$, which is a contradiction since $f(t_{n-2})$ is supposed to be bounded above by y_{n-2}. Thus, the polynomial

$$\pi_n(z) - f(z) = (\pi_{n,n} - a_n)z^n + \sum_{\nu=1}^{n-1} (\pi_{n,n-\nu} - a_{n-\nu})\, z^{n-\nu} - a_0,$$

which has has all its zeros in $\mathcal{E}$, has at least one in $\{z \in \mathbb{C} : |z| < 1\}$. Since $\mathcal{E} \subset \{z \in \mathbb{C} : |z| \le 1\}$ this implies that

$$|a_0| < \pi_{n,n} - a_n,$$

and so (21) holds. □

5. Bounds for the other coefficients. Generalizing Theorem A, Wladimir Markoff ([8]; see also [5]) found sharp bounds for all the Maclaurin coefficients of a polynomial f of degree at most n in terms of its maximum modulus on $[-1, 1]$.

Theorem E. *Let $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ be a polynomial of degree at most n such that $|f(x)| \le 1$ for $-1 \le x \le 1$. Then, $|a_{n-2\mu}|$ is bounded above by the modulus of the corresponding coefficient of T_n for $\mu = 0, \dots, \lfloor n/2 \rfloor$, and $|a_{n-1-2\mu}|$ is bounded above by the modulus of the corresponding coefficient of T_{n-1} for $\mu = 0, \dots, \lfloor (n-1)/2 \rfloor$.*

Erdős [4, p. 1176] remarked that if the coefficients of $f(z) := \sum_{\nu=0}^{n} a_\nu\, z^\nu$ are *all real*, then $\sum_{\nu=0}^{n} |a_\nu|$ is maximal for $f(z) \equiv \pm T_n(z)$. According to him, Szegő proved that $|a_{2k}| + |a_{2k+1}|$ is maximal for $f(z) \equiv \pm T_n(z)$, which is stronger than the observation made by Erdős himself. Unfortunately, Szegő never published his proof. Erdős learned about the result orally (see [4, p. 1176]) from Szegő. It would be nice to know Szegő's proof of his result.

A few years ago (see [3, Theorem 3]) we proved the following result, which covers that of Szegő.

Theorem F. *Let $x_0 < \dots < x_n$ be $n+1$ real numbers, and let $y_0, \dots, y_n$ be a sequence of $n+1$ non-negative numbers, where we suppose that $x_\nu = -x_{n-\nu}$ and $y_\nu = y_{n-\nu}$ for $\nu = 0, \dots, n$, and that $\sum_{\nu=0}^{n} y_\nu > 0$. In addition, let $F(x) := \sum_{\mu=0}^{\lfloor n/2 \rfloor} A_{n-2\mu} x^{n-2\mu}$ be the unique polynomial of degree n such that $F(x_\nu) = (-1)^{n-\nu} y_\nu$ for $\nu = 0, \dots, n$. Furthermore, let $f(x) := \sum_{\nu=0}^{n} a_\nu\, x^\nu$ be a real polynomial of degree at most n, whose modulus does not exceed that of F at the points $x_0 < \dots < x_n$, that is*

$$|f(x_\nu)| \le y_\nu = |F(x_\nu)| \qquad (\nu = 0, \dots, n).$$

Then,

$$|a_{n-2k}| + |a_{n-2k-1}| \le |A_{n-2k}| \qquad \left(k = 0, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor\right). \tag{22}$$

Theorem F says, in particular, that if $T_n(x) := \sum_{\nu=0}^{\lfloor n/2 \rfloor} t_{n,\nu}\, x^{\nu}$ is the Chebyshev polynomial of the first kind of degree n then, for any real polynomial $f(x) := \sum_{\nu=0}^{n} a_\nu\, x^\nu$ of degree n satisfying (11), we have

$$|a_{n-2k}| + |a_{n-2k-1}| \le |t_{n,n-2k}| \qquad \left(k = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right). \tag{23}$$

Not only T_n but all the ultraspherical polynomials $P_n^{(\lambda)}$ have the properties the polynomial F of Theorem F is required to have for (22) to hold. This observation suggests an inequality, more general than (23), involving the coefficients of $P_n^{(\lambda)}$ for any $\lambda > -1/2$ (see Corollary 4). In order to present such a generalization of (23) we need to recall some basic facts about the polynomials $P_n^{(\lambda)}$. This will come in handy in connection with certain extensions (see Theorem 3 and Corollary 5) of the well-known L^2 analogues of Chebyshev's inequality, which we also intend to discuss in this paper.

6. The polynomials $P_n^{(\lambda)}$ and an extension of (23). For $\lambda \in (-\frac{1}{2}, 0) \cup (0, \infty)$, the ultraspherical polynomials $P_n^{(\lambda)}$ are given ([17], see (4.7.1) on p. 81 and (4.7.31) on p. 85) by

$$\begin{aligned} P_n^{(\lambda)}(x) &= \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + \frac{1}{2})} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x) \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n - k + \lambda)}{\Gamma(\lambda)\Gamma(k+1)\Gamma(n - 2k + 1)} (2x)^{n-2k}. \end{aligned} \tag{24}$$

Thus $P_n^{(\lambda)}$ is a multiple of the Jacobi polynomial $P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}$. The multiplicative factor

$$\frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(2\lambda)} \frac{\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + \frac{1}{2})},$$

which depends on λ and n, vanishes for $\lambda = 0$. It is known ([17, p. 82], see (4.7.8)) that

$$\lim_{\lambda \to 0} \lambda^{-1} P_n^{(\lambda)}(x) = \frac{2}{n} T_n(x) \qquad (n = 1, 2, \dots),$$

where T_n is the Chebyshev polynomial of the first kind of degree n. Hence, we may define

$$P_n^{(0)}(x) := \frac{2}{n}T_n(x) \qquad (n = 1, 2, \ldots). \tag{25}$$

It is known ([17, p. 82], see (4.7.15)) that if $\lambda \in (-\frac{1}{2}, 0) \cup (0, \infty)$, then

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} P_n^{(\lambda)}(x) P_m^{(\lambda)}(x)\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}{\{\Gamma(\lambda)\}^2(n+\lambda)\Gamma(n+1)} & \text{if } m = n. \end{cases} \tag{26}$$

In view of (25), the corresponding formula for $\lambda = 0$ is

$$\int_{-1}^{1} P_n^{(0)}(x) P_m^{(0)}(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2\pi}{n^2} & \text{if } m = n \geq 1. \end{cases} \tag{26$'$}$$

We find it convenient to define

$$P_n^{(\lambda)*}(x) := \begin{cases} \Gamma(\lambda)\sqrt{\frac{(n+\lambda)\Gamma(n+1)}{2^{1-2\lambda}\pi\Gamma(n+2\lambda)}}P_n^{(\lambda)}(x) & \text{if } \lambda > -\frac{1}{2},\ \lambda \neq 0, \\ \frac{n}{\sqrt{2\pi}}P_n^{(0)}(x) & \text{if } \lambda = 0. \end{cases} \tag{27}$$

Because of (26) and (26$'$), the polynomials $\{P_n^{(\lambda)*}\}, \lambda > -1/2$ are orthonormal in the sense that

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} P_n^{(\lambda)*}(x) P_m^{(\lambda)*}(x)\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases} \tag{28}$$

For any given $\lambda > -1/2$ and any $n \geq 2$, let $x_{n,1}(\lambda) < \ldots < x_{n,n-1}(\lambda)$ be the extrema of $P_n^{(\lambda)*}$ in $(-1, 1)$. Then, Theorem F, applied with

$$x_0 := -1, x_\nu := x_{n,\nu}(\lambda) \ \text{ for } \ \nu = 1, \ldots, n-1, \ \ x_n := 1$$

and

$$y_\nu := \left|P_n^{(\lambda)*}(x_\nu)\right| \qquad (\nu = 0, \ldots, n),$$

gives us the following result.

Corollary 4. *For any given $\lambda > -1/2$ and any integer $n \geq 2$, let*

$$P_n^{(\lambda)*}(x) := \sum_{\nu=0}^{n} p_{n,\nu}^{(\lambda)*} x^\nu \tag{29}$$

be the ultraspherical polynomial of degree n, as defined in (27). Also let $x_1 := x_{n,1}(\lambda), \ldots, x_{n-1} := x_{n,n-1}(\lambda)$ be the extrema of $P_n^{(\lambda)*}$ in $(-1,1)$, arranged in increasing order. Furthermore, let $f(x) := \sum_{\nu=0}^{n} a_\nu\, x^\nu$ be a real polynomial of degree at most n such that

$$|f(x_\nu)| \le \left|P_n^{(\lambda)*}(x_\nu)\right| \qquad (\nu = 0, \ldots, n), \tag{30}$$

where $x_0 := -1$ and $x_n := 1$. Then,

$$|a_{n-2k}| + |a_{n-2k-1}| \le |p_{n,n-2k}^{(\lambda)*}| \qquad \left(k = 0, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor\right). \tag{31}$$

7. A refined L^2 analogue of Chebyshev's inequality.

In Corollary 4 we require $|f(x)|$ to be dominated by $|P_n^{(\lambda)*}(x)|$ at the points $x_0, x_1, \ldots, x_n$. How large can $|a_{n-2k}| + |a_{n-2k-1}|$ be if $|f(x)|$ is not necessarily bounded by $|P_n^{(\lambda)*}(x)|$ at any specific points of $[-1,1]$ but in some average sense, like

$$\frac{\int_{-1}^{1} w(x)|f(x)|^2 \mathrm{d}x}{\int_{-1}^{1} w(x)\mathrm{d}x} \le \frac{\int_{-1}^{1} w(x)|P_n^{(\lambda)*}(x)|^2 \mathrm{d}x}{\int_{-1}^{1} w(x)\mathrm{d}x}$$

for some appropriate weight function w? We are able to shed some light on this question in the case where $w(x) := (1-x^2)^{\lambda-\frac{1}{2}}$, the weight whose relevance is clear from (28).

As in (29), let $p_{n,n}^{(\lambda)*}$ denote the leading coefficient in the Maclaurin expansion of $P_n^{(\lambda)*}(x)$. Then, from (27), (24), (25) and (4), it follows that

$$p_{n,n}^{(\lambda)*} = 2^n \sqrt{\frac{\Gamma(n+\lambda+1)\Gamma(n+\lambda)}{2^{1-2\lambda}\pi\Gamma(n+1)\Gamma(n+2\lambda)}} \qquad \left(\lambda > -\frac{1}{2}\right), \tag{32}$$

where the formula proves to be in order for $\lambda = 0$ because $P_n^{(0)} := (2/n)T_n$.

Theorem 3. *For any given $\lambda > -1/2$ and any integer $n \ge 2$, let $p_{n,n}^{(\lambda)*}$ denote the coefficient of x^n in the Maclaurin expansion of the ultraspherical polynomial $P_n^{(\lambda)*}(x)$, as defined in (27). Furthermore, let $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ be a polynomial of degree n with coefficients in $\mathbb{C}$ such that*

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} |f(x)|^2 \mathrm{d}x \le \int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} \left|P_n^{(\lambda)*}(x)\right|^2 \mathrm{d}x = 1. \tag{33}$$

Then,

$$|a_n|^2 + 4\,\frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}\,|a_{n-1}|^2 \le \left\{p_{n,n}^{(\lambda)*}\right\}^2. \tag{34}$$

The inequality is best possible in the sense that the coefficient of $|a_{n-1}|^2$ *in* (34) *cannot be replaced by any number larger than*

$$4\,\frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}.$$

It is well-known that the sharp upper bound for each of the two terms

$$|a_n|^2 \ \text{ and } \ 4\,\frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}\,|a_{n-1}|^2,$$

appearing on the left-hand side of (34) is $\left\{p_{n,n}^{(\lambda)*}\right\}^2$. So apparently, inequality (34) is not to be sneezed at. Nevertheless, in view of (31), it would be interesting to know the largest admissible value of ε for the inequality

$$|a_n| + \varepsilon|a_{n-1}| \le p_{n,n}^{(\lambda)*}$$

to be true for any polynomial $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ satisfying the hypothesis of Theorem 3. The following corollary of Theorem 3 gives the sharp upper bound for $|a_n| + \varepsilon|a_{n-1}|$ in terms of $\varepsilon > 0$. The upper bound is larger than $p_{n,n}^{(\lambda)*}$ for any $\varepsilon > 0$.

Corollary 5. *For any given* $\lambda > -1/2$ *and any integer* $n \ge 2$, *let* $p_{n,n}^{(\lambda)*}$ *denote the coefficient of* x^n *in the Maclaurin expansion of the ultraspherical polynomial* $P_n^{(\lambda)*}(x)$, *as defined in* (27). *Furthermore, let* $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ *be a polynomial of degree* n, *with coefficients in* $\mathbb{C}$, *satisfying* (33). *Then,*

$$|a_n| + \varepsilon|a_{n-1}| \le \sqrt{1 + \frac{\varepsilon^2}{4}\,\frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1)}}\; p_{n,n}^{(\lambda)*} \qquad (\varepsilon > 0). \tag{35}$$

The inequality is sharp. It cannot be improved even if the coefficients of f *are all real.*

P r o o f o f T h e o r e m 3. For any given $\lambda > -1/2$ there exist constants b_ν such that

$$f(x) = \sum_{\nu=0}^{n} b_\nu P_\nu^{(\lambda)*}(x).$$

Since $P_\nu^{(\lambda)*}$ is odd or even according as n is odd or even, respectively, we note that $p_{n,n-1}^{(\lambda)*} = 0$. Because of this fact, comparing the coefficients in the two expansions of $f(x)$ and taking (32) into account, we see that

$$a_n = p_{n,n}^{(\lambda)*}\, b_n = 2^n \sqrt{\frac{\Gamma(n+\lambda+1)\Gamma(n+\lambda)}{2^{1-2\lambda}\pi\Gamma(n+1)\Gamma(n+2\lambda)}}\; b_n \tag{36}$$

and

$$a_{n-1} = p_{n-1,n-1}^{(\lambda)*}\, b_{n-1} = 2^{n-1} \sqrt{\frac{\Gamma(n+\lambda)\Gamma(n+\lambda-1)}{2^{1-2\lambda}\pi\Gamma(n)\Gamma(n+2\lambda-1)}}\; b_{n-1}. \tag{37}$$

The orthonormality of $\{P_0^{(\lambda)*}, P_1^{(\lambda)*}, P_2^{(\lambda)*}, \ldots\}$, given in (28), implies that

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} |f(x)|^2 \mathrm{d}x = \sum_{\nu=0}^{n} |b_\nu|^2 \geq |b_n|^2 + |b_{n-1}|^2, \tag{38}$$

where, for $n \geq 3$, the inequality becomes an equality if and only if $b_0, \ldots, b_{n-2}$ are all zero, that is, if and only if $f(x) := b_n P_n^{(\lambda)*}(x) + b_{n-1} P_{n-1}^{(\lambda)*}(x)$. From (33) and (38) it follows that

$$|b_n|^2 + |b_{n-1}|^2 \leq 1. \tag{39}$$

By (37), we have

$$4\,\frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}\, |a_{n-1}|^2 = 2^{2n}\, \frac{\Gamma(n+\lambda+1)\Gamma(n+\lambda)}{2^{1-2\lambda}\pi\Gamma(n+1)\Gamma(n+2\lambda)}\, |b_{n-1}|^2,$$

which, in conjunction with (36), shows that

$$|a_n|^2 + 4\,\frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}\, |a_{n-1}|^2 = \left(|b_n|^2 + |b_{n-1}|^2\right) \left\{p_{n,n}^{(\lambda)*}\right\}^2.$$

This, in view of (39), gives us the desired inequality (34).

In order to see that (34) is sharp let us consider, for any $t \in [0,1]$, the polynomial

$$f(x) := t\mathrm{e}^{\mathrm{i}\alpha} P_n^{(\lambda)*}(x) + \sqrt{1-t^2}\,\mathrm{e}^{\mathrm{i}\beta} P_{n-1}^{(\lambda)*}(x) \qquad (\alpha \in \mathbb{R},\ \beta \in \mathbb{R})\,. \tag{40}$$

Then

$$\begin{aligned}\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}|f(x)|^2\mathrm{d}x &= t^2\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\left|P_n^{(\lambda)*}(x)\right|^2\mathrm{d}x\\ &\qquad + (1-t^2)\int_{-1}^{1}(1-x^2)^{\lambda-\frac{1}{2}}\left|P_{n-1}^{(\lambda)*}(x)\right|^2\mathrm{d}x\\ &= 1\end{aligned}$$

by (28). Considering the Maclaurin expansion $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ of the polynomial defined in (40), we see that

$$a_n = t\mathrm{e}^{\mathrm{i}\alpha}p_{n,n}^{(\lambda)*} \quad \text{and} \quad a_{n-1} = \sqrt{1-t^2}\,\mathrm{e}^{\mathrm{i}\beta}p_{n-1,n-1}^{(\lambda)*}.$$

Hence

$$\begin{aligned}|a_n|^2 + {} & 4\,\frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}\,|a_{n-1}|^2\\ &= t^2\left\{p_{n,n}^{(\lambda)*}\right\}^2 + (1-t^2)\,4\,\frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}\,\left\{p_{n-1,n-1}^{(\lambda)*}\right\}^2\\ &= t^2\left\{p_{n,n}^{(\lambda)*}\right\}^2 + (1-t^2)\left\{p_{n,n}^{(\lambda)*}\right\}^2 = \left\{p_{n,n}^{(\lambda)*}\right\}^2.\end{aligned}$$

The above calculations also show that (34) would fail if its left-hand side was replaced by $|a_n|^2 + c\,|a_{n-1}|^2$ with any

$$c = c(n,\lambda) > 4\,\frac{(n+\lambda)(n+\lambda-1)}{n(n+2\lambda-1)}. \qquad \square$$

Proof of Corollary 5. From (34) it follows that $|a_n| \le p_{n,n}^{(\lambda)*}$ and then

$$|a_{n-1}| \le \frac{1}{2}\sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1))}}\sqrt{\left\{p_{n,n}^{(\lambda)*}\right\}^2 - |a_n|^2}.$$

Hence $|a_n| + \varepsilon|a_{n-1}| \le \varphi(|a_n|)$, where

$$\varphi(u) := u + \frac{\varepsilon}{2}\sqrt{\frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1))}}\sqrt{\left\{p_{n,n}^{(\lambda)*}\right\}^2 - u^2}$$

for $0 \le u \le p_{n,n}^{(\lambda)*}$. The function φ has only one critical point

$$u = \frac{1}{\sqrt{1 + \frac{1}{4}\frac{n(n+2\lambda-1)}{(n+\lambda)((n+\lambda-1)}\varepsilon^2}}\,p_{n,n}^{(\lambda)*}$$

in $(0, p_{n,n}^{(\lambda)*})$ and there it has a local maximum. Simple calculations then lead us to the estimate for $|a_n| + \varepsilon |a_{n-1}|$ that is given in (35).

To see that (35) cannot be improved even if $f(x)$ is real for all real x, let

$$\delta := \frac{p_{n,n}^{(\lambda)*}}{p_{n-1,n-1}^{(\lambda)*}} = 2\sqrt{\frac{(n+\lambda+1)(n+\lambda)}{n(n+2\lambda-1)}}$$

and, for any $\varepsilon > 0$, consider the real polynomial

$$f(x;\varepsilon) := \frac{\delta}{\sqrt{\varepsilon^2+\delta^2}}\, P_n^{(\lambda)*}(x) + \frac{\varepsilon}{\sqrt{\varepsilon^2+\delta^2}}\, P_{n-1}^{(\lambda)*}(x) = \sum_{\nu=0}^{n} a_\nu x^\nu .$$

Then,

$$a_n = \frac{\delta}{\sqrt{\varepsilon^2+\delta^2}}\, p_{n,n}^{(\lambda)*}$$

and

$$a_{n-1} = \frac{\varepsilon}{\sqrt{\varepsilon^2+\delta^2}}\, p_{n-1,n-1}^{(\lambda)*} = \frac{\varepsilon}{\delta\,\sqrt{\varepsilon^2+\delta^2}}\, p_{n,n}^{(\lambda)*} .$$

Clearly,

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} |f(x;\varepsilon)|^2 \mathrm{d}x = \frac{\delta^2}{\delta^2+\varepsilon^2} + \frac{\varepsilon^2}{\delta^2+\varepsilon^2} = 1$$

and

$$|a_n| + \varepsilon|a_{n-1}| = \frac{\delta}{\sqrt{\varepsilon^2+\delta^2}}\, p_{n,n}^{(\lambda)*} + \frac{\varepsilon^2}{\delta\,\sqrt{\varepsilon^2+\delta^2}}\, p_{n,n}^{(\lambda)*} = \sqrt{1+\frac{\varepsilon^2}{\delta^2}}\, p_{n,n}^{(\lambda)*} .$$

This shows that (35) becomes an equality for the polynomial $f(x;\varepsilon)$. $\square$

8. The L^p mean of f with Chebyshev weight. Let $\Phi(z) := \sum_{\mu=0}^{m} c_\mu z^\mu$ be a polynomial of degree $m \geq 1$. In addition, let $A(z) := z^m + 1$. It was shown by one of us [10, Theorem 2] that for any $p \in [1, \infty)$, we have

$$(41) \qquad |c_0| + |c_m| \leq \frac{2}{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |A(\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta\right)^{1/p}} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\Phi\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p \mathrm{d}\theta\right)^{1/p} .$$

If $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree n then

$$\Phi(z) := z^n f\left(\frac{z+z^{-1}}{2}\right) = \sum_{\nu=0}^{2n} c_\nu z^\nu$$

is a polynomial of degree $2n$, where

$$c_0 = c_{2n} = \frac{1}{2^n} a_n,$$

and (41) may be applied with $m = 2n$ to obtain

$$\begin{aligned} |a_n| &\le \frac{2^{n-1}}{\left(\frac{1}{\pi}\int_0^\pi |\cos n\theta|^p \mathrm{d}\theta\right)^{1/p}} \left(\frac{1}{\pi}\int_0^\pi |f(\cos\theta)|^p \mathrm{d}\theta\right)^{1/p} \\ &= \frac{2^{n-1}}{\left(\frac{1}{\pi}\int_0^\pi |T_n(x)|^p \frac{\mathrm{d}x}{\sqrt{1-x^2}}\right)^{1/p}} \left(\frac{1}{\pi}\int_{-1}^1 |f(x)|^p \frac{\mathrm{d}x}{\sqrt{1-x^2}}\right)^{1/p} \end{aligned} \tag{42}$$

for any $p \in [1, \infty)$. Note that this result is a direct consequence of (41) and that the restriction imposed on p comes solely from the restriction on p in that inequality. Hence, (42) should hold for any p for which (41) may turn out to be true. In [13, Theorem 3], it was shown that (41) holds not only for $p \in [1, \infty)$ but also for $p \in (0, 1)$. Hence, so does (42). Thus, the following result holds. Other references for this result are [6, p. 183] and [15, p. 120].

Theorem G. *For any polynomial $f(x) := \sum_{\nu=0}^n a_\nu x^\nu$ of degree n and any $p \in (0, \infty)$, we have*

$$|a_n| \le 2^{n-1} \left(\frac{\pi\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\right)^{1/p} \left(\frac{1}{\pi}\int_{-1}^1 |f(x)|^p \frac{\mathrm{d}x}{\sqrt{1-x^2}}\right)^{1/p}. \tag{43}$$

The inequality is sharp for all $p \in (0, \infty)$.

It may be noted that

$$\int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int_0^\pi \mathrm{d}\theta = \pi$$

and that

$$\left(\frac{1}{\pi}\int_{-1}^1 |f(x)|^p \frac{\mathrm{d}x}{\sqrt{1-x^2}}\right)^{1/p} \to \max_{-1\le x\le 1} |f(x)| \text{ as } x \to \infty.$$

Hence, (43) is a generalization of (5).

In looking for a generalization (of the case $\lambda = 0$) of Theorem 3 in the spirit of Theorem G we did not find anything of interest for $p \in (2, \infty)$ or for $p \in (0, 1]$. We do have the following result for values of $p \in (1, 2]$.

Theorem 4. *For any* $p \in (1, 2]$ *let*

$$p' := \frac{p}{p-1}.$$

Then, for any polynomial $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ *of degree* $n \geq 2$ *with coefficients in* $\mathbb{C}$ *and any* $p \in (1, 2]$, *we have*

$$\left(|a_n|^{p'} + |2a_{n-1}|^{p'}\right)^{1/p'} \leq 2^{n-1}\, 2^{1/p} \left(\frac{1}{\pi}\int_{-1}^{1} |f(x)|^p \frac{\mathrm{d}x}{\sqrt{1-x^2}}\right)^{1/p}. \tag{44}$$

Proof. As the first step, we write $f(x) = \sum_{\nu=0}^{n} b_\nu T_\nu(x)$. Then, clearly

$$a_n = 2^{n-1} b_n \quad \text{and} \quad a_{n-1} = 2^{n-2} b_{n-1}. \tag{45}$$

Let

$$g(\theta) := f(\cos\theta) = \sum_{\nu=0}^{n} b_\nu \cos \nu\theta = \sum_{\nu=-n}^{n} c_\nu \mathrm{e}^{\mathrm{i}\nu\theta},$$

where

$$c_0 = b_0 \quad \text{and} \quad c_{-\nu} = c_\nu = \frac{1}{2} b_\nu \quad (\nu = 1, \ldots, n). \tag{46}$$

By a well-known result due to Hausdorff–Young (see [22, p. 101]), if g belongs to $L^p(0, 2\pi)$ for some $p \in (1, 2]$ and

$$c_k := \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{d}\theta \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

then with $p' := p/(p-1)$, we have

$$\left(\sum_{k=-\infty}^{\infty} |c_k|^{p'}\right)^{1/p'} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |g(\theta)|^p \mathrm{d}\theta\right)^{1/p}.$$

Hence, taking (46) into account, we obtain

$$\begin{aligned}\frac{1}{2^{1/p}}\left(|b_n|^{p'}+|b_{n-1}|^{p'}\right)^{1/p'} &= \left\{2\left(\frac{|b_n|}{2}\right)^{p'}+2\left(\frac{|b_{n-1}|}{2}\right)^{p'}\right\}^{1/p'}\\ &\leq \left(\sum_{\nu=-n}^{n}|c_\nu|^{p'}\right)^{1/p'}\\ &\leq \left(\frac{1}{2\pi}\int_0^{2\pi}|g(\theta)|^p\mathrm{d}\theta\right)^{1/p} = \left(\frac{1}{\pi}\int_0^{\pi}|g(\theta)|^p\mathrm{d}\theta\right)^{1/p}\\ &= \left(\frac{1}{\pi}\int_{-1}^{1}|f(x)|^p\frac{\mathrm{d}x}{\sqrt{1-x^2}}\right)^{1/p}.\end{aligned}$$

In view of (45), this is equivalent to (44). □

9. Polynomials satisfying $\int_{-\infty}^{\infty}|f(x)|^2\mathrm{e}^{-x^2}\mathrm{d}x \leq 1$. In this last section we wish to present the analogue of (34) for polynomials $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ of degree n for which $\int_{-\infty}^{\infty}|f(x)|^2\mathrm{e}^{-x^2}\mathrm{d}x \leq 1$. For this we need to recall certain facts about Hermite polynomials [19, p. 52].

There is a unique polynomial $\sum_{\nu=0}^{n} a_\nu x^\nu$ of degree n, with prescribed $a_n \neq 0$, that satisfies the differential equation

$$y'' - 2xy' + 2ny = 0.$$

The one whose leading term is $2^n x^n$ is denoted by H_n and is called the Hermite polynomial of degree n. Hermite polynomials of odd degree are odd and those of even degree are even. The first four Hermite polynomials are

$$H_0 = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x,$$

and for $n = 4, 5, 6, \ldots$ the Maclaurin expansion of H_n is

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \cdots.$$

The Hermite polynomials are orthogonal, with

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)\mathrm{e}^{-x^2}\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n,\\ 2^n \times n!\sqrt{\pi} & \text{if } m = n.\end{cases}$$

Hence, the polynomials

$$H_n^*(x) := \frac{1}{\sqrt{2^n n!}\pi^{1/4}}H_n(x) \qquad (n = 0, 1, 2, \ldots), \tag{47}$$

are orthonormal in the sense that

$$\int_{-\infty}^{\infty} H_n^*(x)H_m^*(x)\mathrm{e}^{-x^2}\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n. \end{cases} \tag{48}$$

Let $H_m^*(x) := \sum_{\mu=0}^{m} h_{m,\mu}^* x^\mu$ be the Maclaurin expansion of $H_m^*(x)$, and note that

$$h_{m,m}^* = \frac{1}{\pi^{1/4}}\sqrt{\frac{2^m}{m!}} \qquad (m = 0, 1, 2, \ldots). \tag{49}$$

Now, we are ready to prove the following analogue of Theorem 3.

Theorem 5. *Let $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ be a polynomial of degree n with coefficients in $\mathbb{C}$ such that*

$$\int_{-\infty}^{\infty} |f(x)|^2\mathrm{e}^{-x^2}\mathrm{d}x \leq 1. \tag{50}$$

Then,

$$|a_n|^2 + \frac{2}{n}\,|a_{n-1}|^2 \leq \frac{2^n}{n!\sqrt{\pi}}. \tag{51}$$

The inequality is best possible in the sense that the coefficient of $|a_{n-1}|^2$ in (51) cannot be replaced by any number larger than $2/n$.

Here it may be mentioned that the sharp upper bound for each of the two terms

$$|a_n|^2 \ \text{ and } \ \frac{2}{n}\,|a_{n-1}|^2,$$

appearing on the left-hand side of (51) is also $2^n/(n!\sqrt{\pi})$.

P r o o f o f T h e o r e m 5. Let $f(x) = \sum_{m=0}^{n} \beta_m H_m^*(x)$ be the Hermite–Fourier expansion of f in terms of $H_0^*, \ldots, H_n^*$. Then

$$\sum_{\nu=0}^{n} a_\nu x^\nu \equiv \sum_{m=0}^{n} \beta_m \sum_{\mu=0}^{m} h_{m,\mu}^* x^\mu. \tag{52}$$

Because of the fact that $h_{n,n-1}^* = 0$, when we compare the coefficients of x^n and of x^{n-1} on the two sides of (52), and take (49) into account, we obtain

$$a_n = h_{n,n}^*\beta_n = \frac{1}{\pi^{1/4}}\sqrt{\frac{2^n}{n!}}\beta_n \tag{53}$$

and

$$a_{n-1} = h^*_{n-1,n-1}\beta_{n-1} = \frac{1}{\pi^{1/4}}\sqrt{\frac{2^{n-1}}{(n-1)!}}\beta_{n-1}. \tag{54}$$

Formula (48), applied in conjunction with condition (50), which f satisfies, implies that

$$|\beta_n|^2 + |\beta_{n-1}|^2 \le 1. \tag{55}$$

Using (53) and (54) in (55), we obtain

$$|a_n|^2 + \frac{2}{n}\,|a_{n-1}|^2 = \left(|\beta_n|^2 + |\beta_{n-1}|^2\right)\frac{2^n}{n!\sqrt{\pi}} \le \frac{2^n}{n!\sqrt{\pi}},$$

which proves (51).

It is easily checked that (51) becomes an equality for any polynomial of the form

$$f_t(x) := t\mathrm{e}^{\mathrm{i}\alpha}H^*_n(x) + \sqrt{1-t^2}\mathrm{e}^{\mathrm{i}\beta}H^*_{n-1}(x) \qquad (\alpha\in\mathbb{R}, \beta\in\mathbb{R}), \tag{56}$$

where t can be any number in $[0,1]$.

For any t in $[0,1)$, the polynomial f_t appearing in (56) shows that the coefficient of $|a_{n-1}|^2$ in (51) cannot be replaced by any number larger than $2/n$. $\square$

By Schwarz's inequality,

$$|a_n| + \varepsilon|a_{n-1}| \le \sqrt{1+\varepsilon^2\frac{n}{2}}\sqrt{|a_n|^2 + \frac{2}{n}|a_{n-1}|^2},$$

and so Theorem 5 readily implies the following result

Corollary 6. *Let $f(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ be a polynomial of degree n, with coefficients in $\mathbb{C}$, satisfying* (50). *Furthermore, let $h^*_{n,n}$ denote the coefficient of x^n in the Maclaurin expansion of the polynomial $H^*_n(x)$ defined in* (47). *Then*

$$|a_n| + \varepsilon|a_{n-1}| \le \sqrt{1+\varepsilon^2\frac{n}{2}}\frac{1}{\pi^{1/4}}\sqrt{\frac{2^n}{n!}}. \tag{57}$$

The example

$$f(x) := \frac{\delta}{\sqrt{\varepsilon^2+\delta^2}}H^*_n(x) + \frac{\varepsilon}{\sqrt{\varepsilon^2+\delta^2}}H^*_{n-1}(x),\ \delta := \sqrt{\frac{2}{n}}$$

shows that inequality (57) is sharp, and that it cannot be improved even if the coefficients of f are all real.

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