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ON AN ODE RELEVANT FOR THE GENERAL THEORY OF THE HYPERBOLIC CAUCHY PROBLEM

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Dedicated to Professor Vesselin Petkov on the occasion of his 65th birthday

Abstract. In this paper we study an ODE in the complex plane. This is a key step in the search of new necessary conditions for the well posedness of the Cauchy Problem for hyperbolic operators with double characteristics.

1. Introduction and statements. The purpose of this paper is to study the existence of solutions, bounded on the real axis, for an ordinary differential equation (ode). It is a well known fact that many problems in the theory of PDEs reduce to the study of an ode; the problem that motivates us here is the well posedness of the Cauchy problem for a class of hyperbolic operators with double characteristics.

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More specifically it is known, see e.g. [3] and [1], that the Cauchy problem in the presence of double characteristics is not well posed if there are bicharacteristic curves issuing from a simple characteristic point and tending to some double characteristic point. We say that in this case the Hamilton field gives rise to an unstable dynamical system.

In order to show that the Cauchy problem in such a case is not well posed in C^∞ one has to construct an asymptotic solution violating the a priori estimates implied by the well posedness of the Cauchy problem. This is done by showing that a certain ode has a solution which is bounded on the real axis (actually exponentially decreasing at infinity) if the complex parameter appearing in the equation is suitably chosen. Sometimes this parameter is called a nonlinear eigenvalue.

From a geometrical point of view we know that the Hamilton system is unstable if the principal symbol of the operator is transversally non degenerate and there is a Jordan block of size 4 in the canonical form of the Hamilton matrix, which can roughly be thought of as dH_p, where p is the principal symbol and H_p its Hamilton field. There are though other more complex situations where one may have an unstable Hamilton vector field, exhibiting higher degeneracies or a non constant rank of the symplectic form.

In this paper we study the equation

$$u''(x) = (x^{2k+1} + zx^k)u(x), \tag{1.1}$$

where $z \in \mathbb{C}$ and $k = 1, 2$. The case $k = 1$ corresponds to a neat transversally non degenerate situation, while $k > 1$ corresponds to higher order degeneracies.

It is a striking phenomenon that the proof for $k > 1$, actually $k = 2$, is much more difficult than that where $k = 1$. The whole proof boils down to showing that the Wronskian of two linearly independent solutions—an entire function of z—vanishes. In the $k = 1$ case some simple complex analysis is enough, while if $k = 2$ we must estimate the order of the entire function and use some supplementary arguments. This is done following an idea of Christ [6] (see also [8]).

The statement of our result is given in Theorem 3.2.2 at the end of the paper. We have no doubts that an analogous result should hold for arbitrary k, but as of now the proof seems to be much longer and difficult.

2. An upper bound for the order of an entire function. Aim of this section is to study the exponential order of growth of the Wronskian of equation (1.1), which can be written as

$$P_z(u) = \left(-\frac{d^2}{dx^2} + p_z(x)\right) u(x), \tag{2.1}$$

with $p_z(x) = x^{2k+1} + zx^k$. We define:

$$\phi_z(x) = \frac{2}{2k+3} x^{\frac{2k+3}{2}} + zx^{\frac{1}{2}}. \tag{2.2}$$

Then $\phi_z'(x) = x^{\frac{2k+1}{2}} + \frac{1}{2} zx^{-\frac{1}{2}}$. To keep the notation simple we omit the z parameter and write just u, P, p, ϕ. We see that $p(x) = \phi'^2(x) - \frac{1}{4} z^2 x^{-1}$. Let

$$G(x) = \frac{1}{\phi'^{1/2}(x)} e^{-\phi(x)}$$

and

$$A = A(z) = |z|^{\frac{2}{1-\delta}},$$

where $\delta < 1$ and will be chosen suitably later. Let $\psi = \log G = -\phi - \frac{1}{2} \log \phi'$ and $D = \frac{d}{dx} + \psi'$, $\tilde{D} = -\frac{d}{dx} + \psi'$, where we choose the principal branch of the complex logarithm. Then we have:

$$D\tilde{D} = \left(\frac{d}{dx} + \psi'\right)\left(-\frac{d}{dx} + \psi'\right) = -\frac{d^2}{dx^2} + \psi'' + \psi'^2, \tag{2.3}$$

where $\psi' = -\phi' - \frac{1}{2}\frac{\phi''}{\phi'}, \psi'' = -\phi'' - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)'$.

So that:

$$\begin{aligned} \psi'' + \psi'^2 &= -\phi'' - \frac{1}{2}\left(\frac{\phi''}{\phi'}\right)' + \left(\phi' + \frac{1}{2}\frac{\phi''}{\phi'}\right)^2 \\ &= E + \phi'^2 = E + p(x) + \frac{1}{4} z^2 x^{-1}, \end{aligned} \tag{2.4}$$

where we put $E = -\frac{1}{2}\left(\frac{\phi''}{\phi'}\right)' + \frac{1}{4}\left(\frac{\phi''}{\phi'}\right)^2$.

Thus

$$D\tilde{D} = -\frac{d^2}{dx^2} + p(x) + E + \frac{1}{4} z^2 x^{-1} = P + E + \frac{1}{4} z^2 x^{-1}. \tag{2.5}$$

Let us compute

$$PG = \left(-\frac{d^2}{dx^2} + p\right)\left(\frac{1}{\phi'^{1/2}(x)} e^{-\phi(x)}\right).$$

We have:

$$-\frac{d^2}{dx^2}\left(\frac{1}{\phi'^{1/2}(x)} e^{-\phi(x)}\right)$$
$$= -\left(\frac{1}{(\phi')^{1/2}}\right)'' e^{-\phi} + 2\phi'\left(\frac{1}{(\phi')^{1/2}}\right)' e^{-\phi} - \frac{e^{-\phi}}{(\phi')^{1/2}}(\phi'^2 - \phi'')$$

And thus

$$\left(-\frac{d^2}{dx^2} + p(x)\right) G \tag{2.6}$$
$$= -\left(\frac{1}{(\phi')^{1/2}}\right)'' e^{-\phi} + 2\phi'\left(\frac{1}{(\phi')^{1/2}}\right)' e^{-\phi} - \frac{e^{-\phi}}{(\phi')^{1/2}}(\phi'^2 - \phi'')$$
$$+ \phi'^2 \frac{1}{(\phi')^{1/2}} e^{-\phi} - \frac{1}{4} z^2 x^{-1} \frac{1}{(\phi')^{1/2}} e^{-\phi}$$
$$= -\left(\frac{1}{(\phi')^{1/2}}\right)'' e^{-\phi} - \frac{1}{4} z^2 x^{-1} \frac{1}{(\phi')^{1/2}} e^{-\phi}$$
$$= -(\phi')^{1/2}\left(\frac{1}{(\phi')^{1/2}}\right)'' G - -\frac{1}{4} z^2 x^{-1} G$$

Now

$$\frac{\left(\frac{1}{(\phi')^{1/2}}\right)''}{\left(\frac{1}{(\phi')^{1/2}}\right)} = (\phi')^{1/2}\left(\frac{1}{(\phi')^{1/2}}\right) = \mathscr{O}(x^{-2}).$$

So that

$$|PG| \le C|x|^{-2}|G(x) + \frac{1}{4}|z|^2|x|^{-1}|G(x).$$

Let $A < x < y$ and denote by $u_y(x)(= u(x))$ the solution of:

$$\begin{cases} Pu = PG, & A < x < y \\ u(y) = u'(y) = 0. \end{cases} \tag{2.7}$$

Set

$$\begin{aligned} v &= \tilde{D}u \\ e^{-\psi}\frac{d}{dx}e^{\psi}v &= Dv = D\tilde{D}u \\ &= Pu + Eu + \frac{1}{4}z^2x^{-1}u, \end{aligned}$$

then, because $v(y) = 0$, for $A \le s \le y$ we may write:

$$e^{\psi(s)}v(s) = \int_y^s e^{\psi(t)}(PG(t) + Eu + \frac{1}{4}z^2t^{-1}u)dt. \tag{2.8}$$

From (2.8) we then deduce that

$$|v(s)| \le |e^{-\psi(s)}| \int_y^s |e^{\psi(t)}|[|PG(t)| + |Eu| + \frac{1}{4}|z^2|t^{-1}|u|]dt.$$

Lemma 2.1. *If* $x \ge |z|^{\frac{2}{1-\delta}}$ *we have*

$$|\phi'(x)| \sim x^{\frac{2k+1}{2}}.$$

The proof is straightforward and is left to the reader. Thus, due to Lemma 2.1 we have that:

$$\begin{aligned} |e^{-\psi}| &= |\frac{1}{G}| = |(\phi')^{1/2}||e^{\phi}| \lesssim x^{\frac{2k+1}{4}}|e^{\phi}| \\ |e^{\psi}| &= |G| \lesssim x^{-\frac{2k+1}{4}}|e^{-\phi}|, \end{aligned}$$

with suitable constants.

The function v can be then estimated as follows:

$$\begin{aligned} |v(s)| \le s^{\frac{2k+1}{4}}|e^{\phi}| \int_s^y t^{-\frac{2k+1}{4}}|e^{-\phi(t)}|[Ct^{-2}|G(t) \\ + \frac{1}{4}|z|^2t^{-1}|G(t)| + Ct^{-2}|u(t)| + \frac{1}{4}|z|^2t^{-1}|u(t)|]dt = \sum_1^4 I_j, \end{aligned} \tag{2.9}$$

and

$$
\begin{aligned}
I_1 &= Cs^{\frac{2k+1}{4}}|e^{\phi}(s)|\int_s^y t^{-\frac{2k+1}{4}}|e^{-\phi(t)}|Ct^{-2}|G(t)dt \\
&\leq Cs^{\frac{2k+1}{4}}|e^{\phi}(s)|\int_s^y t^{-\frac{2k+1}{4}}|e^{-\phi(t)}|t^{-\frac{2k+1}{4}-2}|e^{-\phi(t)}|dt \\
&\leq Cs^{\frac{2k+1}{4}}|e^{\phi}(s)|\int_s^y t^{-\frac{2k+1}{2}-2}|e^{-2\phi(t)}|dt \\
&\leq Cs^{\frac{2k+1}{4}}|e^{\phi}(s)|\int_s^y t^{-\frac{2k+1}{2}-2}\frac{1}{(-2\operatorname{Re}\phi)'}(-2\operatorname{Re}\phi(t))'e^{-2\operatorname{Re}\phi(t)}dt.
\end{aligned}
$$

As above $(\operatorname{Re}\phi)' \sim \operatorname{Re}\phi' \sim x^{\frac{2k+1}{2}}$, so that

$$
\begin{aligned}
I_1 &\leq Cs^{\frac{2k+1}{4}}|e^{\phi}(s)|\int_s^y t^{-(2k+1)-2}(-2\operatorname{Re}\phi(t))'e^{-2\operatorname{Re}\phi(t)}dt \\
&\leq Cs^{\frac{2k+1}{4}}|e^{\phi}(s)|s^{-(2k+1)-2}|e^{-2\phi}(s)| \\
&= s^{-\frac{3(2k+1)}{4}-2}|e^{-\phi}(s)|.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= C|z|^2s^{\frac{2k+1}{4}}|e^{\phi}(s)|\int_s^y t^{-\frac{2k+1}{4}}|e^{-\phi}(t)|t^{-1}|G(t)dt \\
&\leq C|z|^2s^{\frac{2k+1}{4}}|e^{\phi}(s)|\int_s^y t^{-\frac{2k+1}{2}-1}|e^{-2\phi}(t)|dt \\
&\leq C|z|^2s^{\frac{2k+1}{4}}|e^{\phi}(s)|\int_s^y t^{-\frac{2k+1}{2}-1}\frac{1}{(-2\operatorname{Re}\phi)'}(-2\operatorname{Re}\phi)'e^{-2\operatorname{Re}\phi(t)}dt \\
&\leq C|z|^2s^{\frac{2k+1}{4}}|e^{\phi}(s)|s^{-(2k+1)-1}|e^{-2\phi(s)}| \\
&= C|z|^2s^{-\frac{3(2k+1)}{4}-1}|e^{-\phi(s)}|.
\end{aligned}
$$

But $|z|^2 \leq s^{1-\delta}$, so that

$$I_2 \leq Cs^{-\frac{3(2k+1)}{4}-\delta}|e^{-\phi(s)}|.$$

Let us now estimate I_3:

$$
\begin{aligned}
I_3 &= Cs^{\frac{2k+1}{4}}|e^{\phi(s)}|\int_s^y t^{-\frac{2k+1}{4}}|e^{-\phi(t)}|t^{-2}|u(t)|dt \\
&= Cs^{\frac{2k+1}{4}}|e^{\phi(s)}|\int_s^y t^{-\frac{2k+1}{4}-2}|e^{-\phi(t)}u(t)|dt.
\end{aligned}
$$

Finally:

$$I_4 = Cs^{\frac{2k+1}{4}}|e^{\phi(s)}| \int_s^y |z|^2 t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}u(t)|dt.$$

As above $|z|^2 \le s^{1-\delta}$ so that:

$$I_4 \le Cs^{\frac{2k+1}{4}+1-\delta}|e^{\phi(s)}| \int_s^y t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}u(t)|dt.$$

Altogether:

$$\begin{aligned}\sum_1^4 I_j &\le Cs^{-\frac{3(2k+1)}{4}-\delta}|e^{-\phi(s)}| \\ &+ Cs^{\frac{2k+1}{4}}|e^{\phi(s)}| \int_s^y t^{-\frac{2k+1}{4}-2}|e^{-\phi(t)}u(t)|dt \\ &+ Cs^{\frac{2k+1}{4}+1-\delta}|e^{\phi(s)}| \int_s^y t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}u(t)|dt.\end{aligned}$$

Since $|z| \ge 1$ and hence $s \ge 1$ the third quantity dominates the second:

$$\begin{aligned}(2.10) \qquad |v(s)| \le \sum_1^4 I_j \le \; & C\Big\{ s^{-\frac{3(2k+1)}{4}-\delta}|e^{-\phi(s)}| \\ + \; & s^{\frac{2k+1}{4}+1-\delta}|e^{\phi(s)}| \int_s^y t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}u(t)|dt \Big\}.\end{aligned}$$

Let us estimate the function u for $A \le x \le y$:

$$\begin{aligned}|u(x)| &\le |e^{\psi(x)}| \int_x^y |e^{-\psi(s)}v(s)|ds \\ &\le x^{-\frac{2k+1}{4}}|e^{-\phi(x)}| \int_x^y s^{\frac{2k+1}{4}}|e^{\phi(s)}||v(s)|ds \\ &\le Cx^{-\frac{2k+1}{4}}|e^{-\phi(x)}| \int_x^y s^{\frac{2k+1}{4}-\frac{3(2k+1)}{4}-\delta}ds \\ &+ Cx^{-\frac{2k+1}{4}}|e^{-\phi(x)}| \int_x^y s^{\frac{2k+1}{4}+\frac{2k+1}{4}+1-\delta}|e^{2\phi(s)}|ds \\ &\cdot \int_s^y t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}u(t)|dtds \\ &= J_1 + J_2.\end{aligned}$$

$$\begin{aligned} J_1 &\le C|\frac{1}{(\phi')^{1/2}}e^{-\phi}|\int_x^y s^{-\frac{2k+1}{2}-\delta}ds \\ &\le Cx^{-\frac{2k+1}{2}+1-\delta}|G(x)|. \end{aligned}$$

$$\begin{aligned} J_2 &= Cx^{-\frac{2k+1}{4}}|e^{-\phi(x)}|\int_x^y ds\ s^{\frac{2k+1}{2}+1-\delta}|e^{2\phi(s)}|\cdot \\ &\cdot\int_s^y t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}u(t)| \\ &\le C|G(x)|\int_x^y dt\left(\int_x^y ds\ s^{\frac{2k+1}{2}+1-\delta}|e^{2\phi(s)}|\right) \\ &\cdot t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}|. \end{aligned}$$

Since $\phi'(s)\sim s^{\frac{2k+1}{2}}$ the inner integral can be estimated by

$$t^{1-\delta}|e^{2\phi(t)},$$

so that

$$J_2 \le C|G(x)|\int_x^y t^{-\frac{2k+1}{4}-1+1-\delta}|e^{\phi(t)}||u(t)|dt.$$

Hence, defining

$$B_\alpha = \sup_{[A,y]}\frac{|u(x)|}{x^{-\alpha}|G(x)|},$$

for a suitable $\alpha>0$, we have

$$\begin{aligned} |u(x)|\le J_1+J_2 &\le C\Bigg\{x^{-\frac{2k+1}{4}-\delta}|G(x)| \\ &\quad +|G(x)|\int_x^y t^{-\frac{2k+1}{4}-\delta}|e^{\phi(t)}||u(t)|dt\Bigg\} \\ &= C|G(x)|\Bigg\{x^{-\frac{2k+1}{2}+1-\delta} \\ &\quad +\int_x^y t^{-\frac{2k+1}{4}-\delta}\frac{|u(t)|}{t^{-\alpha}|\frac{1}{(\phi')^{1/2}}e^{-\phi(t)}|}t^{-\alpha}t^{-\frac{2k+1}{4}}dt\Bigg\} \end{aligned}$$

$$\leq \quad C|G(x)|\left\{x^{-\frac{2k+1}{2}+1-\delta} + \int_x^y t^{-\frac{2k+1}{2}-\alpha-\delta} B dt\right\}$$

$$\leq \quad C|G(x)|\left\{x^{-\frac{2k+1}{2}+1-\delta} + B_\alpha x^{-\frac{2k+1}{2}-\alpha-\delta+1}\right\}$$

$$= \quad C|G(x)|x^{-\frac{2k+1}{2}+1-\delta}[1 + B_\alpha x^{-\alpha}].$$

By choosing

$$\alpha = \frac{2k+1}{2} + \delta - 1 > 0$$

we obtained that $B_\alpha \leq C\frac{1}{1-cx^{-\alpha}}$. Thus we proved the following result:

Proposition 2.1. *Let* $\delta > -\dfrac{2k+1}{2} + 1$. *For* $|z|^{\frac{2}{1-\delta}} \leq x \leq y$ *we have the estimate:*

(2.11) $$|u_y(x)| \leq Cx^{-\alpha}|G(x)|,$$

with a constant C *independent of* z *and* y.

Define u_y to be such that:

$$\begin{cases} P_z u_y(x) = P_z G(x) \\ u_y(y) = u'_y(y) = 0 \end{cases}$$

As before define $\alpha = \dfrac{2k+1}{2} + \delta - 1 > 0$, $k = 1, 2, \ldots$.

For $A \leq x \leq y_1 \leq y_2$ we want to find an upper bound for

$$|u_{y_1}(x) - u_{y_2}(x)|$$

It has already been proved in Proposition 2.1 that

$$|u_{y_2}(y_1)| \leq Cy_1^{-\alpha}|G(y_1)|.$$

Furthermore, from the estimate (2.10) we have:

$$\begin{aligned}
|\tilde{D}u_{y_2}(s)| &\leq Cs^{-\frac{3(2k+1}{4}-\delta}|e^{-\phi(s)}| + Cs^{\frac{2k+1}{4}+1-\delta}|e^{\phi(s)}| \\
&\quad \cdot \int_s^{y_2} t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}u_{y_2}(t)|dt \\
&\leq Cs^{-\frac{2k+1}{2}-\delta}|G(s)| + Cs^{\frac{2k+1}{4}+1-\delta}|e^{\phi(s)}| \\
&\quad \cdot \int_s^{y_2} t^{-(2k+1)-\delta}|e^{-2\phi(t)}|dt \\
&\leq Cs^{-\frac{2k+1}{2}-\delta}|G(s)| + Cs^{-\frac{5(2k+1)}{4}+1-2\delta}|e^{-\phi(s)}| \\
&\leq Cs^{-\frac{2k+1}{2}-\delta}|G(s)|,
\end{aligned}$$

since $0 \geq -\dfrac{3(2k+1)}{4} + 1 - \delta$, for $k = 1, 2, \ldots$.

Set

$$w_1 = u_{y_1} - u_{y_2} \qquad v = \tilde{D}w_1.$$

Since $u_{y_1}(y_1) = u'_{y_1}(y_1) = 0$, we have:

$$\begin{aligned}
|w_1(y_1)| = |u_2(y_1)| &\leq Cy_1^{-\alpha}|G(y_1)| \\
&\leq Cy^{-\alpha-\frac{2k+1}{4}}|e^{-\phi(y_1)}| \\
|v(y_1)| = |\tilde{D}u_{y_2}(y_1)| &\leq Cy_1^{-\frac{2k+1}{2}-\delta}|G(y_1)| \\
&\leq y_1^{-\frac{3(2k+1)}{4}-\delta}|e^{-\phi(y_1)}|.
\end{aligned}$$

Now:

$$\begin{aligned}
e^{-\psi}\frac{d}{ds}e^{\psi}v &= Dv = D\tilde{D}w_1 \\
&= Pw_1 + Ew_1 + \frac{1}{4}z^2x^{-1}w_1 \\
&= Ew_1 + \frac{1}{4}z^2x^{-1}w_1,
\end{aligned}$$

so that, for $A \leq s \leq y_1$,

$$\begin{aligned}
|e^{\psi(s)}v(s)| &\leq |e^{\psi(y_1)}v(y_1)| + \int_s^{y_1} |e^{\psi(t)}(Ew_1 + \frac{1}{4}z^2t^{-1}w_1)|dt \\
&\leq Cy_1^{-(2k+1)-\delta}|e^{-2\phi(y_1)}| + \int_s^{y_1} t^{-\frac{2k+1}{4}-2}|e^{-\phi(t)}||w_1|dt \\
&\quad + \frac{1}{4}z^2 \int_s^{y_1} t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}||w_1|dt
\end{aligned}$$

On the other hand, since $-e^{\psi}d_x e^{-\psi}w_1 = v$, we have:

$$\begin{aligned} |e^{-\psi(x)}w_1(x)| &\leq |e^{-\psi(y_1)}w_1(y_1)| \\ &\quad + \int_x^{y_1} |e^{-\psi(s)}v(s)|ds \\ &= |e^{-\psi(y_1)}w_1(y_1)| + \int_x^{y_1} |e^{-2\psi(s)}||e^{\psi(s)}v(s)|ds. \end{aligned}$$

We estimate the two pieces of the above expression:

$$|e^{-\psi(y_1)}w_1(y_1)| \leq \frac{1}{|G(y_1)|}Cy_1^{-\alpha}|G(y_1)| = Cy_1^{-\alpha}.$$

$$\begin{aligned} \int_x^{y_1} |e^{-2\psi(s)}||e^{\psi(s)}v(s)|ds &\leq C\int_x^{y_1} |e^{-2\psi(s)}|y_1^{-(2k+1)-\delta}|e^{-2\phi(y_1)}|ds \\ &+ C\int_x^{y_1} |e^{-2\psi(s)}|\int_s^{y_1} t^{-\frac{2k+1}{4}-2}|e^{-\phi(t)}w_1(t)|dtds \\ &+ C|z|^2\int_x^{y_1} |e^{-2\psi(s)}|\int_s^{y_1} t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}w_1(t)|dtds = \sum_1^3 H_j. \end{aligned}$$

Let us consider the different terms H_j:

$$\begin{aligned} H_1 &\leq Cy_1^{-(2k+1)-\delta}|e^{-2\phi(y_1)}|\int_x^{y_1} s^{\frac{2k+1}{2}}|e^{2\phi(s)}|ds \\ &\leq Cy_1^{-(2k+1)-\delta}|e^{-2\phi(y_1)}||e^{2\phi(y_1)}| \\ &= Cy_1^{-(2k+1)-\delta}. \end{aligned}$$

$$\begin{aligned} H_2 &= C\int_x^{y_1} t^{-\frac{2k+1}{4}-2}|e^{-\phi(t)}w_1(t)|\int_x^t |e^{-2\psi(s)}|dsdt \\ &\leq C\int_x^{y_1} t^{-\frac{2k+1}{4}-2}|e^{\phi(t)}w_1(t)|dt, \end{aligned}$$

arguing as above for the inner integral.

$$\begin{aligned} H_3 &\leq C|z|^2C\int_x^{y_1} t^{-\frac{2k+1}{4}-1}|e^{-\phi(t)}w_1(t)|\int_x^t |e^{-2\psi(s)}|dsdt \\ &\leq C|z|^2\int_x^{y_1} t^{-\frac{2k+1}{4}-1}|e^{\phi(t)}w_1(t)|dt. \end{aligned}$$

Since $|z|^2 \leq x^{1-\delta} \leq t^{1-\delta}$ we see that $H_2 \leq H_1$. Thus, since $\alpha = \dfrac{2k+1}{2} - 1 + \delta$,

$$\begin{aligned} |e^{-\psi(x)} w_1(x)| &\leq C y_1^{-\alpha} + C \int_x^{y_1} t^{-\frac{2k+1}{4}-\delta} |e^{\phi(t)} w_1(t)| dt \\ &\leq C y_1^{-\alpha} + C \int_x^{y_1} t^{-\frac{2k+1}{2}-\delta} |e^{-\psi(t)} w_1(t)| dt. \end{aligned}$$

Set

$$B_1 = \sup_{A \leq x \leq y_1} \frac{|w_1(x)|}{|G(x)|}.$$

Then we have:

$$B_1 \leq C y_1^{-\alpha} + C \int_x^{y_1} B_1 t^{-\alpha-1} dt \leq C y_1^{-\alpha} + \frac{B_1}{2},$$

which implies

$$B_1 \leq C y_1^{-\alpha}.$$

We have thus proved the following

Proposition 2.2. *Let* $\delta > -\dfrac{2k+1}{2} + 1$. *Then for* $A(z) \leq x \leq y$ *we have the inequality*

$$|u_{y_1}(x) - u_{y_2}(x)| \leq C y_1^{-\alpha} |G(x)|, \tag{2.12}$$

where the positive constant C *is independent of* y_1, y_2 *and* z.

Using both Proposition 2.1 and Proposition 2.2 we now proceed to prove an upper bound for the exponential order of the Stokes coefficient.

Denote by u^+ a solution of $Pu^+ = 0$ subdominant e.g. in a sector containing the positive real axis. Then we know that

$$u^+(x,z) = x^{-\frac{2k+1}{4}} e^{-\phi_z(x)} (1 + \mathscr{O}(x^{-1})) \tag{2.13}$$

Proposition 2.3. *Let* $A(z) = C_0 |z|^{\frac{2}{1-\delta}}$, *where* $C_0 > 0$ *and* $k \geq 2$. *Then there exists a constant* $C \in]0,1[$ *and independent of* z, *such that*

$$|u^+(x,z) - x^{-\frac{2k+1}{4}} e^{-\phi_z(x)}| \leq C |x|^{-\frac{2k+1}{4}} |e^{-\phi_z(x)}|, \tag{2.14}$$

and

$$|{u^+}'(x,z) + x^{\frac{2k+1}{4}} e^{-\phi_z(x)}| \leq C |x|^{\frac{2k+1}{4}} |e^{-\phi_z(x)}| \tag{2.15}$$

for $x \geq A(z)$.

P r o o f. We prove only the second estimate, since the first is trivial. We write u instead of u^+ for the sake of simplicity. Let $w(x) = G(x) - u(x)$. Then $u' = (G - W)' = G' + \tilde{D}w - \psi' w$. Thus

$$u' + x^{\frac{2k+1}{4}} e^{-\phi(x)} = G' + x^{\frac{2k+1}{4}} e^{-\phi(x)} + \tilde{D}w - \psi' w. \tag{2.16}$$

It is straightforward to verify that

$$|G' + x^{\frac{2k+1}{4}} e^{-\phi(x)}| \leq \theta |x|^{\frac{2k+1}{4}} |e^{-\phi(x)}|.$$

Let us consider the other two terms of the espression above. Due to (2.10) proved above and Proposition 2.1, we have that

$$\begin{aligned} |\tilde{D}w(x)| &\leq C|x|^{-\frac{3}{4}(2k+1)-\delta}|e^{-\phi(x)}| \\ &\quad + C|x|^{\frac{1}{4}(2k+1)+1-\delta}|e^{\phi(x)}| \int_x^y t^{-\frac{1}{4}(2k+1)-1}|e^{-\phi(t)}u(t)|dt \\ &\leq C|x|^{-\frac{3}{4}(2k+1)-\delta}|e^{-\phi(x)}| \\ &\quad + C|x|^{\frac{1}{4}(2k+1)+1-\delta}|e^{\phi(x)}| \int_x^y t^{-\frac{3}{2}(2k+1)-\delta}|t^{\frac{1}{2}(2k+1)}e^{-2\phi(t)}|dt \\ &\leq C|x|^{-\frac{3}{4}(2k+1)-\delta}|e^{-\phi(x)}| + C|x|^{-\frac{5}{4}(2k+1)+1-2\delta}|e^{-\phi(x)}| \\ &\leq C|x|^{-\frac{3}{4}(2k+1)-\delta}|e^{-\phi(x)}|, \end{aligned}$$

because $\alpha > 0$. Now the latter quantity above is

$$|x|^{-(2k+1)-\delta}|x|^{\frac{1}{4}(2k+1)}$$

and the first factor is small if $k \geq 2$ for every δ for which $\alpha = \alpha(\delta) > 0$.

Let us consider the third term in (2.16): $\psi' w$. Because of Proposition 2.1, we may write

$$|\psi' w(x)| \leq C|x|^{-\alpha}|x|^{\frac{1}{4}(2k+1)}|e^{-\phi(x)}|,$$

and this ends the proof of the proposition. □

An analogous estimate can be derived for any subdominant solution in its Stokes sector (see next section for more details).

We may now state the main result of this section.

Theorem 2.1. *Let* $k \geq 2$. *Let* $A(z) = C_0|z|^{\frac{2}{1-\delta}}$ *be as above. Denote by* W *the Wronskian of* u_0 *and* u_k, *where* u_j *denotes the subdominant solution in the sector of amplitude* $2/(2k+3)\pi$ *and centered at the line of argument* $2j/(2k+3)\pi$.

Then

$$|W(z)| \leq Ce^{C_1|z|^{\frac{2k+3}{1-\delta}}}$$

for suitable positive constants C and C_1.

Proof. We argue on $u = u_0$. The same argument applied to the ather solution allows us to achieve the proof.

First notice that

$$\begin{cases} |u(A)| \leq CA^{-\frac{2k+1}{4}}|e^{-\phi(A)}| \\ |u'(A)| \leq CA^{\frac{2k+1}{4}}|e^{-\phi(A)}| \end{cases}$$

where $C > 0$ does not depend on z. Now define

$$E(x,z) = |z|^{2\frac{2k+1}{1-\delta}}|u(x)|^2 + |u'(x)|^2.$$

Due to the above properties we easily see that

$$\frac{d}{dx}E(x) \leq C|z|^{\frac{2k+1}{1-\delta}}E(x),$$

which implies that

$$E(0,z) \leq CE(A,z)e^{A|z|^{\frac{2k+1}{1-\delta}}}. \tag{2.17}$$

This allows us to conclude that

$$|W(z)| \leq Ce^{C_1|z|^{\frac{2k+3}{1-\delta}}}.$$

This ends the proof of the theorem. □

Since $\alpha > 0$ implies that $1 - \delta < \dfrac{2k+1}{2}$. Thus we may always assume that $W(z)$ is an entire function of order < 3.

3. Proof of the Main Result. We recall from [9] the following facts relative to equation (1.1):

(a) the complex plane can be divided into $2k+3$ sectors $\mathscr{S}_j$, $j = 0, \ldots, 2k+2$ of amplitude $\dfrac{2\pi}{2k+3}$: to each sector one can associate a subdominant solution $\mathscr{Y}_j(x;z)$of the equation (i.e. a solution exponentially decreasing at ∞).

(b) $\forall j = 0, \dots, 2k+3$ we have the Stokes connection formula:

$$\mathscr{Y}_j(x;z) = c_j(z)\mathscr{Y}_{j+1}(x;z) - \omega\mathscr{Y}_{j+2}(x;z), \tag{3.1}$$

where it is understood that $\mathscr{Y}_{2k+3} = \mathscr{Y}_0$ and $\mathscr{Y}_{2k+4} = \mathscr{Y}_1$.

(c) Here we put $c_j(z) = c_0(G^j(z))$, where

$$c_0(z) = \frac{W_{0,2}(z)}{W_{1,2}(z)}$$

and $G^j(z) = \omega^{-(k+1)j}z$, with $\omega = \exp\dfrac{2i\pi}{2k+3}$. Moreover we recall that $c_0(z)$ is not a constant function, since $c'_j(0) \neq 0$.

(d) In the above expression $W_{0,2}(z)$ denotes the Wronskian determinant of $\mathscr{Y}_0, \mathscr{Y}_2$ and $W_{0,1}(z)$ that of $\mathscr{Y}_0, \mathscr{Y}_1$. Furthermore it is easily checked that $W_{1,2}(z) = \omega^{\frac{2k-3}{4}}$. On the other hand $W_{0,2}(z)$ coincides modulo a constant with that estimated in Theorem 2.1.

(e) Define

$$S_j(z) = \begin{bmatrix} c_j(z) & 1 \\ -\omega & 0 \end{bmatrix} \quad j = 0, \dots, 2k+2,$$

then

$$S_{2k+2}(z)S_{2k+1}(z)\cdots S_1(z)S_0(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(f) The solution $\mathscr{Y}_0(x;z)$ has in the sector $\mathscr{S} = \left\{x \in \mathbb{C} | |\arg x| < \dfrac{3}{2k+5}\right\}$ the following asymptotic development:

$$\mathscr{Y}_0(x;z) \sim x^{-\frac{2k+1}{4}}(1 + \mathscr{O}(x^{-1/2}))e^{-E(x;z)},$$

where $E(x;z) = \frac{2}{2k+3}x^{\frac{2k+3}{2}} + zx^{\frac{1}{2}}$.

From (3.1) we see that:

$$\mathscr{Y}_0 = A_j(z)\mathscr{Y}_j - \omega A_{j-1}(z)\mathscr{Y}_{j+1},$$

where the $A_j(z)$ denote entire functions made up from the Stokes coeffiecients c_j, whose explicit expression can be given as follows:

$$A_l(z) = \sum_{j=0}^{[\frac{l}{2}]}(-\omega)^j \sum_{k_1,\dots,k_j=0}^{l-1} c_0c_1\cdots\hat{c}_{k_1}\hat{c}_{k_1+1}\cdots\hat{c}_{k_j}\hat{c}_{k_j+1}\cdots c_{l-2}c_{l-1}, \tag{3.2}$$

where by $\hat{c}_{k_j}$ we mean that the corresponding term has been omitted. (3.2) now yields in the special case of $k = 1, 2$ that:

$$A_1(z) = c_0(z), \tag{3.3}$$

and that:

$$A_2(z) = c_0(z)c_1(z) - \omega. \tag{3.4}$$

The aim of this section is to prove that $A_1(z)$ has at least one zero in $\mathbb{C}$ when $k = 1$ and that if $k = 2$ the same holds for $A_2(z)$.

3.1. The case $k = 1$. This case has been essentially already treated in [1] and for the sake of completeness we briefly recall the argument.

From the holonomy equation in (d) we obtain the functional equation for $A_1(z)$

$$A_1(z) + \omega^2 A_1(\omega z)A_1(\omega^4 z) - \omega^3 = 0, \quad \forall z \in \mathbb{C}.$$

Suppose that $A_1(z) \neq 0$ for all $z \in \mathbb{C}$. Then from the above equation that it follows that $A_1(z) \neq \omega^3$ for all $z \in \mathbb{C}$. Since $A_1(z)$ is an entire function Picard's Little Theorem implies that $A_1(z)$ would be constant because $A_1(z)$ avoids two distinct values 0 and ω^3. But this contradicts what has been stated in item (c) above.

3.2. The case $k = 2$. Here we prove that (3.4) has at least one zero in $\mathbb{C}$.

In fact if we assume that $A_2(z) \neq 0$ $\forall z \in \mathbb{C}$ we may conclude that $c_0(z)$ has in infinite number of zeros in $\mathbb{C}$ due to the Borel-Hadamard lemma, see e.g. [5]. Recalling that $c_0(0) \neq 0$ and the Hadamard representation theorem (see e.g. [10] Theorem 5.1) we have:

$$c_0(z) = e^{g(z)} \prod_1^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}(\frac{z}{a_n})^2}. \tag{3.2.1}$$

Here $g(z)$ is a polynomial with degree less or equal than 2 due to Theorem 2.1. On the other hand, because of our assumption

$$A_2(z) = e^{p(z)}$$

where $p(z)$ is another polynomial whose degree is less or equal than 2. Let us now study the equation $A_2(z) = e^{p(z)} = -\omega$. Taking the logarithm of both sides we get for the zeros:

$$p(z) = i\pi\frac{9}{7} + 2i\pi h, \quad h \in \mathbb{Z}$$

We argue in the case when p has degree 2. The case when the degree is 1 is similar but simpler. Let $p(z) = \alpha z^2 + \beta z + \gamma$, with $\alpha \neq 0$: from the preceding equation and the fact that, due to item c above, $c_1(z) = c_0(\omega^4 z)$, we obtain these two asymptotic represention for the complex numbers $a_h, h \in \mathbb{Z}$:

$$\begin{aligned} \alpha a_h^2 &\sim 2i\pi h'(h) \\ \alpha\omega^6 a_h^2 &\sim 2i\pi h''(h), \end{aligned} \tag{3.2.2}$$

where $h'(h), h''(h) \in \mathbb{Z}$ and $h'(h), h''(h) \to \infty$ if h tends to infinity. Since $\operatorname{Im}\omega^6 \neq 0$ it is clear from (3.2.2) that we have a contradiction. Thus we have proved the following Theorem:

Theorem 3.2.2. *The equation (1.1) has a bounded rapidly decreasing solution defined on the whole real line for suitable values of the complex number z.*

Proof. It is a straightforward consenquence of the previous argument and of the fact that, due to the P-T invariance properties of equation (1.1), it is always possible to choose the zero of $A_2(z)$ in a such a way that its imaginary part is strictly negative. (See e.g. [4]).

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