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AN ITERATIVE PROCEDURE FOR SOLVING NONSMOOTH GENERALIZED EQUATION

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Abstract. In this article, we study a general iterative procedure of the following form

$$0 \in f(x_k) + F(x_{k+1}),$$

where f is a function and F is a set valued map acting from a Banach space X to a linear normed space Y, for solving generalized equations in the nonsmooth framework.

We prove that this method is locally Q-linearly convergent to x^* a solution of the generalized equation

$$0 \in f(x) + F(x)$$

if the set-valued map

$$[f(x^*) + g(\cdot) - g(x^*) + F(\cdot)]^{-1}$$

is Aubin continuous at $(0, x^*)$, where $g : X \to Y$ is a function, whose Fréchet derivative is $L-$Lipschitz.

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1. Introduction. In this study we present an iterative procedure for solving nonsmooth generalized equations of the form:

$$\text{(1)} \qquad \text{find} \quad x \in X \quad \text{such that} \quad 0 \in f(x) + F(x),$$

where f is a fuction and F is a set-valued map acting from a Banach space X to the linear normed space Y.

The generalized equations were introduced by S. M. Robinson in the 1970's as a general tool for describing, analyzing, and solving different problems in a unified manner, for a survey of earlier results see [12]. For example, when $F = \{0\}$, (1) is an equation; when F is the positive orthant in $\mathbf{R}^n$, (1) is a system of inequalities; when F is the normal cone to a convex and closed set in X, (1) represent variational inequalities. For other examples, the reader could refer to [2].

To solve (1), in [2] and [3] A. L. Dontchev introduced a Newton type sequence of the form

$$\text{(2)} \qquad 0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1}), \qquad k = 0, 1, \dots,$$

where $\nabla f(x_k)$ is the Fréchet derivative of f at the point x_k. He also proved the stability of the method (2), the Kantorovich-type theorem and the convergence of an approximate Newton-type method. The main tool used for obtaining a Q-quadratically convergence to a solution x^* of the generalized equation (1) is the Aubin continuity of $(f+F)^{-1}$ and the Lipschitz property of the Fréchet derivative ∇f.

A. Pietrus, in [9], extended this study to the function f whose Fréchet derivative satisfies the Hölder condition, he showed that the convergence is superlinear and proved, in [10], the stability of the method (2) in this mild differentiability context. When the function f do not possesses Fréchet derivative one cannot use the classical approximations, based on Taylor expansion. To overcome this difficulty, M. H. Geoffroy and A. Pietrus, in [7], introduced an extension of the concept of point-based approximation, introduced by S. Robinson [13], so called (n, α)-point-based approximation and established local convergence theorem. More precisely, M. H. Geoffroy and A. Pietrus considered the following method

$$0 \in A(x_k, x_{k+1}) + F(x_{k+1})$$

where $A : X \times X \to Y$ is a (n, α)-point-based approximation for f. Although this method does not require any smoothness property on f several existing methods are subsumed within this relation when f is smooth; for example it recovers a Newton-type method (2).

In this paper we present a more different approach. We study the local convergence of the method

$$0 \in f(x_k) + F(x_{k+1}) \tag{3}$$

under some mild conditions for the function f and the set-valued map F. We do not suppose that the function f possesses Fréchet derivative or any kind of approximation; in this study f is only a Lipschitz function.

This paper is organized as follows: in Section 2, we recall a few preliminary results, in Section 3, we give an extension of the Basic Majorant theorem which has been proved in [11] and in Section 4, we prove that the method (3) is locally convergent.

Throughout this paper all the norms are denoted by $\|\cdot\|$. The distance from a point $x \in X$ to the set $a \subset X$ is $\operatorname{dist}(x, A) = \inf\{\|x - y\| : y \in A\}$. The inverse F^{-1} of the map F is defined as $F^{-1} = \{x \in X : y \in F(x)\}$ and $\operatorname{graph} F$ is the set $\{(x, y) \in X \times Y : y \in F(x)\}$. We denote by $B_a(x)$ the closed ball centered at x with radius a.

2. Preliminaries.

In this section, we collect some definitions and results that we shall need to prove our results. We employ the following concept introduced by Aubin [1].

Definition 2.1. *A set-valued map $\Gamma : Y \to 2^X$ is said to be $M-$ pseudo-Lipschitz arround $(y_0, x_0) \in \operatorname{graph} F$ if there exist neighborhoods V of y_0 and U of x_0 such that*

$$\sup_{x \,\in\, \Gamma(y_1) \,\cap\, U} \operatorname{dist}(x, \Gamma(y_2)) \leq M\|y_1 - y_2\|$$

for every y_1, $y_2 \in V$.

Equivalently, Γ is $M-$pseudo-Lipschitz arround $(y_0, x_0) \in \operatorname{graph} \Gamma$ if there exist positive constants a and b such that for every y_1, $y_2 \in B_b(y_0)$ and for every $x_1 \in \Gamma(y_1) \cap B_a(x_0)$ there exists $x_2 \in \Gamma(y_2)$ such that

$$\|x_1 - x_2\| \leq M\|y_1 - y_2\|.$$

Let A and C be two subsets of X, if we denote by $e(C, A)$ the excess from the set A to the set C (semi-distance of Hausdorff)

$$e(C, A) = \sup\{\operatorname{dist}(x, A) : x \in C\}$$

then we have an equivalent definition of $M-$pseudo-Lipschitz set-valued map:

$$e(\Gamma(y_1) \cap B_a(x_0), \Gamma(y_2)) \leq M\|y_1 - y_2\|$$

for all y_1, $y_2 \in B_b(y_0)$.

In [5], the above property is called the Aubin continuity and the maps satisfying this property are called Aubin continuous. In [4], Dontchev and Hager use the above property to establish an inverse mapping theorem for set-valued maps, moreover they prove the following fixed point statement.

Lemma 2.2. *Let (X, ρ) be a complete metric space, let Φ map X to the closed subsets of X, let $\eta_0 \in X$, r and λ be such that $0 \leq \lambda < 1$ and*
(a) $\mathrm{dist}(\eta_0, \Phi(\eta_0)) < r(1-\lambda)$
(b) $e(\Phi(x_1) \cap B_r(\eta_0), \Phi(x_2)) \leq \lambda\rho(x_1, x_2)$ *for all* x_1, $x_2 \in B_r(\eta_0)$.

Then Φ has a fixed point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \Phi(x)$. If Φ is single-valued, then x is the unique fixed point of Φ in $B_r(\eta_0)$.

The previous lemma is a generalization of a fixed point theorem in Ioffe-Tikhomirov [6] where in (b) the excess e is replaced by the Hausdorff distance.

3. Basic Majorant Theorem.

In this section we give a generalization of Rheinboldt's Basic Majorant Theorem [11]. The main tool is the majorizing sequence, due to Kantorovich and Akilov [8].

Definition 3.1. *Let $\{x_k\}$ be a sequence in the metric space (X, ρ). A real non-negative sequence $\{t_k\}$ is called a majorizing sequence for $\{x_k\}$ if*

$$\rho(x_{k+1}, x_k) \leq t_{k+1} - t_k, \quad k = 0, 1, \ldots$$

Note that any majorizing sequence $\{t_k\}$ for $\{x_k\}$ is necessarily nondecreasing.

Lemma 3.2. *Let (X, ρ) be metric space, let $\{t_k\}$, $t_k \in \mathbf{R}$, $t_k \geq 0$ is a majorizing sequence for $\{x_k\}$, $x_k \in X$. Then:*
(i) *For $m > k \geq 0$*

$$\rho(x_m, x_k) \leq t_m - t_k. \tag{4}$$

(ii) *If* $\lim_{k\to\infty} t_k = t^* < +\infty$ *exists, then* $\{x_k\}$ *is a Cauchy sequence in* X. *Moreover, if* X *is complete,* $\lim_{k\to\infty} x_k$ *also exists and*

$$\rho(x^*, x_k) \le t^* - t_k, \quad k = 0, 1, \dots \tag{5}$$

Proof. For $m > k \ge 0$ we have

$$\rho(x_m, x_k) \le \sum_{j=k}^{m-1} \rho(x_{j+1}, x_j) \le \sum_{j=k}^{m-1} (t_{j+1} - t_j) = t_m - t_k.$$

Hence, if $\lim_{k\to\infty} t_k = t^* < +\infty$ exists, then $\{x_k\}$ is a Cauchy sequence in X, and, therefore, if X is complete, $\lim_{k\to\infty} x_k = x^*$ also exists and for $m \to \infty$ we obtain the error estimate (5). This completes the proof. □

The following class of functions shall be used.

Definition 3.3. *A function* $\varphi : Q \subset \mathbf{R}^3 \to \mathbf{R}$ *is said to be of class* $\Gamma^3(Q)$ *if it has the following properties:*

(a) *The domain* Q *is a hypercube* $Q = J_1 \times J_2 \times J_3$, *where each* J_i *is an interval of the form* $[0, a]$, $[0, a)$ *or* $[0, \infty)$.

(b) φ *is non-negative and isotone on* Q, *i.e., if* $(u_1^{(i)}, u_2^{(i)}, u_3^{(i)}) \in Q$, $i = 1, 2$ *and* $u_j^{(1)} \le u_j^{(2)}$, $j = 1, 2, 3$, *then*

$$0 \le \varphi(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}) \le \varphi(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}).$$

(c) φ *is a strictly increasing function in the first argument, i.e., if* $(u_1^{(i)}, u_2, u_3) \in Q$, $i = 1, 2$ *and* $u_1^{(1)} < u_1^{(2)}$, *then*

$$\varphi(u_1^{(1)}, u_2, u_3) < \varphi(u_1^{(2)}, u_2, u_3).$$

Let $\varphi \in \Gamma^3(Q)$, $Q = J_1 \times J_2 \times J_3$. Consider a difference equation of the form

$$t_{k+1} - t_k = \varphi(t_k - t_{k-1},\, t_k,\, t_{k-1}) \tag{6}$$

for given t_0 and t_1. Then the solution $\{t_k\}$ of the difference equation (6) is said to exist for given t_0, t_1, if

$$t_{k+1} - t_k \in J_1, \qquad t_k \in J_2 \cap J_3$$

for all $k \geq 0$.

Using this notation Rheinboldt proved, in [11], a general convergence theorem for the process

$$x_{k+1} = G(x_k), \qquad k = 0, 1, \ldots,$$

where $G : D \subset X \to X$ is an operator on the complete metric space (X, ρ), $\varphi \in \Gamma^3(Q)$, $x_0 \in D$ and

$$\rho(G(G(x)), G(x)) \leq \varphi(\rho(G(x), x), \rho(G(x), x_0), \rho(x, x_0)),$$

whenever $x, G(x) \in D$. Using these assumptions, the convergence of the iterative process $x_{k+1} = G(x_k)$ in X is deduced from the convergence of the majorizing sequence $\{t_k\}$, where $t_0 = 0$, $t_1 = \rho(G(x_0), x_0)$.

Here we prove a generalization of Rheinboldt's theorem for set-valued maps.

Theorem 3.4. *Let (X, ρ) be a complete metric space, let F maps X to the closed subsets of X, let $r > 0$, $\varphi \in \Gamma^3(Q)$ and $x_0 \in X$ be such that*

(a) *for $t_0 = 0$, $t_1 > \operatorname{dist}(F(x_0), x_0)$ the solution $\{t_k\}$ of the difference equation* (6) *exists and $\lim_{k\to\infty} t_k = t^* \leq r$.*

(b) *$e(F(x) \cap B_r(x_0), F(y)) \leq \varphi(\rho(x, y), \rho(y, x_0), \rho(x, x_0))$ whenever $x, y \in B_r(x_0)$.*

(c) *$\lim_{t\uparrow t^*} \varphi(t^* - t, t^*, t) = 0$.*

Then there exists a sequence $\{x_k\}$ such that $x_{k+1} \in F(x_k)$, $k = 0, 1, 2, \ldots$ and the solution of (6) *majorizes $\{x_k\}$. Moreover, $\lim_{k\to\infty} x_k = x^*$ exists, x^* is a fixed point of F in $B_{t^*}(x_0)$, that is, $x^* \in F(x^*) \cap B_{t^*}(x_0)$, and the following estimate holds*

$$\rho(x^*, x_n) \leq t^* - t_n.$$

P r o o f. The proof follows by induction. Since $\operatorname{dist}(F(x_0), x_0) < t_1$, there exists $x_1 \in F(x_0)$, such that

$$\rho(x_1, x_0) < t_1 - t_0 = t_1.$$

It is obvious, that the sequence t_k is nondecreasing, and $t_k \leq t^*$, $k = 0, 1, \ldots$ Hence

$$\rho(x_1, x_0) < t_1 \leq t^* \leq r,$$

i.e., $x_1 \in B_r(x_0)$.

Proceeding by induction, let us assume that there are points $x_1, x_2, \ldots, x_n \in B_r(x_0)$ such that $x_k \in F(x_{k-1})$ and

$$\rho(x_k, x_{k-1}) < t_k - t_{k-1}$$

for $k = 1, 2, \ldots, n$. Then, using condition (a) and (b), we have

$$\begin{aligned} e(F(x_{n-1}) \cap B_r(x_0), F(x_n)) &\le \varphi(\rho(x_n, x_{n-1}), \rho(x_n, x_0), \rho(x_{n-1}, x_0)) \\ &< \varphi(t_n - t_{n-1}, t_n, t_{n-1}) = t_{n+1} - t_n. \end{aligned}$$

This implies that there exists $x_{n+1} \in F(x_n)$ such that

$$\rho(x_{n+1}, x_n) < t_{n+1} - t_n.$$

Further,

$$\rho(x_{n+1}, x_0) \le \sum_{j=0}^{n} \rho(x_{j+1}, x_j) < \sum_{j=0}^{n} (t_{j+1} - t_j) = t_{n+1} \le t^* \le r.$$

Thus, $\{x_n\}$ is majorized by $\{t_n\}$ and, using Lemma 3.2, $\lim_{t\to\infty} x_n = x^*$ also exists and

$$\rho(x^*, x_n) \le t^* - t_n, \qquad n = 0, 1, \ldots$$

Taking $n = 0$ in the last inequality, we have

$$\rho(x^*, x_0) \le t^* \le r$$

or, equivalently, $x^* \in B_{t^*}(x_0)$. By assumption (b)

$$\begin{aligned} \operatorname{dist}(x_n, F(x^*)) &\le e(F(x_{n-1}) \cap B_r(x_0), F(x^*)) \\ &\le \varphi(\rho(x^*, x_{n-1}), \rho(x^*, x_0), \rho(x_{n-1}, x_0)) \\ &\le \varphi(t^* - t_{n-1}, t^*, t_{n-1}). \end{aligned}$$

Hence, by assumption (c)

$$\lim_{n\to\infty} \operatorname{dist}(x_n, F(x^*)) = 0.$$

The triangle inequality implies that

$$\begin{aligned} \operatorname{dist}(x^*, F(x^*)) &\le \rho(x^*, x_n) + \operatorname{dist}(x_n, F(x^*) \\ &\le \rho(x^*, x_n) + \varphi(t^* - t_{n-1}, t^*, t_{n-1}). \end{aligned}$$

which approaches zero as n increases. Since $F(x^*)$ is closed, we conclude that $x^* \in F(x^*)$, i.e., x^* is a fixed point of F in $B_{t^*}(x_0)$. This completes the proof. □

Now, we show that Lemma 2.2 can be derived from Theorem 3.4 as a corollary. Taking

$$\varphi(\rho(y,x),\rho(y,x_0),\rho(x,x_0)) = \lambda\rho(y,x),$$

where $0 \le \lambda < 1$, the difference equation (6) becomes

$$\lambda(t_k - t_{k-1}) = t_{k+1} - t_k. \tag{7}$$

By setting $t_0 = 0$ and $t_1 = r(1-\lambda)$, and using (7), we have

$$t_2 - t_1 = \lambda(t_1 - t_0) = \lambda t_1 = \lambda r(1-\lambda).$$

Hence,

$$t_2 = t_1 + \lambda r(1-\lambda) = r(1-\lambda^2).$$

It is readily seen, by induction, that

$$t_n = r(1-\lambda^n).$$

Now, since $0 \le \lambda < 1$, $\lim_{n\to\infty} t_n = r$ and the condition (a) of Theorem 3.4 is satisfied. The conditions (b) and (c) are obvious.

In the case when

$$\varphi(u-v,u,v) = h(u) - h(v)$$

we call h a first integral of the difference equation (6). Then we can state the following proposition:

Proposition 3.5. *Let $h : [0,\infty) \subset \mathbf{R} \to \mathbf{R}$ be continuous and nondecreasing function, $h(0) > 0$, let t^* is the smallest positive fixed point of h, let $\varphi \in \Gamma^3(Q)$, where $Q = J \times J \times J$, $J = [0,t^*]$ and*

$$\varphi(u-v,u,v) = h(u) - h(v) \tag{8}$$

for all $0 \le v < u$. Then the sequence

$$t_0 = 0,\ t_{n+1} = h(t_n) \tag{9}$$

for $n = 0,1,\dots,$ is the solution of the difference equation (6).

P r o o f. For $n \leq 1$, using (8), we have

$$\varphi(t_n - t_{n-1}, t_n, t_{n-1}) = h(t_n) - h(t_{n-1}) = t_{n+1} - t_n.$$

Hence $\{t_k\}$ is a solution of the difference equation (6).

Now we prove the convergence of the solution $\{t_k\}$. We know that $t_0 = 0 < t^*$. Since h is nondecreasing $h(0) \leq h(t^*)$, i.e. $t_1 \leq t^*$. Proceeding by induction we obtain $t_n \leq t^*$. Therefore, $\{t_k\}$ is convergent and if $\lim_{n\to\infty} t_n = \bar{t} \leq t^*$, by letting $n \to \infty$ in (9) we obtain, by the continuity of h, $h(\bar{t}) = \bar{t}$, and, since t^* is the smallest positive fixed point of h we have $\bar{t} = t^*$, i.e. $\lim_{n\to\infty} t_n = t^*$. This completes the proof. $\square$

Now, from Theorem 3.4 and Proposition 3.5 it is readily obtain the following proposition:

Proposition 3.6. *Let (X, ρ) be a complete metric space, let F maps X to the closed subsets of X, let $r > 0$, $\varphi \in \Gamma^3(Q)$, where $Q = J \times J \times J$, $J = [0, r]$ and $x_0 \in X$ be such that*

(a) *$e(F(x) \cap B_r(x_0), F(y)) \leq \varphi(\rho(x, y), \rho(y, x_0), \rho(x, x_0))$ whenever x, $y \in B_r(x_0)$.*

(b) *there exists a continuous nondecreasing function $h : [0, \infty) \subset \mathbf{R} \to \mathbf{R}$ such that $h(0) > \operatorname{dist}(F(x_0), x_0)$,*

$$\varphi(u - v, u, v) = h(u) - h(v)$$

for all $0 \leq v < u$, and let $t^* \leq r$ is the smallest positive fixed point of h.

Then there exists a sequence $\{x_k\}$ such that $x_{k+1} \in F(x_k)$, $k = 0, 1, 2, \dots$ and the solution of (6) $t_0 = 0$, $t_{n+1} = h(t_n)$, $n = 0, 1, \dots$ majorizes $\{x_k\}$. Moreover, $\lim_{k\to\infty} x_k = x^*$ exists, x^* is a fixed point of F in $B_{t^*}(x_0)$, that is, $x^* \in F(x^*) \cap B_{t^*}(x_0)$, and the folloing estimate holds

$$\rho(x^*, x_n) \leq t^* - t_n.$$

We have already seen that Lemma 2.2 can be derived from Theorem 3.4 as a corollary. In this case the majorizing sequence is $t_n = r(1 - \lambda^n)$. Define $h(t) = \lambda t + (1 - \lambda)r$. Then it is obvious that $h(t_0) = h(0) = (1 - \lambda)r = t_1$. Since $\varphi(u - v, u, v) = \lambda(u - v) = h(u) - h(v)$ and $h(r) = r$, all conditions of Proposition 3.6 are fulfilled. Hence Proposision 3.6 contains as special case Lemma 2.2.

4. Local convergence analysis of method (3). As we mentioned it in the introduction, our purpose is to present an iterative procedure for solving the following nonsmooth generalized equation (1):

$$0 \in f(x) + F(x).$$

From now on, we make the following assumptions (we recall that x^* denotes a solution of (1)):

(H1) $f : X \to Y$ is Lipschitz with a constant K in an open neighborhood Ω of x^*.

(H2) $F : X \to 2^Y$ is a set-valued map with closed graph.

(H3) There exists a Fréchet differentiable in Ω function $g : X \to Y$ such that

$$(f(x^*) + g(\cdot) - g(x^*) + F(\cdot))^{-1}$$

be Aubin continuous at $(0, x^*)$ with a constant M for growth.

(H4) The Fréchet derivative ∇g of g is Lipschitz continuous with a constant L.

(H5) $\|\nabla g(x^*)\| \leq p < 1/M$.

(H6) The constants M, K and p are such that $\dfrac{2MK}{1 - Mp} < 1$.

Then, we can state our main result which reads as follows:

Theorem 4.1. *Let x^* be a solution of* (1) *and suppose that the assumption* (H1)–(H6) *are satisfied. Then for every c such that $\dfrac{2MK}{1 - Mp} < c < 1$ one can find $\delta > 0$ such that for every starting point $x_0 \in B_\delta(x^*)$, there exists a sequence $\{x_k\}$ for* (1)*, defined by* (3)*, which satisfies*

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| \tag{10}$$

that is, the sequence $\{x_k\}$ is Q-linearly convergent to x^*.

Before proving Theorem 4.1, we need to introduce some notations. First, define the set-valued map from X into the subsets of Y by

$$P(x) = f(x^*) + g(x) - g(x^*) + F(x)$$

and the map Φ_0 for x_0 fixed in X by

$$x \to \Phi_0(x) = P^{-1}(f(x^*) + g(x) - g(x^*) - f(x_0))$$

from X to the closed subsets of X.

One can note that $x_1 \in X$ is a fixed point of Φ_0 if and only if

$$f(x^*) + g(x_1) - g(x^*) - f(x_0) \in P(x_1),$$

or, equivalently,

$$0 \in f(x_0) + F(x_1),$$

i.e., x_1 is a solution of the equation (3).

Once x_k is computed, we prove that the map

$$x \to \Phi_k(x) = P^{-1}(f(x^*) + g(x) - g(x^*) - f(x_k))$$

has a fixed point x_{k+1}. This process allows us to show the existence of a sequence $\{x_k\}$ satisfying (3).

Now, we state a result, which is the starting point of our algorithm. It is an efficient tool to prove Theorem 4.1 and reads as follows:

Proposition 4.2. *Under the assumptions of Theorem* 4.1 *there exists* $\delta > 0$ *such that for all* $x_0 \neq x^*$ *and* $x_0 \in B_\delta(x^*)$, *the map*

$$\Phi_0(x) = P^{-1}(f(x^*) + g(x) - g(x^*) - f(x_0))$$

has a fixed point x_1 *in* $B_\delta(x^*)$ *satisfying* $\|x_1 - x^*\| \leq c\|x^* - x_0\|$.

Proof of Proposition 4.2. By hypothesis (H3), there exist positive numbers a and b such that

$$e(P^{-1}(y') \cap B_a(x^*), P^{-1}(y'')) \leq M\|y' - y''\|, \tag{11}$$

whenever y', $y'' \in B_b(0)$. Fix c such that $2MK/(1 - Mp) < c < 1$. Choose $\delta > 0$ such that $B_\delta(x^*) \subset \Omega$, $\|f(x^*) - f(x)\| \leq b/2$, $\|g(x^*) - g(x)\| \leq b/2$ for all $x \in B_\delta(x^*)$ and

$$\delta < \min\left(a; \frac{1 - Mp}{CML}\right). \tag{12}$$

According to the definition of the excess e we have

$$\operatorname{dist}(x^*, \Phi_0(x^*)) \leq e(P^{-1}(0) \cap B_\delta(x^*), P^{-1}(f(x^*) - f(x_0))).$$

Moreover, for all $x_0 \in B_\delta(x^*)$ we have $\|f(x^*) - f(x_0)\| \leq b/2$ which implies that $f(x^*) - f(x_0) \in B_b(0)$.

Then from (11) one has

$$\mathrm{dist}(x^*, \Phi_0(x^*)) \le M\|f(x^*) - f(x_0)\| \le MK\|x^* - x_0\| < c(1 - Mp)\|x^* - x_0\|/2$$

Denote $\alpha = c(1 - Mp)\|x^* - x_0\|/2$. For any $x \in B_\delta(x^*)$ we have

$$\|f(x^*) + g(x) - g(x^*) - f(x_0)\| \le \|f(x^*) - f(x_0)\| + \|g(x) - g(x^*)\| \le b.$$

which implies that $f(x^*) + g(x) - g(x^*) - f(x_0) \in B_b(0)$.

By setting $r_0 := c\|x^* - x_0\| < \delta$, for any x', $x'' \in B_\delta(x^*)$ we obtain

$$\begin{aligned}
&e(\Phi_0(x') \cap B_{r_0}(x^*), \Phi_0(x''))\\
&\quad \le e(P^{-1}(f(x^*) + g(x') - g(x^*) - f(x_0)) \cap B_\delta(x^*),\\
&\qquad P^{-1}(f(x^*) + g(x'') - g(x^*) - f(x_0)))\\
&\quad \le M\|g(x') - g(x'')\|\\
&\quad \le M(\|g(x'') - g(x') - \nabla g(x')(x'' - x')\|\\
&\qquad + \|(\nabla g(x') - \nabla g(x^*))(x'' - x')\| + \|\nabla g(x^*)(x'' - x')\|)\\
&\quad \le M(L\|x'' - x'\|^2/2 + L\|x' - x^*\|.\|x'' - x'\| + p\|x'' - x'\|).
\end{aligned}$$

Let

$$\varphi(w, u, v) = M(Lw^2/2 + Lvw + pw).$$

Then it is easy to show that $\varphi \in \Gamma^3(Q)$, $Q = J \times J \times J$, $J = [0, \delta]$. Denote $L_1 = ML$, $p_1 = Mp$. It is readily seen that

$$\varphi(u - v, u, v) = \psi(u) - \psi(v),$$

where $\psi(t) = L_1t^2/2 + p_1t + \alpha$ and ψ is evidently nondecreasing and has a fixed point

$$t^* = \frac{1 - \sqrt{1 - 2h}}{h} \cdot \frac{\alpha}{1 - p_1},$$

where,using (12),

$$h = \frac{L_1\alpha}{(1 - p_1)^2} \le \frac{1}{2}.$$

Since $c < 1$ we have

$$t^* = \frac{2}{1 + \sqrt{1 - 2h}} \cdot \frac{\alpha}{1 - p_1} \le \frac{2\alpha}{1 - p_1} = c\|x^* - x_0\| < \delta.$$

Since all conditions of Proposision 3.6 are fulfilled, we can deduce the existence of a fixed point $x_1 \in B_\delta(x^*)$ such that

$$\|x_1 - x^*\| < c\|x^* - x_0\|.$$

Then the proof of Proposition 4.2 is complete. □

Proof of Theorem 4.1. We have $x_1 \in B_{r_0}(x^*)$. That is

$$\|x_1 - x^*\| \le r_0 = c\|x^* - x_0\|.$$

Proceeding by induction, keeping x^* and setting $r_k = c\|x^* - x_k\|$, the application of Proposition 4.2 to the map Φ_k gives the existence of a fixed point x_{k+1} for Φ_k in $B_{r_k}(x^*)$, which implies (10).

That completes the proof of the Theorem 4.1. □

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