Mathematica
Balkanica

New Series Vol. 27, 2013, Fasc. 1–2

Another Generalization of Arhangel'skii's Theorem

Nurettin Ergun

Presented by Petăr Kenderov

A. Arhangel'skii has proved his famous inequality $|X| \leq 2^\kappa$ where $\kappa = \chi(X) \cdot L(X)$ for any T_2 space X in 1969. Afterwards the result and proving technique have been seriously improved. We give in this note another improvement of this inequality in the class of T_1 spaces. We utilize a proving technique, defined independently by R. Pol and B. Sapirovskii in 1974 (see [4] and [6]) which is frequently and efficiently used by R. Hodel [3] recently.

MSC 2010: 54A25, 54A35.
Key Words: tightness, local pseudo-character number, Lindelöf number, sequential space.

Introduction

It is well known that in 1922, P. S. Alexandroff has formulated one of the interesting question on cardinal invariants: Does every first countable compact T_2 space have cardinality at most $\mathbf{c} = 2^{\aleph_0}$? This problem has been solved by Alexander Arhangel'skii in 1969, see [1]. He has proved the famous Arhangel'skii inequality, which states that $|X| \leq 2^{\chi(X) \cdot L(X)}$ for any T_2 space X. This inequality and its proving technique have been improved and simplified afterwards, see for example two surveys of R. Hodel [2],[3]. Two well known generalizations for instance, is written in the sequel as Corollary 1 and Corollary 2. We give in this note another generalization of Arhangel'skii Theorem in T_1 spaces.

In this paper $t(X)$ and $L(X)$ denotes the tightness and Lindelöf number of the topological space X respectively . $\psi(x, X)$ on the other hand denotes the local pseudo-character number at the point $x \in X$. One must remember that the last number is meaningful in T_1 spaces; for its definition see [2] for instance. The pseudo-character $\psi(X)$ of any T_1 space X on the other hand is defined as

$\psi(X)=\sup_{x\in X}\psi(x,X)+\omega_0$ and satisfy $\psi(X)\leq o(X)\leq 2^{|X|}$ whenever X is infinite. In 1974 it has been proved by V.Shapirovskii that $|X|\leq 2^{t(X)\cdot\psi(X)\cdot L(X)}$ holds for any T_2 space X. Note that this is an elegant generalization of the Arhangel'skii's inequality since it is known that $t(X)\cdot\psi(X)\leq\chi(X)$ and therefore $2^{t(X)\psi(X)L(X)}\leq 2^{L(X)\chi(X)}$ holds in any T_1 space.

As is well known, for any cardinal number κ we write κ^+ as the well defined cardinal number $\min\{\lambda\in\mathbf{Card}:\kappa<\lambda\}$ which certainly exists and κ is nothing but the well ordered set $W(\kappa)=\{\alpha\in\mathbf{Ord}:\alpha<\kappa\}$. Finally for any set A we write $\mathcal{P}_{\leqslant\kappa}(A)$ instead of $[A]^{\leqslant\kappa}$ i.e. $\mathcal{P}_{\leqslant\kappa}(A)=\{E\subseteq A:|E|\leqslant\kappa\}$. κ satisfies $\kappa\geqslant\aleph_0$ in this paper. Y^X denotes as always the set of the whole functions $f:X\to Y$.

The main result

Theorem. Let X be a T_1 space having the following conditions

1) $t(X)\leqslant\kappa$ and $L(X)\leqslant\kappa$,
2) $\psi(x,X)\leqslant 2^\kappa$ for each $x\in X$,
3) $|\overline{A}|\leqslant 2^\kappa$ for any subset $A\subseteq X$ satisfying $|A|\leqslant\kappa$.

Then we necessarily have $|X|\leqslant 2^\kappa$.

P r o o f. We may suppose that $\kappa<|X|$, since otherwise there is nothing to prove. Notice first that, there is a family of open subsets $\mathcal{G}_x=\{G_{x,\gamma}:\gamma<2^\kappa\}$ for each $x\in X$ such that $\{x\}=\bigcap\mathcal{G}_x=\bigcap_{\gamma<2^\kappa}G_{x,\gamma}$ by the condition 2). Let us construct via transfinite induction the family $\{A_\alpha:\alpha<\kappa^+\}$ of subsets of X with the following properties:

i) $A_\beta\subseteq A_\alpha$ if $\beta<\alpha<\kappa^+$,

ii) $1\leqslant|A_\alpha|\leqslant 2^\kappa$ for any $\alpha<\kappa^+$,

iii) $\overline{A}\subseteq A_\alpha$ for any $A\in\mathcal{P}_{\leqslant\kappa}(\bigcup\limits_{\beta<\alpha}A_\beta)$,

iv) Let $A\in\mathcal{P}_{\leqslant\kappa}(\bigcup\limits_{\beta<\alpha}A_\beta)$. If $\gamma_x<2^\kappa$ is determined for each $x\in A$

and if furthermore $X\neq\bigcup\limits_{x\in A}G_{x,\gamma_x}$ then $A_\alpha-\bigcup\limits_{x\in A}G_{x,\gamma_x}\neq\emptyset$.

In fact let us take as A_0 any non empty subset of X with $|A_0|\leqslant 2^\kappa$. Take any $\alpha<\kappa^+$. If all the $A_\beta\subseteq X$ sets, $\beta<\alpha$, satisfying all the required conditions have already been defined at the earlier steps than define $A_\alpha \subseteq X$ as in the following:

$$A_\alpha = \bigcup\{\overline{A} : A \in \mathcal{P}_{\leqslant\kappa}(\bigcup_{\beta<\alpha} A_\beta)\} \cup \{c(X - \bigcup_{x\in A} G_{x,\gamma_x}) : A \in \mathcal{P}_{\leqslant\kappa}(\bigcup_{\beta<\alpha} A_\beta) \text{ and } X \neq \bigcup_{x\in A} G_{x,\gamma_x}\}.$$

In here c denotes the choice function defined for all nonempty subsets of X, i.e. if $A \neq \emptyset$ then the uniquely determined point $c(A)$ satisfy $c(A) \in A$. Since $\mathrm{card}(\mathcal{P}_{\leqslant\kappa}(\bigcup_{\beta<\alpha} A_\beta)) \leqslant 2^\kappa$ and $\mathrm{card}((W(\kappa^+))^A) \leqslant 2^\kappa$ we easily have $1 \leqslant |A_\alpha| \leqslant 2^\kappa$. A_α evidently satisfy conditions $iii)$ and $iv)$. Notice furthermore we have for any $x \in \bigcup_{\beta<\alpha} A_\beta$

$$\{x\} \subseteq \overline{\{x\}} \subseteq \bigcup\{\overline{A} : A \in \mathcal{P}_{\leqslant\aleph_0}(\bigcup_{\beta<\alpha} A_\beta)\} \subseteq \bigcup\{\overline{A} : A \in \mathcal{P}_{\leqslant\kappa}(\bigcup_{\beta<\alpha} A_\beta)\} \subseteq A_\alpha\ .$$

Hence we get $\bigcup_{\beta<\alpha} A_\beta \subseteq A_\alpha$, i.e. the condition $i)$ is obtained. Therefore the transfinite induction process has been achieved. Define now the closed subset

$$K \ = \bigcup_{\alpha<\kappa^+} A_\alpha \quad \subseteq X$$

Since $t(X) \leqslant \kappa$ it is indeed easy to see that $\overline{K} \subseteq K$ and $|K| \leq 2^\kappa$. Suppose that for a moment $K \neq X$. Then there exists a point $x_0 \in X - K$. Notice that for any covering $\mathcal{U}$ of K where each $U \in \mathcal{U}$ is an open set in X we have a subcovering $\mathcal{U}^* \subseteq \mathcal{U}$ satisfying $K \subseteq \bigcup\mathcal{U}^*$ and $|\mathcal{U}^*| \leqslant \kappa$. Furthermore for any $\alpha < \kappa^+$ and any $x \in A_\alpha(\subseteq K)$ one can define an ordinal $\gamma_x < 2^\kappa$ such that $x_0 \notin G_{x,\gamma_x}$, hence as we have just observed there is a subfamily $\mathcal{G}^* \subseteq \{G_{x,\gamma_x} : x \in K\}$ such that $|\mathcal{G}^*| \leqslant \kappa$ and $K \subseteq \bigcup\mathcal{G}^*$. Define now $E_0 = \{x \in K : G_{x,\gamma_x} \in \mathcal{G}^*\} \subseteq K$ and notice that $|E_0| \leqslant \kappa$. Thus there exists an ordinal $\alpha_0 < \kappa^+$ such that $E_0 \subseteq \bigcup_{\beta<\alpha_0} A_\beta$. Since $x_0 \in X - \bigcup\mathcal{G}^* = X - \bigcup_{x\in E_0} G_{x,\gamma_x} \neq \emptyset$, we necessarily would have the following contradiction:

$$c(X - \bigcup_{x\in E_0} G_{x,\gamma_x}) \in A_{\alpha_0} - \bigcup_{x\in E_0} G_{x,\gamma_x} \subseteq K - \bigcup_{x\in E_0} G_{x,\gamma_x} = K - \bigcup\mathcal{G}^* = \emptyset$$

Hence, we should necessarily have $X = K$ and therefore $|X| = |K| \leqslant 2^\kappa$.

Corollary 1 (Arhangel'skii [1]). Let X be a sequential Lindelöf T_2 space with $\psi(X) \leqslant 2^{\aleph_0}$. Then $|X| \leqslant 2^{\aleph_0}$.

P r o o f: Every sequential space have countable tightness and in any sequential T_2 space we have $|\overline{A}| \leqslant 2^{\aleph_0}$ for any countable subset $A \subseteq X$.

Corollary 2 (Sapirovskii [5]). Every Lindelöf T_2 space X having countable tightness and countable pseudo-character, satisfy $|X| \leqslant 2^{\aleph_0}$.

P r o o f. Let X be a Lindelöf T_2 space having all the properties mentioned in the statement. For any closed $K \subseteq X$ we have $t(K) \leq \aleph_0$ because of $cl_K E = \overline{E} \cap K = \overline{E}$ for any $E \subseteq K$ and $t(X) \leq \aleph_0$. Thus we get $|K| \leq 2^{t(K)\psi(K)L(K)} \leq 2^{\aleph_0}$ by the well known Sapirovskii's inequality hence $|\overline{A}| \leq 2^{\aleph_0}$ for any $A \subseteq X$, so the main theorem is used for $\kappa = \aleph_0$.

Acknowledgement

We sincerely grateful to Prof. R. Hodel from Duke University for his intimate interest and approval of the paper.

References

[1] A. A r h a n g e l' s k i i. On the cardinality of bicompacta satisfying the first axiom of countability. // *Soviet Math. Dokl.*, **10**, 1969, 951–955.

[2] R. H o d e l. Cardinal Functions I. // K. Kunen& J.E. Vaughan, (Eds.), *The Handbook of Set-Theoretical Topology*, North Holland, 1984, 1–61.

[3] R. H o d e l. Arhangel'skii's solution to Alexandrooff's problem: a survey. // *Topology and its Applications*, **153**, 2006, 2199–2217.

[4] R. P o l. Short proofs of two theorems on cardinality of topological spaces. // *Bull. Acad. Pol. Sci.*, **22**, 1974, 1245–1249.

[5] B. S a p i r o v s k i i. On discrete subspaces of topological spaces; weight, tightness and Suslin number. // *Soviet Math. Dokl.*, **13**, 1972, 215–219.

[6] B. S a p i r o v s k i i. Canonical sets and character. Density and weight in compact spaces. // *Soviet Math. Dokl.*, **15**, 1974, 1282–1287.

Department of Mathematics
Faculty of Science and Letters
Marmara University
Üst Göztepe, Kadıköy Istanbul, TURKEY
e-mail: nergun@marmara.edu.tr

Received: Juni 6, 2013