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Extention of Apolarity and Grace Theorem

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The classical notion of apolarity is defined for two algebraic polynomials of equal degree. The main property of two apolar polynomials p and q is the classical Grace theorem: Every circular domain containing all zeros of p contains at least one zero of q and vice versa. In this paper, the definition of apolarity is extended to polynomials of different degree and an extension of the Grace theorem is proved. This leads to simplification of the conditions of several well-known results about apolarity.

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1. Preliminaries

Let $\mathcal{C}$ denote the complex plane and $\mathcal{C}^* := \mathcal{C} \cup \{\infty\}$ its one-point compactification. The set of all polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n), \quad (1.1)$$

with $a_n \neq 0$ is denoted by $\mathcal{P}_n$. The set of zeros of $p \in \mathcal{P}_n$, each counted with its multiplicity, is denoted by $Z(p) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. The closure of $\mathcal{P}_n$ with respect to taking limits of the coefficients is

$$\bar{\mathcal{P}}_n = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_n.$$

The (real) degree of $p \in \bar{\mathcal{P}}_n$ is the highest power of z with non-zero coefficient. If $p \in \bar{\mathcal{P}}_n$ has degree $n - s$ for some $s \in \{0, 1, \ldots, n\}$, we say that p has a zero at ∞ with multiplicity s. Thus, the set $Z(p)$ of the zeros of a polynomial $p \in \bar{\mathcal{P}}_n$ of degree $n - s$ contains the symbol ∞ repeated s times. Thus, every polynomial

$p \in \bar{\mathcal{P}}_n$ has n zeros, counting their multiplicity. We formally write $p(\infty) = 0$ if $p \in \mathcal{P}_{n-s}$ for some $s = 1, 2, \ldots, n$ and $p(\infty) \neq 0$ if $p \in \mathcal{P}_n$.

A (non-degenerate) Möbius transformation

$$T(z) = \frac{az+b}{cz+d}; \quad a, b, c, d \in \mathcal{C}, \quad ad - bc \neq 0$$

maps the extended complex plane $\mathcal{C}^*$ one-to-one on itself, and

$$T(-d/c) = \infty,\ T(\infty) = \frac{a}{c},\ T^{-1}(z) = \frac{dz-b}{-cz+a},\ T^{-1}(a/c) = \infty,\ T^{-1}(\infty) = -\frac{d}{c}.$$

Every Möbius transformation defines an operator on $\bar{\mathcal{P}}_n$ in the following way:

$$T[p](z) := (cz+d)^n p\left(\frac{az+b}{cz+d}\right), \quad p \in \bar{\mathcal{P}}_n. \tag{1.2}$$

The operator $T[p]$ maps the set $\bar{\mathcal{P}}_n$ onto itself. More precisely, we have the following lemma.

Lemma 1.1. *Let $p \in \bar{\mathcal{P}}_n$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the zeros of p, counting multiplicities and zeros at ∞. Then, the zeros of $T[p]$ are*

$$\{T^{-1}(\alpha_1), T^{-1}(\alpha_2), \ldots, T^{-1}(\alpha_n)\}.$$

Proof. Let $p \in \mathcal{P}_{n-s}$, for some $s \in \{0, 1, ..., n-1\}$ and let $\{\alpha_1, \alpha_2, ..., \alpha_{n-s}\}$ be the finite roots of p. We have the following cases:

1) If $p(a/c) \neq 0$, then $T[p] \in \mathcal{P}_n$ and

$$Z(T[p]) = \left\{T^{-1}(\alpha_1), T^{-1}(\alpha_2), \ldots, T^{-1}(\alpha_{n-s}), -\frac{d}{c}, -\frac{d}{c}, \ldots, -\frac{d}{c}\right\}, \tag{1.3}$$

where the zero $-d/c$ has multiplicity s. Informally, for a polynomial of degree $n - s$, with $p(a/c) \neq 0$, the operator T "brings" s zeros from ∞ to $-d/c$, which conforms with the fact $T^{-1}(\infty) = -d/c$. In this case, if $c = 0$ then ∞ is not a zero of p and we must have $s = 0$.

2a) Suppose a/c is a zero of p of multiplicity l and $c \neq 0$. Let α_1, α_2, $\ldots$, α_{n-s-l} be the finite zeros of p different from a/c. Then $T[p] \in \mathcal{P}_{n-l}$ and

$$Z(T[p]) = \left\{T^{-1}(\alpha_1), T^{-1}(\alpha_2), ..., T^{-1}(\alpha_{n-s-l}), -\frac{d}{c}, -\frac{d}{c}, ..., -\frac{d}{c}, \infty, \infty, ..., \infty\right\}, \tag{1.4}$$

where the zero $-d/c$ has multiplicity s and the zero ∞ has multiplicity l. Informally, the operator T "brings" s zeros from ∞ to $-d/c$ and "sends" a/c

to ∞ with multiplicity l. This conforms with the fact $T^{-1}(\infty) = -d/c$ and $T^{-1}(a/c) = \infty$.

2b) If a/c is a zero of p of multiplicity l and $c = 0$, then we must have $l = s$ and

$$Z(T[p]) = \Big\{T^{-1}(\alpha_1), T^{-1}(\alpha_2), \ldots, T^{-1}(\alpha_{n-s}), \infty, \infty, \ldots, \infty\Big\}, \tag{1.5}$$

where the zero ∞ has multiplicity s. That is, $T[p] \in \mathcal{P}_{n-s}$. Informally, the operator T does not "bring" or "send" zeros to ∞. ■

For any $a \in \mathcal{C}$ and $p \in \mathcal{P}_n$, $n \geq 1$, define the polynomial $\mathcal{D}_a(p; z) \in \mathcal{P}_{n-1}$ by

$$\mathcal{D}_a(p; z) = np(z) - (z - a)p'(z). \tag{1.6}$$

The linear operator (1.6) is called *polar derivative* with pole a, see [1, p. 97]. It is clear that

$$\lim_{a\to\infty} \frac{\mathcal{D}_a(p; z)}{a} = p'(z),$$

and we define

$$\mathcal{D}_\infty(p; z) := p'(z).$$

It is important to see how the operator T transforms the zeros of the derivative of p (i.e. the critical points), when $p \in \mathcal{P}_n$. Naturally, we consider the polynomials $\mathcal{D}_a(p; z)$ and $\mathcal{D}_\infty(p; z)$ to be members of $\bar{\mathcal{P}}_{n-1}$.

Lemma 1.2. *Let $p \in \bar{\mathcal{P}}_n$. The critical points of $T[p]$ are*

$$\{T^{-1}(\beta_1), T^{-1}(\beta_2), \ldots, T^{-1}(\beta_{n-1})\},$$

where $\{\beta_1, \beta_2, \ldots, \beta_{n-1}\}$ are the zeros of $\mathcal{D}_{a/c}(p; z)$, counting multiplicities and zeros at infinity.

Proof. Let $p \in \mathcal{P}_{n-s}$, for some $s \in \{0, 1, \ldots, n-1\}$, implying that $\mathcal{D}_{a/c}(p; z) \in \mathcal{P}_{n-s-1}$. Let $\beta_1, \beta_2, \ldots, \beta_{n-s-1}$ be the finite roots of $\mathcal{D}_{a/c}(p; z)$. We consider two cases:

1) If $c \neq 0$, then from (1.2), we calculate

$$\begin{aligned} T[p]'(z) &= nc(cz + d)^{n-1}p(T(z)) + (cz + d)^n p'(T(z))T'(z) \\ &= c(cz + d)^{n-1}\Big(np(T(z)) + \tfrac{1}{c}\tfrac{ad-bc}{cz+d}p'(T(z))\Big) \\ &= (cz + d)^{n-1}\Big(np(T(z)) - (T(z) - a/c)p'(T(z))\Big) \\ &= (cz + d)^{n-1}\mathcal{D}_{a/c}(p; T(z)) \\ &= (cz + d)^s \textstyle\prod_{k=1}^{n-s-1}(z - T^{-1}(\beta_k)). \end{aligned}$$

Thus, the critical points of $T[p]$ are

$$Z(T[p]') = \left\{T^{-1}(\beta_1), T^{-1}(\beta_2), \ldots, T^{-1}(\beta_{n-s-1}), -\frac{d}{c}, -\frac{d}{c}, \ldots, -\frac{d}{c}\right\},$$

where the zero $-d/c$ has multiplicity s. Since $T^{-1}(\infty) = -d/c$, we are done.

2) If $c = 0$, then both a and d are non-zero and observing that

$$T[p]'(z) = ad^{n-1}p'(T(z)) = ad^{n-1}\mathcal{D}_\infty(p; T(z)),$$

we are done. ∎

We may consider the ordinary derivative as a polar derivative with pole ∞. When we are interested in the critical points of the polynomial $T[p]$, we have to get the polar derivative of p with pole $T(\infty) = a/c$.

2. Apolarity

The notion of *apolarity*, see [1, p. 102], is defined as a symmetric relation between two polynomials of the same degree.

Definition 2.1. The polynomials

$$p(z) = \sum_{k=0}^{n} a_k z^k \in \mathcal{P}_n \quad \text{and} \quad q(z) = \sum_{k=0}^{n} b_k z^k \in \mathcal{P}_n$$

are called *apolar*, if

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_{n-k} b_k = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k p^{(n-k)}(0) q^{(k)}(0) = 0. \tag{2.1}$$

The basic relation between the zeros of two apolar polynomials is the classical theorem of Grace, see [1, p. 107]. A circular domain, open or closed, is a proper subset of $\mathcal{C}$, bounded by a circle, or a line.

Theorem 2.2. **(Grace)** *Let p and q be apolar. Then every circular domain containing all the zeros of one of them contains at least one zero of the other.*

It is natural to extend the notion apolarity to polynomials of arbitrary degree.

Definition 2.3. We say that $p, q \in \bar{\mathcal{P}}_n$ are *n-apolar*, or just *apolar*, if

$$\sum_{k=0}^{n} (-1)^k p^{(n-k)}(0) q^{(k)}(0) = 0. \tag{2.2}$$

Such a definition of apolarity does not require the degree of the polynomials to be fixed or the same.

For example, let $p(z) = a_1 z + a_0$ and $q(z) = b_1 z + b_0$. If we consider p and q to be in $\mathcal{P}_1$, then they are 1-apolar if $a_1 b_0 - a_0 b_1 = 0$. If we consider p and q to be in $\mathcal{P}_2$, then they are 2-apolar if $a_1 b_1 = 0$. Finally, they are n-apolar for every $n \geq 3$.

As another example, consider $p(z) = a_2 z^2 + a_1 z + a_0$ and $q(z) = b_1 z + b_0$. If we consider p and q to be in $\mathcal{P}_2$, then they are 2-apolar if $a_2 b_0 - a_1 b_1/2 = 0$. If we consider p and q to be in $\mathcal{P}_3$, then they are 3-apolar if $a_2 b_1 = 0$. Finally, they are n-apolar for every $n \geq 4$.

Statement 2.1. *Let $s \in \{0, 1, \ldots, n-1\}$. If $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_{n-s}$ are n-apolar, then $p^{(l)}$ and q are $(n-l)$-apolar for $l = 0, 1, \ldots, s$.*

Proof. As p and q are n-apolar and $q \in \mathcal{P}_{n-s}$, we have

$$0 = \sum_{k=0}^{n} (-1)^k p^{(n-k)}(0) q^{(k)}(0) = \sum_{k=0}^{n-l} (-1)^k \frac{d^{n-l-k}}{dz^{n-l-k}} p^{(l)}(z)\Big|_{z=0} q^{(k)}(0)$$

for any $l = 1, 2, \ldots, s$. ■

Lemma 2.4. *If $p, q \in \bar{\mathcal{P}}_n$ are apolar, then so are $T[p]$ and $T[q]$.*

The statement of Lemma 2.4 is analogous to the one given in [1, Remark 3.3.4, p. 103]. Only there, it is required that both p and q be of degree n, as well as that both $T[p]$ and $T[p]$ be of degree n. The justification of Lemma 2.4 is similar: Every Möbius transformation T, with $ad - bc \neq 0$, is a composition of transformations of the type $1/z$ and $az + b$ for $a \neq 0$. It is not difficult to see that Lemma 2.4 holds for these two types of transformations.

We formulate an extension of the Grace theorem for two arbitrary polynomials.

Theorem 2.5. *Let p and q in $\bar{\mathcal{P}}_n$ be apolar. Then every circular domain containing all the zeros of one of them contains at least one zero of the other.*

Proof I. If p and q are both of degree n, then the theorem coincides with the classical theorem of Grace. If p and q are both of degree strictly less than n, then both polynomials have ∞ as a zero. Hence, the theorem is true. The more interesting case is when the degree of p is $m < n$, while the degree of q is n. Let $p(z) = \sum_{k=0}^{m} a_k z^k$ and consider the polynomial

$$p_\varepsilon(z) = \varepsilon z^n + \sum_{k=0}^{m} a_k(\varepsilon) z^k = \varepsilon(z - \zeta_1(\varepsilon))(z - \zeta_2(\varepsilon)) \cdots (z - \zeta_n(\varepsilon)).$$

It is possible to choose the coefficients $\{a_k(\varepsilon) : k = 0, \ldots, m\}$ so that $a_k(\varepsilon)$ approaches a_k as ε approaches 0 and so that p_ε is apolar to q. Then, ε approaches

0, m of the zeros $\{\zeta_1(\varepsilon), \zeta_2(\varepsilon), \dots, \zeta_n(\varepsilon)\}$ approach the finite zeros of p, and the rest $n-m$ escape to ∞. The classical Grace theorem is valid for the pair p_ε, q and by continuity it remains valid for the pair p, q.

Proof II. Similarly to the first proof, the interesting case is when the degree of p is $m < n$, while the degree of q is n. The rest follows immediately from the original Grace theorem and Lemma 2.4 after choosing the Möbius transformation T to be such that T^{-1} sends the roots of p and ∞ into $\mathcal{C}$ and referring to Lemma 1.1. ■

Observe that if the degree of q is less than n, then every circular domain, containing all the zeros of q is a half plane or the exterior of a open disk. A simple corollary, which is a priori obvious from the Gauss-Lucas theorem, is the following. Let $p(z) = (z-z_1)(z-z_2)\cdots(z-z_n) = z^n - (z_1+z_2+\cdots+z_n)z^{n-1} + \cdots + a_0$; $n \geq 2$ and $q(z) = z - b$ be apolar to p. According to Definition 2.3, we have $b = (z_1 + z_2 + \cdots + z_n)/n$. The extended Grace theorem says that every half plane having b on its boundary contains at least one of the zeros of p. This is a priori trivial, as b is the center of gravity of the zeros of p.

2.1. Properties of extended apolarity. The apolarity relation, see [1, p. 102-104], has several important properties. In some of them, it is necessary to state that the two polynomials are of the same degree. We will rephrase these properties for the extended definition of apolarity and see that in some instanced the underlying conditions are simplified. Definition 2.3 immediately implies that: (i) Every polynomial of odd degree is apolar to it self. (ii) If two polynomials q_1 and q_2 are both apolar to p and $\lambda_1, \lambda_2 \in \mathcal{C}$, then the polynomial $q = \lambda_1 q_1 + \lambda_2 q_2$ is also apolar to p.

For polynomials $p, q \in \bar{\mathcal{P}}_n$ given by $p(z) = \sum_{k=0}^n a_k z^k$ and $q(z) = \sum_{k=0}^n b_k z^k$, introduce the functional

$$\Lambda_p[q] := n! \sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_{n-k} b_k = \sum_{k=0}^{n} (-1)^k p^{(n-k)}(0) q^{(k)}(0)$$

defined in $\bar{\mathcal{P}}_n$ for every fixed $p \in \bar{\mathcal{P}}_n$. The following lemma is proved in [1, p. 103] for $p, q \in \mathcal{P}_n$.

Lemma 2.6. *Let* $p, q \in \bar{\mathcal{P}}_n$ *and let* $\deg(p) = \lambda$, $\deg(q) = \mu$. *Define*

$$P(z) = p(az+b), \quad G(z) = g(az+b), \quad a \neq 0.$$

Then, $\Lambda_p[q] = a^{-m} \Lambda_P[Q]$, *where* $m := \max\{\lambda, \mu\}$.

Proof. It is easy to see that

$$\Lambda_P[Q] = a^m \sum_{k=0}^{n} (-1)^k p^{(n-k)}(b) q^{(k)}(b) \tag{2.3}$$

and

$$\frac{\partial \Lambda_P[Q]}{\partial b} = a^m \Big(p^{(n+1)}(b)q(b) + (-1)^n p(b)q^{(n+1)}(b) \Big) = 0.$$

The last equality implies that the sum on the right-hand side of (2.3) does not depend on b. Letting $b = 0$ completes the proof. ∎

The proof of Lemma 2.6 shows that for any $b \in \mathcal{C}$ we have

$$\Lambda_p[q] = \sum_{k=0}^{n} (-1)^k p^{(n-k)}(b) q^{(k)}(b) \tag{2.4}$$

The following lemma, see [1, p. 104], suggests that apolarity is connected with common zeros.

Lemma 2.7. *Let $p, q \in \bar{\mathcal{P}}_n$ have a common zero z_1. Suppose the sum of the multiplicities of z_1 as a zero of p and q, respectively, is larger than n. Then, $\Lambda_p[q] = 0$ and so q is apolar to p.*

Proof. Let l and m be the multiplicities of z_1 as a zero of p and q, respectively. By assumption, we have $l + m > n$.

If $z_1 = \infty$, then $p^{(k)}(0) = 0$ for all $k = n - l + 1, \ldots, n$ and $q^{(n-k)}(0) = 0$ for all $k = 0, \ldots, m - 1$. Since $n - l + 1 \leq (m - 1) + 1$, we get $\Lambda_p[q] = 0$.

If $z_1 \in \mathcal{C}$, then using (2.4), we get

$$\Lambda_p[q] = \sum_{k=0}^{n} (-1)^k p^{(k)}(z_1) q^{(n-k)}(z_1),$$

and the proof concludes as in the first case. ∎

The following theorem describes the set of all polynomials $q(z) \in \bar{\mathcal{P}}_n$ which are apolar to a given polynomial $p(z) \in \mathcal{P}_n$ in terms of its zeros. The proof follows the proof of the analogous theorem in [1, p. 104], keeping in mind that $q(z) \in \bar{\mathcal{P}}_n$. Note that while the result in [1, p. 104] requires that the constants $c_{j,l}$ be non-zero, this restriction is now relaxed, thanks to the extended notion of apolarity.

Theorem 2.8. *Let $p(z) = a_n \prod_{j=1}^{k} (z - z_j)^{m_j} \in \mathcal{P}_n$, where $z_1, z_2, \ldots, z_k$ be distinct, and $\{m_j\}_{j=1}^{k}$ be natural numbers with $\sum_{j=1}^{k} m_j = n$. Then, the polynomials*

$$q(z) = \sum_{j=1}^{k} \sum_{l=0}^{m_l - 1} c_{j,l} (z - z_j)^{n-l}, \tag{2.5}$$

where $c_{j,l} \in \mathcal{C}$ for $j = 1, 2, \ldots, k,\ l = 0, 1, \ldots, m_j - 1$, constitute the set of all polynomials $q \in \bar{\mathcal{P}}_n$ apolar to p.

P r o o f. Each polynomial

$$q_j(z) = \sum_{l=0}^{m_l-1} c_{j,l}(z - z_j)^{n-l}, \quad j = 1, 2, \dots, k, \tag{2.6}$$

has z_j as a zero of multiplicity at least $n - m_j + 1$. Hence, Lemma 2.7 implies that $\Lambda_p[q_j] = 0$. Thus, $\Lambda_p[q] = \sum_{j=1}^{k} \Lambda_p[q_j] = 0$ showing that every polynomial of the form (2.5) is apolar to p. It remains to show that there are no other polynomials apolar to p.

The functional $\Lambda_p[q]$ is not identically zero, since for the polynomial

$$h(z) := \sum_{k=0}^{n} \frac{(-1)^k}{k!} p^{(n-k)}(0) z^k,$$

we have

$$h^{(k)}(0) = (-1)^k p^{(n-k)}(0) \quad \text{and} \quad \Lambda_p[h] = \sum_{k=0}^{n} |p^{(n-k)}(0)|^2 > 0.$$

By a standard result from linear algebra, the kernel of $\Lambda_p[q]$, that is, the set of all polynomials $q \in \bar{\mathcal{P}}_n$ satisfying $\Lambda_p[q] = 0$, is a linear subspace V_n of dimension n. A polynomial q, apolar to p, must belong to V_n. By Lemma 2.7, the polynomials

$$q_{j,l}(z) := (z - z_j)^{n-l}, \quad j = 1, 2, \dots, k, \; l = 0, 1, \dots, m_j - 1$$

do all belong to V_n. Since there is total of n polynomials $q_{j,l}(z)$, it is enough to show that they are linearly independent. Assume the contrary that

$$\sum_{j=1}^{k} \sum_{l=0}^{m_j-1} c_{j,l} q_{j,l}(z) \equiv 0$$

with some of the coefficients $c_{j,l}$ being non-zero. Writing the left-hand side as $\sum_{\nu=0}^{n} L_\nu z^\nu$, we must have $L_n = L_{n-1} = \cdots = L_0 = 0$. Since

$$L_\nu = \sum_{j=1}^{k} \sum_{l=0}^{m_j-1} \binom{n-l}{\nu} (-z_j)^{n-l-\nu} c_{j,l}, \quad \nu = 0, 1, \dots, n,$$

where $\binom{n-l}{\nu} = 0$ when $\nu > n - l$, it follows that the system

$$\sum_{j=1}^{k} \sum_{l=0}^{m_j-1} \binom{n-l}{\nu} (-z_j)^{n-l-\nu} u_{j,l} = 0, \quad \nu = n, n-1, \dots, 1, 0$$

of n equation in the n unknowns $u_{j,l}$ has a non-trivial solution. Hence, the determinant Δ of this system must vanish. Its entries are

$$\begin{array}{cccc} 1 & 0 & \dots & 0 \\ -\binom{n}{1} z_j & 1 & \dots & 0 \\ \binom{n}{2} z_j^2 & -\binom{n-1}{1} z_j & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n-1}\binom{n}{n-1} z_j^{n-1} & (-1)^{n-2}\binom{n-1}{n-2} z_j^{n-2} & \dots & (-1)^{n-m_j}\binom{n-m_j+1}{n-m_j} z_j^{n-m_j}, \end{array}$$

where $j = 1, 2, \dots, k$. From $\Delta = 0$ follows that a non-trivial linear combination of the rows is zero. Hence, there exists a polynomial

$$f(z) = \sum_{\nu=0}^{n-1} (-1)^{\nu} \binom{n}{\nu} \lambda_{\nu} z^{\nu},$$

which is not identically zero and such that

$$(-1)^l \frac{n!}{(n-l)!} f^{(l)}(z_j) = 0, \quad j = 1, 2, \dots, k, \ \ l = 0, 1, \dots, m_l - 1.$$

Thus, $f(z)$ would have a total of at least $\sum_{j=1}^{k} m_j = n$ zeros, which is impossible since the degree of $f(z)$ is at most $n-1$. ∎

The next lemma gives a connection between apolarity and polar derivatives.

Lemma 2.9. *Let $p \in \mathcal{P}_n$, $q \in \bar{\mathcal{P}}_n$, and let z^* be a zero of $p(z) = \sum_{k=0}^{n} a_k z^k$. If p and q are apolar, then*

$$p_1(z) = \begin{cases} p(z)/(z - z^*) & \text{if } z^* \in \mathcal{C}, \\ \sum_{k=0}^{n-1} a_k z^k & \text{if } z^* = \infty, \end{cases}$$

is apolar to the polar derivative $\mathcal{D}_{z^*}(q; z)$.

Proof. We have $p_1 \in \mathcal{P}_{n-1}$ and $\mathcal{D}_{z^*}(q; z) \in \bar{\mathcal{P}}_{n-1}$. If $z^* = \infty$, then we have $b_n = 0$ and

$$0 = \sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_{n-k} b_k = \frac{1}{n} \sum_{k=1}^{n} \frac{(-1)^k}{\binom{n-1}{k-1}} a_{n-k} (k b_k).$$

This shows that $\Lambda_{q'}[p_1] = 0$, or that p_1 and $\mathcal{D}_{\infty}(q; z)$ are apolar.

Suppose now, that $z^* \in \mathcal{C}$. Let

$$p(z) = a_n \prod_{j=1}^{k} (z - z_j)^{m_j}, \quad \sum_{j=1}^{k} m_j = n, \ a_n \neq 0,$$

where $z_1, z_2, \ldots, z_k$ are distinct numbers. Then, by Theorem 2.8, $q(z)$ must be of the form $q(z) = \sum_{j=1}^{k} q_j(z)$, where

$$q_j(z) := \sum_{l=0}^{m_j-1} c_{j,l}(z - z_j)^{n-l}, \quad j = 1, 2, \ldots, k.$$

Hence, we may write

$$\mathcal{D}_{z^*}(q; z) = n \sum_{j=1}^{k} q_j(z) - (z - z^*) \sum_{j=1}^{k} q_j'(z) = \sum_{j=1}^{k} \mathcal{D}_{z^*}(q_j; z).$$

Clearly, $\mathcal{D}_{z^*}(q_j; z)$ is a polynomial of degree at most $n - 1$ in z, with zero z_j having multiplicity at least $n - m_j + 1$ when $z_j = z^*$ and at least $n - m_j$ when $z_j \neq z^*$. Therefore, $\mathcal{D}_{z^*}(q_j; z)$ is of the form

$$\mathcal{D}_{z^*}(q_j; z) = \sum_{\nu=0}^{n_j-1} \gamma_{j,\nu}(z - z_j)^{n-1-\nu}, \text{ where } n_j := \begin{cases} m_j & \text{if} \quad z_j \neq z^*, \\ m_j - 1 & \text{if} \quad z_j = z^*. \end{cases}$$

This means, by Theorem 2.8, that $\mathcal{D}_{z^*}(q; z) = \sum_{j=1}^{k} \mathcal{D}_{z^*}(q_j; z)$ is amongst the polynomials apolar to

$$a_n \prod_{j=1}^{k} (z - z_j)^{n_j} = p_1(z),$$

completing the proof. ∎

Combining Lemma 2.9 and Statement 2.1, we obtain the following corollary.

Corollary 2.1. *Let $s \in \{0, 1, \ldots, n-1\}$. Let $p \in \mathcal{P}_n$, $q \in \mathcal{P}_{n-s}$, and z^* be a zero of $p = \sum_{k=0}^{n} a_k z^k$. If p and q are apolar, then*

$$p_1(z) = \begin{cases} p(z)/(z - z^*) & \text{if } z^* \in \mathcal{C}, \\ \sum_{k=0}^{n-1} a_k z^k & \text{if } z^* = \infty, \end{cases}$$

is apolar to the polar derivative $\mathcal{D}_{z^*}(q^{(s)}; z)$.

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