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WARPED PRODUCT SEMI-SLANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

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Abstract. In the present note, it is proved that there donot exist warped product semi-slant submanifolds in a Sasakian manifold other than contact CR-warped product submanifolds and thus the results obtained in [8] are generalized.

1. Introduction. Bishop and O'Neill [1] introduced the notion of warped product manifolds and obtained results which reveal important geometric properties of these manifolds. Many physical applications of these manifolds are recently discovered e.g., the space around a black hole or a massive star is modeled on warped product manifolds. To be more precise, the best relativistic model of the Schwarzchild space time describing the neighbourhoods of stars or black holes is given as a warped product (c.f. [10]). Recently B. Y. Chen [5] initiated the study of warped product CR-submanifolds of Kaehler manifolds. B. Sahin

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[11], extending the study of Chen, proved that there donot exist proper warped product semi-slant submanifolds in a Kaehler manifold. Hesegawa and Mihai [7] initiated the study of contact CR-warped product submanifolds in Sasakian manifolds. They obtained a sharp relationship between the warping function f of a warped product CR-submanifold and the squared norm of the second fundamental form. K. Matsumoto and Mihai [9] studied warped product submanifolds in a Sasakian space form. They also worked out variuous inequalities regarding the squared norm of mean curvature vector of warped products in a Sasakian space form and derived some important applications. In view of the physical applications of these manifolds, the question of existance or non existanc of warped product submanifolds assumes significant in general. In the present note, we have adressed the same problem by studying warped product semi-slant submanifold of a Sasakian manifold.

2. Preliminaries. Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact manifold i.e., a manifold endowed with an almost contact structure (ϕ, ξ, η) where ϕ is a $(1,1)$ tensor field, ξ is a vector field and η is a 1-form such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0.$$

The almost contact structure is said to be normal if on the product manifold $\overline{M} \times R$, the induced almost complex structure J defined by

$$J\left(U, \lambda \frac{d}{dt}\right) = \left(\phi U - \lambda \xi, \eta(U) \frac{d}{dt}\right),$$

is integrable, where U is tangent to $\overline{M}$, t is the coordinate function on R and λ is a smooth function on $\overline{M} \times R$. The condition for an almost contact structure to be normal is equivalent to the vanishing of the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$ where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ.

On an almost contact manifold, there exists a Riemannian metric g which is compatible with the contact structure (ϕ, ξ, η) i.e.,

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.1}$$

from which it can be observed that

$$g(U, \xi) = \eta(U), \tag{2.2}$$

for any $U, V \in T(\overline{M})$. In this case, the Riemannian manifold $(\overline{M}, g)$ is called an almost contact metric manifold. An almost contact metric structure is called a contact metric structure if $d\eta = \Phi$ where Φ is the fundamental 2-form defined by

$$\Phi(U, V) = g(U, \phi V).$$

A normal contact metric manifold is called a Sasakian manifold. It is known that an almost contact metric manifold is Sasakian if and only if

$$(\overline{\nabla}_U \phi)V = -g(U, V)\xi + \eta(V)U, \tag{2.3}$$

and,

$$\overline{\nabla}_U \xi = \phi U, \tag{2.4}$$

where $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$.

Let M be a submanifold of an almost contact metric manifold $\overline{M}$. Then the induced Riemannian metric on M is denoted by the same symbol g and the induced connection by ∇. If $T\overline{M}$ and TM denote the tangent bundle on $\overline{M}$ and on M respectively, then the Gauss and Weingarten formulae are written as:

$$\overline{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.5}$$

$$\overline{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.6}$$

for $U, V \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ denotes the induced connection in the normal bundle $T^{\perp}M$. A_N and h denote the shape operator of the immersion and the second fundamental respectively. The two are related as :

$$g(A_N U, V) = g(h(U, V), N), \tag{2.7}$$

For any $U \in TM$, we write

$$\phi U = TU + FU, \tag{2.8}$$

where TU and FU are respectively the tangential and normal components of ϕU. Similarly, for $N \in T^{\perp}M$, we decompose ϕN into tangential and normal parts as:

$$\phi N = BN + CN. \tag{2.9}$$

The covariant derivatives of T and F are defined as:

$$(\overline{\nabla}_U T)V = \nabla_U TV - T\nabla_U V, \tag{2.10}$$

$$(\overline{\nabla}_U F)V = \nabla^{\perp}_U FV - F\nabla_U V, \tag{2.11}$$

for any $U, V \in TM$.

Making use of equations (2.3)–(2.6) and (2.8)–(2.11), we obtain that

$$(\overline{\nabla}_U T)V = A_{FV}U + Bh(U,V) - g(U,V)\xi + \eta(V)U, \tag{2.12}$$

and,

$$(\overline{\nabla}_U F)V = Ch(U,V) - h(U,TV). \tag{2.13}$$

A submanifold M of an almost contact metric manifold $\overline{M}$ tangential to the structure vector field ξ is called an invariant submanifold if ϕ-preserves all tangent spaces of M, that is $\phi T_x(M) \subset T_x(M)$ for every $x \in M$. A submanifold M tangent to ξ is called an anti-invariant submanifold if ϕ maps a tangent space of M into the normal space, i.e., $\phi T_x(M) \subset T_x^{\perp}(M)$ for all $x \in M$ where $T_x^{\perp}(M)$ denotes the normal space at $x \in M$. The above definitions have been generalized in several ways:

(i) A submanifold M tangent to ξ, is called a contact CR-Submanifold if there exists a differentiable distribution $D : x \longrightarrow D_x \subset T_x(M)$ such that D is invariant with respect to ϕ and the complementry distribution $D^{\perp}$ is anti-invariant with respect to ϕ.

(ii) A submanifold M tangent to ξ is called a slant submanifold if for all non zero vector U tangent to M and independent with ξ, the angle $\theta(U)$ between ϕU and $T_x(M)$ is a constant i.e., it doesnot depend on the choice of $x \in M$ and $U \in T_x(M)$ [8].

(iii) A submanifold M is called a semi-slant submanifold if it is endowed with two orthogonal complementry distributions D and D^0 where D is invariant with respect to ϕ and D^0 is slant i.e., $\theta(U)$ between ϕU and D^0_x is constant for each $U \in D^0_x$ and $x \in M$ [3].

It is clear that invariant and anti-invariant submanifolds are CR-Submanifolds (resp., slant submanifolds) with $D^{\perp} = \{0\}$ (resp. $\theta = 0$) and $D = \{0\}$ (resp. $\theta = \pi/2$). It is also clear that contact CR-submanifolds and slant submanifolds are particular semi-slant submanifolds with $\theta = \pi/2$ and $D = \{0\}$ respectively.

A submanifold M of an almost contact metric manifold $\overline{M}$ is a slant submanifold if and only if

$$T^2 = -\lambda(-I + \eta \otimes \xi), \tag{2.14}$$

for some real number $\lambda \in [0,1]$. Furthermore, in such case if θ is the slant angle of M, then $\lambda = \cos^2\theta$ [4]. Hence,

$$g(TU,TV) = \cos^2\theta(g(U,V) - \eta(U)\eta(V)), \tag{2.15}$$

$$g(FU,FV) = \sin^2\theta(g(U,V) - \eta(U)\eta(V)), \tag{2.16}$$

for any $U, V \in TM$.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. Then the warped product $M_1 \times_f M_2$ is a product manifold $M_1 \times M_2$ endowed with a Riemannian metric g given by

$$g = g_1 + f^2 g_2.$$

More explicitly, if U is a tangent vector on $M = M_1 \times_f M_2$ then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2,$$

where $\pi_i (i = 1,2)$ are the canonical projections of M onto M_1 and M_2 respectively and $d\pi_i$ are their differentials.

Theorem 2.1 [1]. *Let $M_1 \times_f M_2$ be a warped product manifold. Then for any $X, Y \in TM_1$ and $Z, W \in TM_2$,*

(i) $\nabla_X Y \in TM_1$,

(ii) $\nabla_X Z = \nabla_Z X = \left(\dfrac{Xf}{f}\right) Z$,

(iii) $\mathrm{nor}(\nabla_Z W) = -\dfrac{g(Z,W)}{f}\nabla f$,

where $\mathrm{nor}(\nabla_Z W)$ *is the component of* $\nabla_Z W$ *in* TM_1 *and* ∇f *is the gradient of* f *defined by*

$$g(\nabla f, U) = Uf.$$

Corollary 2.1. *On a warped product manifold $M = M_1 \times_f M_2$,*

(i) *M_1 is totally geodesic in M.*

(ii) *M_2 is totally umbilical in M.*

3. Semi-Slant Submanifolds as Warped Products. Let $\overline{M}$ be a Sasakian manifold. Throughout the section, we denote by M_T an invariant submanifold of $\overline{M}$ and M_θ, a slant submanifold of $\overline{M}$ with slant angle θ. Our aim in this section is to study warped product submanifolds $M_T \times_f M_\theta$ and $M_\theta \times_f M_T$. We further assume that the structure vector field ξ is tangential to the underlying warped product submanifolds.

From equations (2.4) and (2.5), we have

$$\nabla_X \xi = \phi X, \tag{3.1}$$

$$h(X, \xi) = 0, \tag{3.2}$$

for any $X \in TM_T$ whereas for $Z \in TM_\theta$, we get

$$\nabla_Z \xi = TZ. \tag{3.3}$$

$$h(Z, \xi) = FZ. \tag{3.4}$$

Matsumoto and Mihai [9] proved

Theorem 3.1 [9]**.** *If $M = M_1 \times_f M_2$ is a warped product submanifold of a Sasakian manifold $\overline{M}$ where M_1 and M_2 are any submanifolds of $\overline{M}$ with ξ tangential to M_2. Then M is a Riemannian product.*

Hence, the possible non trivial semi-slant warped product submanifolds are $M_\theta \times_f M_T$ and $M_T \times_f M_\theta$ with ξ tangential to M_θ and M_T respectively.

For $\theta = \pi/2$ the above warped products are known as warped product contact CR-submanifold and contact CR-warped product submanifolds respectively. Hesegawa and Mihai [7] showed the the warped product contact CR-submanifolds with ξ tangential to M_T are non existant.

Theorem 3.2. *Let $\overline{M}$ be a $(2m+1)$-dimensional Sasakian manifold. Then there do not exist warped product submanifolds $M_\theta \times_f M_T$ on $\overline{M}$ such that M_θ is a slant submanifold tangent to ξ and M_T is an invariant submanifold of $\overline{M}$.*

P r o o f. Let $M = M_\theta \times_f M_T$ be a submanifold of $\overline{M}$ with ξ tangential to M_θ. Then by Theorem 2.1,

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X, \tag{3.5}$$

for any $X \in TM_T$ and $Z \in TM_\theta$. In particular for $Z = \xi$, we get $\xi f = 0$. Now by equations (3.1) and (3.5), it follows that

$$\phi X = \nabla_X \xi = (\xi \ln f)X = 0.$$

Thus M_T can not exist and the Theorem is proved.

Now, we obtain same important relations for later use

Lemma 3.1. *Let $M = M_T \times_f M_\theta$ be a warped product semi-slant submanifold of a Sasakian manifold $\overline{M}$ such that M_T is an invariant submanifold tangent to ξ and M_θ a slant submanifold of $\overline{M}$ with slant angle $\theta \neq 0$, then*

(i) $g(h(X,Y), FZ) = 0.$

(ii) $g(h(X,W), FZ) = g(h(X,Z), FW).$

(iii) $g(h(\phi X, Z), FW) = (X \ln f)g(Z,W).$

for any $X, Y \in TM_T$ and $Z, W \in TM_\theta$

Proof. By Corollary 2.1, M_T is totally geodesic in M and therefore by formula (2.10), $(\overline{\nabla}_X T)Y \in TM_T$ for all $X, Y \in TM_T$. Using this fact in equation (2.12), we deduce that

$$g(Bh(X,Y), Z) = 0,$$

for all $X, Y \in TM_T$ and $Z \in TM_\theta$. The above equation is equivalent to

$$g(h(X,Y), FZ) = 0.$$

This prove (i). Now, by Theorem (2.1),

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z. \tag{3.6}$$

Taking account of (3.6) in (2.10) and (2.12), it follows that

$$A_{FZ}X + Bh(X,Z) = 0,$$

which proves that

$$g(h(X,W), FZ) = g(h(X,Z), FW). \tag{3.7}$$

Further, by equations (2.10) and (3.6), we obtain that

$$(\overline{\nabla}_Z T)X = (TX \ln f)Z - (X \ln f)TZ, \tag{3.8}$$

whereas by formula (2.12)

$$(\overline{\nabla}_Z T)X = Bh(X, Z) + \eta(X)Z. \tag{3.9}$$

From the above two equations, it follows that

$$g(Bh(\phi X, Z), W) = -(X \ln f)g(Z, W) - (TX \ln f)g(TZ, W),$$

which on making use of (3.7) yields

$$g(h(\phi X, Z), FW) = (X \ln f)g(Z, W), \tag{3.10}$$

for $X \in TM_T$ and $Z, W \in TM_\theta$. This proves statement (iii).

Theorem 3.3. *There does not exist a warped product semi-slant submanifold in a Sasakian manifold other than a contact CR-warped product submanifold.*

Proof. Consider a warped product submanifold $M = M_T \times_f M_\theta$ of a Sasakian manifold $\overline{M}$ with ξ tangential to M_T. On replacing X by ϕX in equation (3.10), we get

$$\eta(X)g(h(Z, \xi), FW) - g(h(X, Z), FW) = (\phi X \ln f)g(Z, W).$$

The above relation on making use of formula (3.4) and (2.16) yields

$$g(h(X, Z), FW) = (\eta(X) - \csc^2\theta(\phi X \ln f))g(FZ, FW), \tag{3.11}$$

for any $X \in TM_T$ and $Z, W \in TM_\theta$.

If we denote by D^0 the tangent bundle of M_θ and ν the orthogonal complement of FD^0 i.e.,

$$T^\perp M = FD^0 \oplus \nu,$$

then we may write

$$h(X, Z) = h_{FD^0}(X, Z) + h_\nu(X, Z),$$

where $h_{FD^0}(X, Z) \in FD^0$ and $h_\nu(X, Z) \in \nu$. In view of the above decomposition, equation (3.11) yields

$$h_{FD^0}(X, Z) = (\eta(X) - \csc^2\theta(\phi X \ln f))FZ. \tag{3.12}$$

On the other hand by equations (3.8) and (3.9), we have

$$(TX \ln f)Z - (X \ln f)TZ = Bh(X, Z) + \eta(X)Z.$$

Taking product with TZ and using the formula (2.15), the above equation implies that

$$g(h(X, Z), FTZ) = \cos^2 \theta (X \ln f) g(Z, Z). \tag{3.13}$$

From equations (3.12) and (3.13), it follows that either $\theta = \frac{\pi}{2}$ or $X(\ln f) = 0$. This proves the assertion.

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