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Letter to the Editor

Remarks on Some Inequalities for Polynomials

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In the present article, I point out serious errors in a paper published in *Mathematica Balkanica* three years ago. These errors seem to go unnoticed because some mathematicians are applying the results stated in this paper to prove other results, which should not continue.

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The following result was proved by Govil (see [4, p. 51]).

Theorem A. *Let $p(z)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, and let $q(z) := z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$. If $|p'(z)|$ and $|q'(z)|$ attain maximum at the same point on the circle $|z| = 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)| . \tag{1}$$

The result is best possible with equality holding for the polynomial $p(z) = z^n + k^n$.

Aziz and Ahmad [1] proved that if p satisfies the conditions of Theorem A, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\} . \tag{2}$$

which is stronger than (1).

The following result of Govil, which is clearly related to Theorem A, appears in [5] as Theorem D on p. 184.

Theorem B. *If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)| . \tag{3}$$

In [2] the authors state and I quote: "In this paper, we consider polynomials of the form

$$p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu},\ 1 \leq \mu \leq n$$

and obtain generalization as well as improvement of (1). Also we generalize Theorem B". They state their so-called generalizations of (1), (2) and (3) as follows.

Theorem 1. *Let $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ be a polynomial of degree n, having no zero in $|z| < k$, $k \leq 1$ and $q(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. If $|p'(z)|$ and $|q'(z)|$ become maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^{n-\mu+1}} \max_{|z|=1} |p(z)| . \tag{4}$$

Theorem 2. *Let $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ be a polynomial of degree n, having no zero in $|z| < k$, $k \leq 1$ and $q(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$. If $|p'(z)|$ and $|q'(z)|$ become maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^{n-\mu+1}} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\} . \tag{5}$$

Theorem 2 is supposed to generalize Theorem B.

Theorem 3. *If $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ is a polynomial of degree n, having all its zeros on $|z| = k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{k^{n-2\mu+1} + k^{n-\mu+1}} \max_{|z|=1} |p(z)| . \tag{6}$$

Unfortunately, Theorems 1, 2 and 3 are false. To see that Theorem 1 is invalid, let us consider the example $p(z) := z^n + k^n$. It is of the form $c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu}^{n} c_{n-\nu} z^{n-\nu}$ with

$$c_n = 1,\ c_{n-\nu} = 0 \ \text{ for } \ \nu = \mu, \dots, n-1 \ \text{ and } \ c_0 = k^n\,,$$

where μ can be taken to be any integer in $\{1, 2, \dots, n-1\}$. This polynomial has all its zeros on $|z| = k$. Since $p'(z) = nz^{n-1}$ and $q'(z) = nk^n z^{n-1}$, we see that $|p'(z)|$ and $|q'(z)|$ become maximum at every point on $|z| = 1$. Thus p satisfies the conditions of Theorem 1. Since

$$\max_{|z|=1} |p(z)| = 1 + k^n \quad \text{and} \quad \max_{|z|=1} |p'(z)| = n\,,$$

inequality (4) implies that

$$n \le \frac{n}{1 + k^{n-\mu+1}}(1 + k^n) \qquad (\mu = 1, 2, \dots, n-1\,,$$

which holds if and only if $k^{n-\mu+1} \le k^n$, However, this is manifestly false for any $k \in (0\,,\,1)$ and any $\mu \in \{2, \dots, n-1\}$.

A naive reader might say that in this example $c_{n-\mu}$ is taken to be 0 and that, in Theorem 1, $c_{n-\mu}$ is supposed to be different from zero. No problem, we will modify our example.

Take any $a \in (0\,,\,1)$ and consider the polynomial

$$p(z) := z^n + \delta z^{n-\mu} + a^n\,,$$

where δ is positive and small. Since the zeros of p are continuous functions [7, p. 9] of δ and those of $z^n + a^n$ all lie on $|z| = a$ the polynomial p has all its zeros in $|z| \ge k$, where $|a - k| \to 0$ as $\delta \to 0$. Now, note that

$$p'(z) = nz^{n-1} + \delta(n-\mu)z^{n-\mu-1} \qquad \text{and} \quad q'(z) = a^n n z^{n-1} + \delta\mu z^{\mu-1}\,.$$

So, both $|p'(z)|$ and $|q'(z)|$ become maximum at the same point on $|z| = 1$, namely the point 1. Thus, Theorem 1 applies and would imply that

$$n + \delta(n-\mu) \le \frac{n}{1 + k^{n-\mu+1}}(1 + \delta + a^n)\,,$$

where δ is any small positive number. Letting δ tend to 0, we would obtain

$$n \le \frac{n}{1 + a^{n-\mu+1}}(1 + a^n)\,,$$

where we have used the fact that $k \to a$ as $\delta \to 0$. This last inequality holds if and only if $a^{n-\mu+1} \leq a^n$, which is not true for any $\mu \in \{2, \dots, n-1\}$ since $a \in (0\,,\,1)$.

Now we know that Theorem 1 is incorrect, but then Theorem 2, being stronger than Theorem 1 cannot be true either.

According to the statement of Theorem 3, the coefficients $c_{n-\mu}, \dots, c_1$ can be anything as long as the polynomial p has all its zeros on $|z| = k$. So, there is nothing to prevent us from taking $p(z) = z^n + k^n$ and applying (6) with $\mu = n-1$. We would then obtain

$$n \leq \frac{n}{k^{3-n} + k^2}(1 + k^n)\,,$$

which is true if and only if

$$k^3 + k^{n+2} \leq nk^n\,(1 + k^n).$$

Fixing $k \in (0\,,\,1)$ and letting $n \to \infty$ the left-hand side of this inequality tends to k^3 whereas the right-hand side tends to 0, which is a contradiction. This shows that Theorem 3, as stated by the authors, cannot be true.

The proof of Theorem 3 uses the following faulty statement which appears as Lemma 3 in their paper:

Proposition 1. *If* $p(z) = c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ *is a polynomial of degree* n*, having no zero in* $|z| < k$, $k \leq 1$, *then*

$$k^{n-\mu+1} \max_{|z|=1} |p'(z)| \leq \max_{|z|=1} |q'(z)|\,, \tag{7}$$

where $q(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

In order to show that this is an invalid statement we may once again take any $a \in (0\,,\,1)$ and consider the polynomial $p(z) := z^n + \delta z^{n-\mu} + a^n$, where δ is positive and small. As observed above, it has all its zeros in $|z| \geq k$, where $|a - k| \to 0$ as $\delta \to 0$. Clearly,

$$\max_{|z|=1} |p'(z)| = |p'(1)| = n + \delta(n-\mu) \quad \text{and} \quad \max_{|z|=1} |q'(z)| = |q'(1)| = a^n n + \delta\mu\,.$$

Hence, if Proposition 1 was true then (7) would imply that

$$k^{n-\mu+1}\big(n + \delta(n-\mu)\big) \leq a^n n + \delta\mu\,.$$

Letting δ tend to 0, we would obtain

$$a^{n-\mu+1} n \le a^n n\,.$$

This holds if and only if $a^{n-\mu+1} \le a^n$, which is not true for any μ in $\{2, \ldots, n-1\}$ since $a \in (0\,,\,1)$.

The proof of Proposition 1 ($\equiv$ Lemma 3 in the paper of Dewan and Hans [2]) makes use of Lemma 2 in [2], which in turn uses the the following invalid statement presented by them as Lemma 1.

Proposition 2. *If $p(z) = c_0 z^n + \sum_{\nu=\mu}^{n} c_\nu z^\nu$, $1 \le \mu < n$ is a polynomial of degree n, having all its zeros in the disk $|z| < k$, $k \ge 1$, then for $|z| = 1$*

$$k^{n+\mu-3} \max_{|z|=1} |q'(z)| \le \max_{|z|=1} |p'(k^2 z)|\,, \tag{8}$$

where $q(z) := z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

In order to see that Proposition 2 ($\equiv$ Lemma 1 in [2]) is false, let us take any $b > 1$ and consider the polyomial $p(z) := b^n + \delta z^\mu + z^n$, where δ is positive and small. The polynomial p all its zeros in $|z| < k$, where $|b - k| \to 0$ as $\delta \to 0$. We have

$$p'(k^2 z) = nk^{2n-2} z^{n-1} + \delta \mu k^{2\mu-2} z^{\mu-1} \ \text{ and } \ q'(z) = nb^n z^{n-1} + \delta(n-\mu) z^{n-\mu-1}\,.$$

Clearly then

$$\max_{|z|=1} |p'(k^2 z)| = nk^{2n-2} + \delta\mu k^{2\mu-2} \ \text{ and } \ \max_{|z|=1} |q'(z)| = nb^n + \delta(n-\mu)\,.$$

Hence, if Proposition 2 was true then from (8) we would obtain

$$k^{n+\mu-3}\big(nb^n + \delta(n-\mu)\big) \le nk^{2n-2} + \delta\mu k^{2\mu-2}\,.$$

Letting δ tend to 0, we would obtain $b^2 \le b^{3-\mu}$, which is not true for any $\mu \in \{2, \ldots, n-1\}$ since $b > 1$.

All the three theorems in [2] are wrong because of a serious mistake in the proof of Proposition 2 ($\equiv$ Lemma 1 in [2]). In particular, inequality (2.4) on page 30 of [2] is wrong. It says:

$$k^{n-1} \left| q'\left(\frac{z}{k}\right) \right| \le k^\mu \left| \sum_{\nu=\mu}^{n} \nu c_\nu \,(kz)^{\nu-\mu} \right| \quad \text{for} \quad |z| \ge 1\,. \tag{2.4}$$

They justify this inequality *essentially* as follows. By (2.3), inequality (2.4) holds for $|z| = 1$. We agree with this. Then they seem to consider the function

$$\phi(z) := \frac{k^{n-1}Q'(z)}{k^{\mu}\sum_{\nu=\mu}^{n}\nu c_{\nu}(kz)^{\nu-\mu}}$$

and note that $\sum_{\nu=\mu}^{n}\nu c_{\nu}\,(kz)^{\nu-\mu} \neq 0$ in $|z| > 1$ and so $\phi(z)$ is holomorphic in $|z| > 1$. We agree with this also. Since $|\phi(z)| \leq 1$ for $|z| = 1$ they think that "by maximum modulus principle, $|\phi(z)| \leq 1$ for $|z| > 1$". Not so fast! The maximim modulus principle, in the case they are in, requires the function to tend to a finite limit as $|z|$ tends to infinity but the function $\phi(z)$ to which the maximim modulus principle is being applied tends to infinity as $|z| \to \infty$ if $\mu \in \{2, \ldots, n-1\}$. Thus (2.4) is not true for $\mu \in \{2, \ldots, n-1\}$.

We wonder how Dewan and Hans could have overlooked the fact that in inequality (1) of Govil as well as in inequality (2) of Aziz and Ahmad, equality holds for $p(z) := z^n + k^n$, which is a polynomial of the form $p(z) := c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$. To think that they could improve upon (1) and (2), by considering polynomials which are of the form $p(z) := c_n z^n + \sum_{\nu=\mu}^{n} c_{n-\nu} z^{n-\nu}$, was a hopeless idea to start with. They could do something better than Theorems A and B only if they considered a class of polynomials which did not contain $z^n + k^n$. In fact, there is no *raison d' être* for Theorems 1 and 2. Not only their proofs are wrong, these statements are simply false. The problem with Theorem 3 is of a different nature; namely, its proof uses Lemma 3, which is faulty.

Erroneous though they are, Theorems 1, 2, 3 and Lemmas 1, 2, 3 appearing in [2], have been used, as if they were "true", to draw conclusions, which cannot be taken seriously. For example in [6], the authors use Lemma 3 from [2] (quoted as Lemma 2.1 in [6]) to generalize Theorem 3 of Dewan and Hans from [2]. In [3], Dewan and Ahuja generalize Theorem 3 from [2] to polar derivatives; like Pukhta, Mir and Raja, they use Lemma 3 from [2] in their proof. One can easily construct counter-examples to the result which Dewan and Ahuja think they have proved in [3]. What a terrible waste of time to "generalize" something that is invalid to startwith!

As indicated above, there are people who have already lost time applying and generalizing Theorems 1, 2, 3 and Lemmas 1, 2, 3 of [2]. So, it seems desirable that the truth about these *so-called* theorems and lemmas be made known.

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