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OSTROWSKI TYPE INEQUALITIES OVER SPHERICAL SHELLS

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Abstract. Here are presented Ostrowski type inequalities over spherical shells. These regard sharp or close to sharp estimates to the difference of the average of a multivariate function from its value at a point.

1. Introduction. The famous Ostrowski's inequality (1938), see [4], is

$$\left|\frac{1}{b-a}\int_a^b f(y)dy - f(x)\right| \leq \left(\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right)(b-a)\|f'\|_\infty,$$

for $f \in C^1([a,b]), x \in [a,b]$, and it is a sharp inequality.

This was generalized from intervals to boxes in $\mathbb{R}^N, N \geq 1$, see [3], [2], p. 507. Here we establish Ostrowski type inequalities over spherical shells.

We present first our sharp results for the radial functions, then we move to the non-radial case. We use the polar method.

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Our estimates in both cases involve radial derivatives of arbitrary order of the engaged function.

At the end we give the connection of radial derivative to the ordinary partial derivative of the function.

2. Results. We make

Remark 2.1. Let A be a *spherical shell* $\subseteq \mathbb{R}^N, N \geq 1$, *i.e.* $A := B(0,R_2) - \overline{B(0,R_1)}, 0 < R_1 < R_2$.
Here the ball $B(0,R) := \{x \in \mathbb{R}^N : |x| < R\}, R > 0$, where $|\cdot|$ is the Euclidean norm , also $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ is the unit sphere in $\mathbb{R}^N$ with surface area $\omega_N := \dfrac{2\pi^{N/2}}{\Gamma(N/2)}$. For $x \in \mathbb{R}^N - \{0\}$ one can write uniquely $x = r\omega$, where $r > 0$, $\omega \in S^{N-1}$.

Let $f \in C^1(\bar{A})$. We assume first that f is radial i.e $f(x) = g(r)$, where $r = |x|$, $R_1 \leq r \leq R_2$. Clearly here $g \in C^1([R_1, R_2])$.

In general it holds $\left\| \dfrac{\partial f}{\partial r} \right\|_\infty \leq \| \nabla f \|_\infty$, with equality in the radial case.

For $F \in C(\bar{A})$ we have

$$\int_A F(x)dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) d\omega.$$

We notice that

$$\frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} s^{N-1} ds = 1, \tag{2.1}$$

and

$$Vol(A) = \frac{\omega_N (R_2^N - R_1^N)}{N}. \tag{2.2}$$

Let $x \in A$. Then by using the polar method we obtain

$$\left| f(x) - \frac{\int_A f(y)dy}{Vol(A)} \right| = \left| f(x) - \frac{N \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right) d\omega}{\omega_N (R_2^N - R_1^N)} \right| \tag{2.3}$$

$$= \left| g(r) - \frac{N \int_{S^{N-1}} \left(\int_{R_1}^{R_2} g(s) s^{N-1} ds \right) d\omega}{\omega_N (R_2^N - R_1^N)} \right| \tag{2.4}$$

$$(2.5)\qquad = \left| g(r) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right|$$

$$= \left| \left(\frac{N}{R_2^N - R_1^N} \right) \left(\int_{R_1}^{R_2} g(r) s^{N-1} ds - \int_{R_1}^{R_2} g(s) s^{N-1} ds \right) \right|$$

$$(2.6)\qquad = \left(\frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} \left(g(r) - g(s) \right) s^{N-1} ds \right|$$

$$\leq \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} \left| g(r) - g(s) \right| s^{N-1} ds \leq$$

$$(2.7)\qquad \left(\frac{N}{R_2^N - R_1^N} \right) \left\| g' \right\|_\infty \int_{R_1}^{R_2} \left| r - s \right| s^{N-1} ds$$

$$(2.8)\qquad = \left(\frac{N}{R_2^N - R_1^N} \right) \left\| g' \right\|_\infty \left[\int_{R_1}^{r} (r-s) s^{N-1} ds + \int_{r}^{R_2} (s-r) s^{N-1} ds \right]$$

$$= \left(\frac{N}{R_2^N - R_1^N} \right) \left\| g' \right\|_\infty \left[r \left(\frac{2r^N - (R_1^N + R_2^N)}{N} \right) \right.$$

$$(2.9)\qquad \left. + \left(\frac{R_1^{N+1} + R_2^{N+1} - 2r^{N+1}}{N+1} \right) \right].$$

So me have established our first main result.

Theorem 2.1. *Let $f \in C^1(\bar{A})$ be radial, i.e. $f(x) = g(r)$, $R_1 \leq r \leq R_2$, $x \in \bar{A}$.*

Then

$$\left| f(x) - \frac{\int_A f(y) dy}{Vol(A)} \right| = \left| g(r) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s) s^{N-1} ds \right| \leq$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \left\| g' \right\|_\infty \left[r \left(\frac{2r^N - (R_1^N + R_2^N)}{N} \right) \right.$$

$$+\left(\frac{R_1^{N+1}+R_2^{N+1}-2r^{N+1}}{N+1}\right)\Bigg] \tag{2.10}$$

$$=\left(\frac{N}{R_2^N-R_1^N}\right)\|\nabla f\|_\infty\left[|x|\left(\frac{2|x|^N-(R_1^N+R_2^N)}{N}\right)\right.$$

$$\left.+\left(\frac{R_1^{N+1}+R_2^{N+1}-2|x|^{N+1}}{N+1}\right)\right]. \tag{2.11}$$

Optimality comes next

Theorem 2.2. *Inequality* (2.10) *is sharp. More precisely*

(i) *it is asymptotically attained by* $g^*(z):=|z-r|^\alpha$, $1<\alpha\le k$, *when* $0<r<R$.

(ii) *It is attained by* $g^*(z)=(z-R_1)$, *when* $r=R_1$.

(iii) *It is attained by* $g^*(z)=(z-R_2)$, *when* $r=R_2$.

Proof. (i) We see that

$$g^{*'}(z)=\alpha|z-r|^{\alpha-1}sign(z-r)$$

and

$$\|g^{*'}\|_\infty=\alpha(max\{R_2-r,\,r-R_1\})^{\alpha-1},$$

along with $g^*(r)=0$.

We observe that

$$L.H.S(2.10)=\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}|s-r|^\alpha\,s^{N-1}ds$$

$$\xrightarrow{\alpha\to1}\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}|s-r|\,s^{N-1}ds$$

$$=\left(\frac{N}{R_2^N-R_1^N}\right)\left[r\left(\frac{2r^N-(R_1^N+R_2^N)}{N}\right)+\left(\frac{R_1^{N+1}+R_2^{N+1}-2r^{N+1}}{N+1}\right)\right]. \tag{2.12}$$

We also have

$$R.H.S(2.10) = \left(\frac{N}{R_2^N - R_1^N}\right) \alpha(\max\{R_2 - r,\, r - R_1\})^{\alpha-1}$$

$$\left[r\left(\frac{2r^N - (R_1^N + R_2^N)}{N}\right) + \left(\frac{R_1^{N+1} + R_2^{N+1} - 2r^{N+1}}{N+1}\right)\right] \xrightarrow{\alpha \to 1}$$

$$\left(\frac{N}{R_2^N - R_1^N}\right) \left[r\left(\frac{2r^N - (R_1^N + R_2^N)}{N}\right) + \left(\frac{R_1^{N+1} + R_2^{N+1} - 2r^{N+1}}{N+1}\right)\right]. \tag{2.13}$$

That is

$$\lim_{\alpha \to 1} L.H.S(2.10) = \lim_{\alpha \to 1} R.H.S(2.10),$$

proving sharpness for the case.

(*ii*) We have $g^*(R_1) = 0$ and $\|g^{*'}\|_\infty = 1$.
Thus

$$L.H.S(2.10) = \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} (s - R_1) s^{N-1} ds$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right) \left[\left(\frac{R_2^{N+1} - R_1^{N+1}}{N+1}\right) - R_1 \left(\frac{R_2^N - R_1^N}{N}\right)\right] = R.H.S\,(2.10), \tag{2.14}$$

proving the attainability for the case.

(*iii*) We have $g^*(R_2) = 0$, and $\|g^{*\prime}\|_\infty = 1$.
Hence

$$L.H.S(2.10) = \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} (R_2 - s) s^{N-1} ds =$$

$$\left(\frac{N}{R_2^N - R_1^N}\right) \left[R_2 \left(\frac{R_2^N - R_1^N}{N}\right) - \left(\frac{R_2^{N+1} - R_1^{N+1}}{N+1}\right)\right] = R.H.S\,(2.10), \tag{2.15}$$

proving attainability of last case. □

We would line to rewrite Theorem 2.2 for the equivalent inequality (2.10) in terms of f. We have

Theorem 2.2*. *Let $x \in \bar{A}$. Inequality* (2.10) *is sharp as follows:*
(i) Let $0 < |x| < R$, then it is asymptotically attained by

$$f^*(w) := |\ |w| - |x|\ |^{\alpha},\ 1 < \alpha \le k.$$

(ii) Let $|x| = R_1$, then it is asymptotically attained by

$$f^*(w) = |w| - R_1.$$

(iii) Let $|x| = R_2$, then it is asymptotically attained by

$$f^*(w) = |w| - R_2.$$

We continue from Remark 2.1 into

Remark 2.2. Now $f \in C^n(\bar{A})$, $n \in \mathbb{N}$, again radial such that $f(x) = g(r)$, where $r = |x|$, $x \in \bar{A}$, $R_1 \le r \le R_2$. Hence $g \in C^n([R_1, R_2])$.
Using the polar method we obtain again

$$E := \left| f(x) - \frac{\int_A f(y)dy}{Vol(A)} \right| = \left(\frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} (g(r) - g(s)) s^{N-1} ds \right|. \tag{2.16}$$

Let $s, r \in [R_1, R_2]$, then by Taylor's formula we get

$$g(s) - g(r) = \sum_{k=1}^{n-1} \frac{g^{(k)}(r)}{k!} (s - r)^k + R_{n-1}(r, s), \tag{2.17}$$

where

$$R_{n-1}(r, s) := \int_r^s \left(g^{(n-1)}(t) - g^{(n-1)}(r) \right) \frac{(s-t)^{n-2}}{(n-2)!} dt. \tag{2.18}$$

As in [2, p. 500], we find

$$|R_{n-1}(r, s)| \le \frac{\|g^{(n)}\|_{\infty,[R_1,R_2]}}{n!} |s - r|^n, \tag{2.19}$$

$\forall\ s, r \in [R_1, R_2]$.

Therefore

$$E=\left(\frac{N}{R_2^N-R_1^N}\right)\left|\int_{R_1}^{R_2}\left(\sum_{k=1}^{n-1}\frac{g^{(k)}(r)}{k!}(s-r)^k+R_{n-1}(r,s)\right)s^{N-1}ds\right| \tag{2.20}$$

$$\leq\left(\frac{N}{R_2^N-R_1^N}\right)\left[\sum_{k=1}^{n-1}\frac{|g^{(k)}(r)|}{k!}\left|\int_{R_1}^{R_2}s^{N-1}(s-r)^k ds\right|+\int_{R_1}^{R_2}s^{N-1}|R_{n-1}(r,s)|ds\right]\leq$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\left[\sum_{k=1}^{n-1}\frac{|g^k(r)|}{k!}\left|\sum_{m=0}^{k}\binom{k}{m}(-1)^m r^m\right.\right.$$

$$\left.\left.\left(\frac{R_2^{N+k-m}-R_1^{N+k-m}}{N+k-m}\right)\right|+\frac{\|g^{(n)}\|_\infty}{n!}\int_{R_1}^{R_2}s^{N-1}|s-r|^n ds\right]. \tag{2.21}$$

But one finds that

$$\int_{R_1}^{R_2}s^{n-1}|s-r|^n ds=$$

$$\sum_{m=0}^{n}\binom{n}{m}(-1)^m\left[r^{n-m}\left(\frac{r^{m+N}-R_1^{m+N}}{m+N}\right)+r^m\left(\frac{R_2^{N+n-m}-r^{N+n-m}}{N+n-m}\right)\right]. \tag{2.22}$$

Putting all the above together we have derived

Theorem 2.3. *Let $f\in C^n(\bar{A}), n\in\mathbb{N}$, be radial, i.e. $f(x)=g(r), R_1\leq r\leq R_2, x\in\bar{A}$.*

Then

$$\left|f(x)-\frac{\int_A f(y)dy}{Vol(A)}\right|=\left|g(r)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}g(s)s^{N-1}ds\right|\leq$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\left[\sum_{k=1}^{n-1}|g^{(k)}(r)|\left|\sum_{m=0}^{k}\frac{(-1)^m r^m}{m!(k-m)!}\left(\frac{R_2^{N+k-m}-R_1^{N+k-m}}{N+k-m}\right)\right|+\right.$$

$$\left\|g^{(n)}\right\|_\infty \left[\sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} \left[r^{n-m}\left(\frac{r^{m+N}-R_1^{m+N}}{m+N}\right) + r^m\left(\frac{R_2^{N+n-m}-r^{N+n-m}}{N+n-m}\right)\right]\right] \tag{2.23}$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right)\left\{\sum_{k=1}^{n-1}\left|\frac{\partial^k f(x)}{\partial r^k}\right| \left|\sum_{m=0}^{k}\frac{(-1)^m |x|^m}{m!(k-m)!}\left(\frac{R_2^{N+k-m}-R_1^{N+k-m}}{N+k-m}\right)\right| + \left\|\frac{\partial^n f}{\partial r^n}\right\|_{\infty,\bar{A}}\cdot\right.$$

$$\left.\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\left[|x|^{n-m}\left(\frac{|x|^{m+N}-R_1^{m+N}}{m+N}\right) + |x|^m\left(\frac{R_2^{N+n-m}-|x|^{N+n-m}}{N+n-m}\right)\right]\right]\right\}. \tag{2.24}$$

We give

Corollary 2.1. *Let* $f \in C^n(\bar{A})$, $n \in \mathbb{N}$, *be radial i.e.* $f(x) = f(r)$, $R_1 \leq r \leq R_2$, $x \in \bar{A}$. *Assume* $\dfrac{\partial^i f(x_0)}{\partial^i r^i} = g^{(i)}(r_0) = 0, i = 1, \ldots, n-1,$ *for* $r_0 \in [R_1, R_2]$, $x_0 = r_0\omega \in \bar{A}$, $\omega \in S^{N-1}$.
Then

$$\left|f(x_0) - \frac{\int_A f(y)dy}{Vol(A)}\right| = \left|g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right)\int_{R_1}^{R_2} g(s)s^{N-1}ds\right| \leq$$

$$\frac{N\|g^{(n)}\|_\infty}{R_2^N - R_1^N}\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\right.$$

$$\left.\left[r_0^{n-m}\left(\frac{r_0^{m+N}-R_1^{m+N}}{m+N}\right) + r_0^m\left(\frac{R_2^{N+n-m}-r_0^{N+n-m}}{N+n-m}\right)\right]\right] \tag{2.25}$$

$$= \frac{N\|\frac{\partial^n f}{\partial r^n}\|_{\infty,\overline{A}}}{R_2^N - R_1^N}\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\right.$$

$$\left.\left[|x_0|^{n-m}\left(\frac{|x_0|^{m+N}-R_1^{m+N}}{m+N}\right)+|x_0|^m\left(\frac{R_2^{N+n-m}-|x_0|^{N+n-m}}{N+n-m}\right)\right]\right]. \tag{2.26}$$

We also have the extreme cases.

Corollary 2.2. *Let $f \in C^n(\bar{A})$, $n \in \mathbb{N}$, be radial ; $f(x) = g(r)$, $r \in [R_1, R_2]$, $x \in \bar{A}, x = r\omega$. Assume that $\frac{\partial^i f(x)}{\partial r^i}$, $i = 1, \ldots, n-1$, are zero on $\partial B(0, R_1)$, i.e. $g^{(i)}(R_1) = 0$, $i = 1, \ldots, n-1$. Then for $x \in \partial B(0, R_1)$ we have*

$$\left|f(x) - \frac{\int_A f(y)dy}{Vol(A)}\right| = \left|g(R_1) - \frac{N}{R_2^N - R_1^N}\int_{R_1}^{R_2} g(s)s^{N-1}ds\right|$$

$$\leq \left(\frac{N}{R_2^N - R_1^N}\right)\|g^{(n)}\|_\infty\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}R_1^m\left(\frac{R_2^{N+n-m}-R_1^{N+n-m}}{N+n-m}\right)\right] \tag{2.27}$$

(2.28)
$$= \left(\frac{N}{R_2^N - R_1^N}\right)\left\|\frac{\partial^n f}{\partial r^n}\right\|_{\infty,\bar{A}}\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}R_1^m\left(\frac{R_2^{N+n-m}-R_1^{N+n-m}}{N+n-m}\right)\right].$$

Another extreme case follows.

Corollary 2.3. *Let $f \in C^n(\bar{A})$, $n \in \mathbb{N}$, be radial; $f(x) = g(r)$, $r \in [R_1, R_2]$, $x \in \bar{A}; x = r\omega$. Assume that $\frac{\partial^i f(x)}{\partial r^i}, i = 1, \ldots, n-1$, are zero on $\partial B(0, R_2)$, i.e., $g^{(i)}(R_2) = 0, i = 1, \ldots, n-1$. Then for $x \in \partial B(0, R_2)$ we have*

$$\left|f(x) - \frac{\int_A f(y)dy}{Vol(A)}\right| = \left|g(R_2) - \frac{N}{R_2^N - R_1^N}\int_{R_1}^{R_2} g(s)s^{N-1}ds\right|$$

$$\leq \left(\frac{N}{R_2^N - R_1^N}\right)\|g^{(n)}\|_\infty\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}R_2^{n-m}\left(\frac{R_2^{m+N}-R_1^{m+N}}{m+N}\right)\right] \tag{2.29}$$

(2.30)

$$= \left(\frac{N}{R_2^N - R_1^N}\right) \left\| \frac{\partial^n f}{\partial r^n} \right\|_{\infty, \bar{A}} \left[\sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} R_2^{n-m} \left(\frac{R_2^{m+N} - R_1^{m+N}}{m+N} \right) \right].$$

Optimality follows.

Proposition 2.1. *Inequality* (2.25) *is sharp, namely it is attained by*

$$g^*(y) := (y - r_0)^n, \quad y, \quad r_0 \in [R_1, R_2], \ \ \textit{when } n \ \textit{ is even} .$$

P r o o f. Notice

$$g^{*(j)}(r_0) = 0, \quad j = 0, 1, \ldots, \quad n-1 \ \text{ and } \ \|g^{*(n)}\|_\infty = n!.$$

We see

$$L.H.S(2.25) = \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} (s - r_0)^n s^{N-1} ds =$$

$$\left(\frac{N}{R_2^N - R_1^N}\right) \left[\sum_{m=0}^{n} \binom{n}{m} (-1)^m r_0^m \left(\frac{R_2^{N+n-m} - R_1^{N+n-m}}{N+n-m} \right) \right]. \tag{2.31}$$

Next we observe that

$$R.H.S\,(2.25) = \left(\frac{N}{R_2^N - R_1^N}\right) \left[\sum_{m=0}^{n} \binom{n}{m} (-1)^m \right.$$

$$\left.\left[\left(\frac{r_0^{n+N} - r_0^{n-m} R_1^{m+N}}{m+N} \right) + \left(\frac{r_0^m R_2^{N+n-m} - r_0^{N+n}}{N+n-m} \right) \right] \right] \tag{2.32}$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right) \left\{ \sum_{m=0}^{n} \binom{n}{m} (-1)^m \left[\frac{r_0^m R_2^{N+n-m}}{N+n-m} - \frac{r_0^{n-m} R_1^{m+N}}{m+N} \right] \right.$$

$$\left. + r_0^{n+N} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \left[\frac{1}{N+m} - \frac{1}{N+n-m} \right] \right\} \tag{2.33}$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right)\left\{\sum_{m=0}^{n}\binom{n}{m}(-1)^m \frac{r_0^m R_2^{N+n-m}}{N+n-m} - \right.$$

$$\left.\sum_{m=0}^{n}\binom{n}{m}(-1)^m \frac{r_0^{n-m} R_1^{m+N}}{m+N}\right\} \tag{2.34}$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right)\left\{\sum_{m=0}^{n}\binom{n}{m}(-1)^m \frac{r_0^m R_2^{N+n-m}}{N+n-m} - \right.$$

$$\left.\sum_{m=0}^{n}\binom{n}{m}(-1)^m r_0^m \frac{R_1^{N+n-m}}{N+n-m}\right\} \tag{2.35}$$

$$= \left(\frac{N}{R_2^N - R_1^N}\right)\left\{\sum_{m=0}^{n}\binom{n}{m}(-1)^m r_0^m \frac{(R_2^{N+n-m} - R_1^{N+n-m})}{N+n-m}\right\}$$

$$= L.H.S \quad (2.25). \tag{2.36}$$

That is proving the claim. □

The other optimal case follows.

Proposition 2.2. *Inequality* (2.25) *is sharp , namely it is asymptotically attained by* $g^*(y) := |y - r_0|^{n-1+\alpha}$, y, $r_0 \in [R_1, R_2]$, $1 < \alpha \le T$, *in the case of* n *is odd.*

Proof. It holds $g^{*(k)}(r_0) = 0$, $k = 0, 1, \ldots, n-1$.
Also we have

$$g^{*(n)}(y) = (n-1+\alpha)(n-2+\alpha)\cdots(\alpha+1)\alpha|y - r_0|^{\alpha-1} sign(y - r_0).$$

That is

$$\left|g^{*(n)}(y)\right| = \left(\prod_{j=1}^{n}(n - j + \alpha)\right)|y - r_0|^{\alpha-1},$$

and

$$\left\| g^{*(n)} \right\|_\infty = \left(\prod_{j=1}^{n} (n-j+\alpha) \right) (max\{R_2 - r_0, r_0 - R_1\})^{\alpha-1} \tag{2.37}$$

Consequently,

$$L.H.S(2.25) = \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} |s-r_0|^{n-1+\alpha} s^{N-1} ds \xrightarrow{\alpha \to 1}$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} |s-r_0|^{n} s^{N-1} ds \tag{2.38}$$

$$= \left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \sum_{m=0}^{n} \binom{n}{m} (-1)^m \left[r_0^{n-m} \left(\frac{r_0^{m+N} - R_1^{m+N}}{m+N} \right) \right.\right. \tag{2.39}$$

$$\left.\left. + r_0^m \left(\frac{R_2^{N+n-m} - r_0^{N+n-m}}{N+n-m} \right) \right] \right\} =: \mu. \tag{2.40}$$

Next we find

$$R.H.S(2.25) = \left(\frac{N}{R_2^N - R_1^N} \right) \frac{\left(\prod_{j=1}^{n} (n-j+\alpha) \right) (\max\{R_2 - r_0, r_0 - R_1\})^{\alpha-1}}{n!}$$

$$\left\{ \sum_{m=0}^{n} \binom{n}{m} (-1)^m \left\{ r_0^{n-m} \left(\frac{r_0^{m+N} - R_1^{m+N}}{m+N} \right) + r_0^m \left(\frac{R_2^{N+n-m} - r_0^{N+n-m}}{N+n-m} \right) \right\} \right\} \tag{2.41}$$

$$\xrightarrow{\alpha \to 1} \left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \sum_{m=0}^{n} \binom{n}{m} (-1)^m \right.$$

$$\left. \left\{ r_0^{n-m} \left(\frac{r_0^{m+N} - R_1^{m+N}}{m+N} \right) + r_0^m \left(\frac{R_2^{N+n-m} - r_0^{N+n-m}}{N+n-m} \right) \right\} \right\} = \mu. \tag{2.42}$$

I.e. $L.H.S$ (2.25), $R.H.S$(2.25) $\to \mu$, proving asymptotic attainability and sharpness of (2.25). $\square$

Optimality of external inequality (2.25) follows.

Proposition 2.3. *External inequality* (2.25) *is asymptotically attained, that is sharp as follows:*

(i) when n is even, then optimal function is $f^*(w) := (|w| - |x_0|)^n$, $w \in \bar{A}$.

(ii) when n is odd, then optional function is $f^*(w) :=\mid |w| - |x_0| \mid^{n-1+\alpha}$, $1 < \alpha \leq T$, $w \in \bar{A}$.

Similarly we obtain

Proposition 2.4. *Inequalities* (2.23), (2.27) *and* (2.29) *are asymptotically attained, therefore sharp, as inequality* (2.25).

A simple but general result follows.

Theorem 2.4. *Let $\emptyset \neq \mathcal{R}$ be a convex bounded region of $\mathbb{R}^N, N \geq 1$. Let $f \in C^1(\overline{\mathcal{R}})$. Then*

$$\left| f(x) - \frac{\int_{\mathcal{R}} f(y)dy}{Vol(\mathcal{R})} \right| \leq \frac{\|\nabla f\|_\infty}{Vol(\mathcal{R})} \int_{\mathcal{R}} |x-y|dy, x \in \overline{\mathcal{R}}. \tag{2.43}$$

P r o o f. We see that

$$\left| f(x) - \frac{\int_{\mathcal{R}} f(y)dy}{Vol(\mathcal{R})} \right| = \frac{1}{Vol(\mathcal{R})} \left| f(x)Vol(\mathcal{R}) - \int_{\mathcal{R}} f(y)dy \right| =$$

$$\frac{1}{Vol(\mathcal{R})} \left| \int_{\mathcal{R}} (f(x) - f(y))dy \right| \leq \tag{2.44}$$

$$\frac{1}{Vol(\mathcal{R})} \int_{\mathcal{R}} |f(x) - f(y)|dy =$$

(z belongs to the line segment from $x \, to \, y$)

$$\frac{1}{Vol(\mathcal{R})} \int_{\mathcal{R}} |\nabla f(z) \cdot (x-y)|dy \tag{2.45}$$

$$\leq \frac{1}{Vol(\mathcal{R})} \int_{\mathcal{R}} |\nabla f(z)| \cdot |x-y| dy \leq \frac{\|\nabla f\|_\infty}{Vol(\mathcal{R})} \int_{\mathcal{R}} |x-y| dy. \tag{2.46}$$

□

Specializing on the shell and sphere me have

Proposition 2.5. *Let $f \in C^1(\bar{A})$, or $f \in C^1(\overline{B(0,R)})$, $R>0$. Then*

(i)

$$\left| f(x) - \frac{N\Gamma(\frac{N}{2})}{2\pi^{N/2}(R_2^N - R_1^N)} \int_A f(y) dy \right| \leq$$

$$\frac{N\Gamma(\frac{N}{2})}{2\pi^{N/2}(R_2^N - R_1^N)} \|\nabla f\|_\infty \int_A |x-y|\, dy, \quad x \in \bar{A}; \tag{2.47}$$

also it holds

(ii)

$$\left| f(x) - \frac{N\Gamma(\frac{N}{2})}{2\pi^{N/2} R^N} \int_{B(0,R)} f(y) dy \right| \leq \frac{N\Gamma(\frac{N}{2})}{2\pi^{N/2} R^N} \|\nabla f\|_\infty \int_{B(0,R)} |x-y|\, dy,$$

$x \in \overline{B(0,R)}$.

More precise Ostrowski type inequalities for general, not necessarily radial functions, follow.

Theorem 2.5. *Let $f \in C^1(\bar{A})$, $x \in \bar{A}$, $x = r\omega$, $r>0$. Then*

$$\left| f(x) - \frac{\int_A f(y) dy}{Vol(A)} \right| \leq \left| f(x) - \frac{\int_{S^{N-1}} f(r\omega) d\omega}{\omega_N} \right|$$

$$+ \left(\frac{N}{R_2^N - R_1^N} \right) \left\| \frac{\partial f}{\partial r} \right\|_\infty \left[|x| \left(\frac{2|x|^N - (R_1^N + R_2^N)}{N} \right) \right.$$

$$\left. + \left(\frac{R_1^{N+1} + R_2^{N+1} - 2|x|^{N+1}}{N+1} \right) \right] \tag{2.48}$$

$$\leq \left| f(x) - \frac{\Gamma(\frac{N}{2}) \int_{S^{N-1}} f(r\omega) d\omega}{2\pi^{N/2}} \right| + \left(\frac{N}{R_2^N - R_1^N} \right) \|\nabla f\|_\infty$$

$$\left[|x|\left(\frac{2|x|^N-(R_1^N+R_2^N)}{N}\right)+\left(\frac{R_1^{N+1}+R_2^{N+1}-2|x|^{N+1}}{N+1}\right)\right]. \tag{2.49}$$

P r o o f. Applying internal (2.10) to $f(r\omega)$ we obtain

$$\left|f(r\omega)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2} f(s\omega)s^{N-1}ds\right|\leq$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\left\|\frac{\partial f(r\omega)}{\partial r}\right\|_{\infty,r\in[R_1,R_2]}\left[r\left(\frac{2r^N-(R_1^N+R_2^N)}{N}\right)\right.$$

$$\left.+\left(\frac{R_1^{N+1}+R_2^{N+1}-2r^{N+1}}{N+1}\right)\right] \tag{2.50}$$

$$\leq\left(\frac{N}{R_2^N-R_1^N}\right)\left\|\frac{\partial f}{\partial r}\right\|_{\infty,\bar{A}}\left[|x|\left(\frac{2|x|^N-(R_1^N+R_2^N)}{N}\right)\right.$$

$$\left.+\left(\frac{R_1^{N+1}+R_2^{N+1}-2|x|^{N+1}}{N+1}\right)\right]. \tag{2.51}$$

Therefore

$$\left|\frac{\int_{S^{N-1}} f(r\omega)d\omega}{\omega_N}-\frac{N}{\omega_N(R_2^N-R_1^N)}\int_{S^{n-1}}\left(\int_{R_1}^{R_2} f(s\omega)s^{N-1}ds\right)d\omega\right|$$

$$\leq\left(\frac{N}{R_2^n-R_1^N}\right)\left\|\frac{\partial f}{\partial r}\right\|_{\infty}\left[|x|\left(\frac{2|x|^N-(R_1^N+R_2^N)}{N}\right)\right.$$

$$\left.+\left(\frac{R_1^{N+1}+R_2^{N+1}-2|x|^{N+1}}{N+1}\right)\right], \tag{2.52}$$

proving the claim. □

We continue with

Theorem 2.6. *Let* $f \in C^n(\bar{A})$, $n \in \mathbb{N}$, $x \in \bar{A}$, $x = r\omega$, $r > 0$. *Then*

$$\left| f(x) - \frac{\int_A f(y)dy}{Vol(A)} \right| \leq \left| f(x) - \frac{\Gamma(\frac{N}{2}) \int_{S^{N-1}} f(r\omega)d\omega}{2\pi^{N/2}} \right|$$

$$+ \left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \frac{\Gamma(\frac{N}{2})}{2\pi^{N/2}} \left\{ \sum_{k=1}^{n-1} \left(\int_{S^{N-1}} \left| \frac{\partial^k f(r\omega)}{\partial r^k} \right| d\omega \right) \right.\right.$$

$$\left.\left| \sum_{m=0}^{k} \frac{(-1)^m |x|^m}{m!(k-m)!} \left(\frac{R_2^{N+k-m} - R_1^{N+k-m}}{N+k-m} \right) \right| \right\} +$$

$$\left\| \frac{\partial^n f}{\partial r^n} \right\|_\infty \left[\sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} \left[|x|^{n-m} \left(\frac{|x|^{m+N} - R_1^{m+N}}{m+N} \right) \right.\right.$$

$$\left.\left.\left. +|x|^m \left(\frac{R_2^{N+n-m} - |x|^{N+n-m}}{N+n-m} \right) \right] \right] \right\}. \tag{2.53}$$

Proof. Applying internal (2.23) to $f(r\omega)$ we get

$$\left| f(r\omega) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right| \leq$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \sum_{k=1}^{n-1} \left| \frac{\partial^k f(r\omega)}{\partial r^k} \right| \left| \sum_{m=0}^{k} \frac{(-1)^m \, |x|^m}{m! \, (k-m)!} \right.\right.$$

$$\left.\left(\frac{R_2^{N+k-m} - R_1^{N+k-m}}{N+k-m} \right) \right| + \left\| \frac{\partial^n f}{\partial r^n} \right\|_{\infty, \bar{A}} \left[\sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} \right.$$

$$\left.\left.\left[|x|^{n-m} \left(\frac{|x|^{m+N} - R_1^{m+N}}{m+N} \right) + |x|^m \left(\frac{R_2^{N+n-m} - |x|^{N+n-m}}{N+n-m} \right) \right] \right] \right\} \tag{2.54}$$

Hence it holds

$$\left|\frac{\Gamma(\frac{N}{2})}{2\pi^{N/2}}\int_{S^{N-1}} f(r\omega)d\omega - \frac{1}{Vol(A)}\int_A f(y)dy\right| \leq$$

$$\left(\frac{N}{R_2^N - R_1^N}\right)\left\{\frac{\Gamma(N/2)}{2\pi^{N/2}}\left\{\sum_{k=1}^{n-1}\left(\int_{S^{N-1}}\left|\frac{\partial^k f(r\omega)}{\partial r^k}\right| d\omega\right)\right.\right.$$

$$\left.\left|\sum_{m=0}^{k}\frac{(-1)^m|x|^m}{m!(k-m)!}\left(\frac{R_2^{N+k-m} - R_1^{N+k-m}}{N+k-m}\right)\right|\right\}$$

$$+\left\|\frac{\partial^n f}{\partial r^n}\right\|_{\infty,\bar{A}}\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\left[|x|^{n-m}\left(\frac{|x|^{m+N} - R_1^{m+N}}{m+N}\right)\right.\right.$$

$$\left.\left.\left.+|x|^m\left(\frac{R_2^{N+n-m} - |x|^{N+n-m}}{N+n-m}\right)\right]\right]\right\}, \tag{2.55}$$

proving the claim. □

We also give

Proposition 2.6. *Let $f \in C^n(\bar{A})$, $n \in \mathbb{N}$, such that $\frac{\partial^i f}{\partial r^i}$, $i = 1, \ldots, n-1$, are zero on $\partial B(0, r_0)$, $r_0 \in (R_1, R_2)$. Then for $x_0 \in \partial B(0, r_0)$ we have*

$$\left|f(x_0) - \frac{\int_A f(y)dy}{Vol(A)}\right| \leq \left|f(x_0) - \frac{\Gamma(N/2)}{2\pi^{N/2}}\int_{S^{N-1}} f(r_0\omega)d\omega\right|$$

$$+\left(\frac{N}{R_2^N - R_1^N}\right)\left\|\frac{\partial^n f}{\partial r^n}\right\|_\infty\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\left[|x_0|^{n-m}\left(\frac{|x_0|^{m+N} - R_1^{m+N}}{m+N}\right)\right.\right.$$

$$\left.\left.+|x_0|^m\left(\frac{R_2^{N+n-m} - |x_0|^{N+n-m}}{N+n-m}\right)\right]\right]. \tag{2.56}$$

Proof. Applying internal (2.25) to $f(r\omega)$ we find

$$\left|f(r_0\omega) - \left(\frac{N}{R_2^N - R_1^N}\right)\int_{R_1}^{R_2} f(s\omega)s^{N-1}ds\right|$$

$$\leq \left(\frac{N}{R_2^N - R_1^N}\right)\left\|\frac{\partial^n f}{\partial r^n}\right\|_{\infty,\bar{A}}\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\right.$$

$$\left.\left[|x_0|^{n-m}\left(\frac{|x_0|^{m+N} - R_1^{m+N}}{m+N}\right) + |x_0|^m\left(\frac{R_2^{N+n-m} - |x_0|^{N+n-m}}{N+n-m}\right)\right]\right]. \tag{2.57}$$

Therefore

$$\left|\frac{\Gamma(\frac{N}{2})}{2\pi^{N/2}}\int_{S^{N-1}} f(r_0\omega)d\omega - \frac{1}{Vol(A)}\int_A f(y)dy\right| \leq$$

$$\left(\frac{N}{R_2^N - R_1^N}\right)\left\|\frac{\partial^n f}{\partial r^n}\right\|_\infty\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}\left[|x_0|^{n-m}\left(\frac{|x_0|^{m+N} - R_1^{m+N}}{m+N}\right)\right.\right.$$

$$\left.\left.+|x_0|^m\left(\frac{R_2^{N+n-m} - |x_0|^{N+n-m}}{N+n-m}\right)\right]\right]. \tag{2.58}$$

Claim is clear. $\square$

We present the extreme cases.

Proposition 2.7. *Let $f \in C^n(\bar{A})$, $n \in \mathbb{N}$ such that $\frac{\partial^i f}{\partial r^i}$, $i = 1, \ldots, n-1$, are zero on $\partial B(0, R_1)$. Then for $x_0 \in \partial B(0, R_1)$ we get*

$$\left|f(x_0) - \frac{\int_A f(y)dy}{Vol(A)}\right| \leq \left|f(x_0) - \frac{\Gamma(\frac{N}{2})}{2\pi^{N/2}}\int_{S^{N-1}} f(R_1\omega)d\omega\right|$$

$$+\left(\frac{N}{R_2^N - R_1^N}\right)\left\|\frac{\partial^n f}{\partial r^n}\right\|_\infty\left[\sum_{m=0}^{n}\frac{(-1)^m}{m!(n-m)!}R_1^m\left(\frac{R_2^{N+n-m} - R_1^{N+n-m}}{N+n-m}\right)\right] \tag{2.59}$$

Proof. By internal (2.27). □

We finish our main results with

Proposition 2.8. *Let $f \in C^n(\bar{A})$, $n \in \mathbb{N}$, such that $\frac{\partial^i f}{\partial r^i}$, $i = 1, \ldots, n-1$, are zero on $\partial B(0, R_2)$. Then for $x_0 \in \partial B(0, R_2)$ we find*

$$\left| f(x_0) - \frac{\int_A f(y)dy}{Vol(A)} \right| \leq \left| f(x_0) - \frac{\Gamma \frac{(N)}{2}}{2\pi^{N/2}} \int_{S^{N-1}} f(R_2\omega)d\omega \right|$$

$$+ \left(\frac{N}{R_2^N - R_1^N} \right) \left\| \frac{\partial^n f}{\partial r^n} \right\|_\infty \left[\sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!} R_2^{n-m} \left(\frac{R_2^{m+N} - R_1^{m+N}}{m+N} \right) \right]. \tag{2.60}$$

Proof. By internal (2.29). □

The radial derivatives appearing in the right hand sides of our inequalities can be expressed and estimated by regular partial derivatives in terms of $x_1, \ldots, x_N$. Please see Addendum next.

3. Addendum.

I. Let $u \in C^n(\overline{B(0,R)})$, the open ball $B(0,R) \subseteq \mathbb{R}^N$, n, $N \in \mathbb{N}$. Here $x = (x_1, \cdots, x_N) \in \overline{B(0,R)}$ and the *radial derivative* of u is given also by

$$\frac{\partial u(x)}{\partial r} = \nabla u(x) \cdot \frac{x}{|x|}, \quad x \neq 0.$$

I.e.

$$\frac{\partial u(x)}{\partial r} = \frac{1}{|x|} \left(\sum_{i=1}^{N} \frac{\partial u(x)}{\partial x_i} x_i \right), \quad x \neq 0. \tag{3.1}$$

In general for $1 \leq l \leq n$ it holds (by induction) that

$$\frac{\partial^l u(x)}{\partial r^l} = \frac{1}{|x|^l} \left[\sum_{\substack{k_1;\ldots;k_N, \\ \sum_{j=1}^N k_j = l, k_j \in \mathbb{Z}_+}} \frac{l!}{\Pi_{j=1}^N k_j!} \cdot \frac{\partial^l u(x)}{\Pi_{j=1}^N \partial^{k_j} x_j} \Pi_{j=1}^N x_j^{k_j} \right], \ x \neq 0. \tag{3.2}$$

E.g. when $n = N = 2$ we get

$$\frac{\partial^2 u(x)}{\partial r^2} = \frac{1}{|x|^2}\left[\frac{\partial^2 u(x)}{\partial x_1^2}x_1^2 + 2\frac{2\partial^2 u(x)}{\partial x_1 \partial x_2}x_1 x_2 + \frac{\partial^2 u(x)}{\partial x_2^2}x_2^2\right], \ x \neq 0. \tag{3.3}$$

Thus we have

$$\left|\frac{\partial^l u(x)}{\partial r^l}\right| \leq \sum_{k_1;\ldots;k_N, \sum_{j=1}^N k_j = l, k_j \in \mathbb{Z}_+} \frac{l!}{\Pi_{j=1}^N k_j!} \cdot \left|\frac{\partial^l u(x)}{\Pi_{j=1}^N \partial^{k_j} x_j}\right|, \ x \neq 0, \tag{3.4}$$

or better in brief,

$$\left|\frac{\partial^l u(x)}{\partial r^l}\right| \leq \left(\sum_{i=1}^N \left|\frac{\partial}{\partial x_i}\right|\right)^l (u(x)), \tag{3.5}$$

all $x \in \overline{B(0,R)} - \{0\}$, all $1 \leq l \leq n$.

So if all the u partial derivatives vanish then the corresponding radial derivative is zero. Consequently, from (3.5) it holds for the essential suprema $\|\cdot\|_\infty$ that

$$\left\|\frac{\partial^l u}{\partial r^l}\right\|_{\infty, \overline{B(0,R)}} \leq \left(\sum_{i=1}^N \left\|\frac{\partial}{\partial x_i}\right\|_{\infty, \overline{B(0,R)}}\right)^l (u) < +\infty. \tag{3.6}$$

II. Continuing and specializing in the study of higher order radial derivatives of radial functions.

Let now u be radial i.e. $u(x) = g(|x|) = g(r); \quad x = r\omega, \quad r \in \mathbb{R}_+, \quad \omega \in S^{N-1}$. Here we will assume $x \neq 0$ and $N = 2$. Let further $x_1, x_2 \neq 0$, then by chain rule one has

$$g'(r) = \frac{\partial u}{\partial x_1}\frac{r}{x_1} = \frac{\partial u}{\partial x_2}\frac{r}{x_2}, \ \text{ where } r = \sqrt{x_1^2 + x_2^2}. \tag{3.7}$$

I.e.

$$g'(r) = \frac{1}{2}\left[\frac{\partial u}{\partial x_1}\frac{|x|}{x_1} + \frac{\partial u}{\partial x_2}\frac{|x|}{x_2}\right]. \tag{3.8}$$

So one has

$$|\nabla u(x)| = |g'(r)| = \left|\frac{\partial u(x)}{\partial r}\right|, \tag{3.9}$$

for any $x \in \overline{B(0,R)}$.

But if u radial, not necessarily $\frac{\partial u}{\partial x_j}$ is radial. Here we put $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Again for $x_1, x_2 \neq 0$ and via chain rule we obtain

$$g''(r) = \frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2 \partial x_1} \tan\,\theta = \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_1 \partial x_2} \cot\,\theta. \tag{3.10}$$

That is

$$g''(r) = \frac{1}{2}\left[\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}(\tan\,\theta + \cot\,\theta)\right]. \tag{3.11}$$

Or better, by using the Laplacian Δ we get

$$g''(r) = \frac{1}{2}\Delta u(x) + \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\csc(2\,\theta), \tag{3.12}$$

or

$$g''(r) = \frac{1}{2}\left(\Delta u(x) + \frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\frac{|x|^2}{x_1 x_2}\right), \tag{3.13}$$

$x_1, x_2 \neq 0$.
Similarly , one has that

$$g'''(r) = \frac{\partial^3 u(x)}{\partial x_1^3}\cos\theta + 2\frac{\partial^3 u(x)}{\partial x_2 \partial x_1^2}\sin\theta + \frac{\partial^3 u(x)}{\partial x_2^2 \partial x_1}\frac{\sin^2\theta}{\cos\theta}, \quad x_1, x_2 \neq 0. \tag{3.14}$$

Also we find

$$g'''(r) = 2\frac{\partial^3 u(x)}{\partial x_1 \partial x_2^2}\cos\,\theta + \frac{\partial^3 u(x)}{\partial x_2^3}\sin\theta + \frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2}\frac{\cos^2\theta}{\sin\theta}, \quad x_1, x_2 \neq 0. \tag{3.15}$$

That is, by (3.14) and (3.15) we have

$$g'''(r) = \frac{1}{2}\left[\frac{\partial^3 u(x)}{\partial x_1^3}\cos\theta + \frac{\partial^3 u(x)}{\partial x_2^3}\sin\theta + \right.$$

$$\left. 2\left(\frac{\partial^3 u(x)}{\partial x_2 \partial x_1^2}\sin\,\theta + \frac{\partial^3 u(x)}{\partial x_1 \partial x_2^2}\cos\theta\right) + \frac{\partial^3 u(x)}{\partial x_2^2 \partial x_1}\frac{\sin^2\theta}{\cos\theta} + \frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2}\frac{\cos^2\theta}{\sin\theta}\right], \tag{3.16}$$

$$x_1, x_2 \neq 0.$$

Clearly, it holds

$$g'''(r) = \frac{1}{2}\left[\frac{\partial^3 u(x)}{\partial x_1^3}\frac{x_1}{|x|} + \frac{\partial^3 u(x)}{\partial x_2^3}\frac{x_2}{|x|} + \right.$$

$$\left.\frac{\partial^3 u(x)}{\partial x_1^2 \partial x_2}\left(\frac{x_1^2}{x_2|x|} + 2\frac{x_2}{|x|}\right) + \frac{\partial^3 u(x)}{\partial x_2^2 \partial x_1}\left(\frac{x_2^2}{x_1|x|} + 2\frac{x_1}{|x|}\right)\right], x_1, x_2 \neq 0. \tag{3.17}$$

Not general formula, as in (3.2), can be derived for $g^{(l)}(r)$, $l \in \mathbb{N}$, in the radial case. Of course (3.2) is valid for both non-radial and radial cases. Notice in (3.17), (3.10), (3.14) and (3.15) are used less number of terms than in the corresponding non-radial cases.

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