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INTERVAL OSCILLATION FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH A DAMPING TERM

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Abstract. It is the purpose of this paper to give oscillation criteria for the second order nonlinear differential equation with a damping term

$$(a(t)y'(t))' + p(t)y'(t) + q(t)|y(t)|^{\alpha-1}y(t) = 0, \quad t \geq t_0,$$

where $\alpha \geq 1$, $a \in C^1([t_0,\infty);(0,\infty))$ and $p,q \in C([t_0,\infty);\mathbb{R})$. Our results here are different, generalize and improve some known results for oscillation of second order nonlinear differential equations that are different from most known ones in the sencse they are based on the information only on a sequence of subintervals of $[t_0,\infty)$, rather than on the whole half-line and can be applied to extreme cases such as $\int_{t_0}^{\infty} q(t)\,dt = -\infty$. Our results are illustrated with an example.

1. Introduction. We are concerned with the oscillatory behavior of the second order nonlinear differential equation with a damping term

$$\left(a(t)y'(t)\right)' + p(t)y'(t) + q(t)|y(t)|^{\alpha-1}y(t) = 0, \tag{1.1}$$

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where $\alpha \geq 1$, $a \in C^1([t_0,\infty);(0,\infty))$ and $p,q \in C([t_0,\infty);\mathbb{R})$. Our attention is restricted to those solutions of equation (1.1) that satisfy $\sup\{|y(t)| : t \geq t_1\} > 0$. We make a standing hypothesis that (1.1) does possess such solutions. By a solution of equation (1.1) we mean a continuously differentiable function $y(t) : [t_0,t_1) \to \mathbb{R}$, $t_1 > t_0$ such that $a(t)y'(t)$ is continuously and differentiable for $t \in [t_0,t_1)$ and satisfies (1.1) for all $t \in [t_0,t_1)$. In the sequel it will be always assumed that solutions of equation (1.1) exist for any $t_0 \geq 0$. A solution of equation (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory.

In the last few decades, there has been increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of different classes of second order differential equations [1–43]. In particular, much work has been done on the following particular cases of (1.1):

$$y''(t) + q(t)y(t) = 0, \tag{1.2}$$

$$\left(r(t)y'(t)\right)' + q(t)y(t) = 0, \tag{1.3}$$

$$y''(t) + q(t)|y(t)|^{\alpha-1}y(t) = 0. \tag{1.4}$$

An important tool in the study of the oscillatory behavior of solutions of these equations is the averaging technique which goes back as far as the classical result of Wintner [34] where it was proved that (1.2) is oscillatory if

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(u)\,duds = \infty. \tag{1.5}$$

Hartman [15] proved that the limit in (1.5) cannot be replaced by the limit supremum and proved that the condition

$$-\infty < \liminf_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(u)\,duds < \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}q(u)\,duds \leq \infty, \tag{1.6}$$

implies that every solution of (1.2) is oscillatory. Kamenev [18] improved Wintner's result by proving that the condition

$$\lim_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n q(s)\,ds = \infty, \quad \text{for some integer } n > 1, \tag{1.7}$$

is sufficient condition for the oscillation of (1.2). Yan [39] proved that if

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n q(s)\,ds < \infty, \quad \text{for some integer } n > 1,$$

and there exists a function ϕ on $[t_0, \infty)$ satisfying

$$\int_{t_0}^{\infty} \phi_+^2(t)\, dt = \infty \quad \text{where } \phi_+(t) = \max\{\phi(t), 0\},$$

and

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n q(s)\, ds > \sup_{u\geq t_0} \phi(u), \tag{1.8}$$

then every solution of equation (1.2) is oscillatory. Philos [27] further improved Kamenev's result by proving the following: Suppose there exist continuous functions

$$H,\ h : \mathbb{D} := \{(t,s) : t \geq s \geq t_0\} \to \mathbb{R},$$

such that

$$H(t,t) = 0, \quad t \geq t_0, \quad H(t,s) > 0, \quad t > s \geq t_0,$$

and H has a continuous and nonpositive partial derivative on $\mathbb{D}$ with respect to the second variable and satisfies

$$\frac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)}. \tag{1.9}$$

Further, suppose that

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\, q(s) - \frac{1}{4} h^2(t,s) \right] ds = \infty. \tag{1.10}$$

Then every solution of equation (1.2) is oscillatory. We note, however, that when $q(t) = \dfrac{\gamma}{t^2}$, (1.2) reduces to the well-known Euler–Cauchy equation

$$u''(t) + \frac{\gamma}{t^2} u(t) = 0, \quad t \geq 1, \tag{1.11}$$

to which none of the above mentioned oscillation criteria is applicable. In fact, the Euler–Cauchy equation (1.11) is oscillatory if $\gamma > \dfrac{1}{4}$, and nonoscillatory if $\gamma \leq \dfrac{1}{4}$, see [19]. For oscillation of equation (1.3), Leighton [20] proved that, if

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t)\, dt = \infty, \tag{1.12}$$

then every solution of equation (1.3) is oscillatory. Willett [32] used the transformation

$$\tau = \left(\int_t^{\infty} \frac{ds}{r(s)} \right)^{-1} \quad \text{and} \quad u(t) = \tau^{-1}(y(t)),$$

to establish a new version of Leighton's criterion and obtained the following oscillation criterion. If

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} q(t) \left(\int_{t}^{\infty} \frac{ds}{r(s)} \right)^2 dt = \infty, \tag{1.13}$$

then every solution of (1.3) is oscillatory. We note, however, that the oscillation criteria of Leighton and Willett are not applicable to the equation

$$\left(t^2 u'(t)\right)' + \gamma u(t) = 0, \quad t > 0, \tag{1.14}$$

where γ is a positive constant. Kong [19], Li [21], Li and Yeh [26], Rogovechenkov [28], and Yu [42] used the generalized Riccati technique and have given several sufficient conditions for oscillation of (1.3) which can be applied to (1.14); in fact every solution of (1.14) oscillates if $\gamma > \frac{1}{4}$, (see [26], [27]). In [35], Wong extended the mentioned results of equation (1.2) to equation (1.4) and showed that equation (1.4) is oscillatory, for every $\alpha > 0$, if

$$\liminf_{t\to\infty} \int_{t_0}^{t} q(s)\, ds > 0,$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds = \infty, \quad \text{for some integer } n > 2.$$

Most oscillation results involve the interval of $q(t)$ and hence require the information of $q(t)$ on the entire half-line $[t_0, \infty)$. It is difficult to apply them to the cases where $q(t)$ has a "bad" behavior on a big part of $[t_0, \infty)$, e.g., when $\int_{t_0}^{\infty} q(t)\, dt = -\infty$. However, from the Sturm Separation Theorem (see [16]), we see that oscillation is only an interval property, i.e., if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \to \infty$, such that for each i there exists a solution of equation (1.2) that has at least two zeros in $[a_i, b_i]$, then every solution of equation (1.2) is oscillatory, no matter how "bad" equation (1.2) is on the remaining parts of $[t_0, \infty)$. El-Sayed [10] established an interval criterion for oscillation of a forced second order equation, but the result is not very sharp, because a comparison with equations of constant coefficient is used in the proof.

As pointed out in Kong [19], oscillation is an interval property, that is, it is more reasonable to investigate solutions on an infinite set of bounded intervals. Therefore, the problem is to find oscillation criteria which use only the information about the involved functions on these intervals; outside of these intervals the behavior of the functions is irrelevant. Such type of criteria are referred to as interval oscillation criteria. The first beautiful interval criteria in this direction was due to Kong [19], who employed the technique in the work of Philos [27]

for the second-order differential equations, and presented sharp several interval oscillation criteria for equation (1.2) involving the Kamenev's type condition.

Recently, Li and Agarwal [23] extended Kong's criterion to the equation

$$\left(a\left(t\right)y'\left(t\right)\right)' + p\left(t\right)y'\left(t\right) + q\left(t\right)f\left(y\left(t\right)\right) = 0,$$

and also, Li [24], Zheng [43], and other extended Kong's criterion to more general equation under restriction

$$f'\left(y\right) \geq \mu > 0 \quad \text{or} \quad \frac{f\left(y\right)}{y} \geq \mu > 0, \quad \text{for} \quad y \neq 0. \tag{1.15}$$

Note that assumption (1.15) does not allow f to be of superlinear or sunlinear growth. Also, some other results can be found in ([22], [40]).

Motivated by idea of El-Sayed [10], Kong [19] and Elabbasy *et al [9]* in this paper, we use the generalized Riccati transformation technique to establish interval oscillation criteria for equation (1.1), that is, criteria given by the behavior of equation (1.1) only on a sequence of subintervals of $[t_0, \infty)$ and can be apply to extreme cases such as $\int_{t_0}^{\infty} q(t)dt = -\infty$. Finally, an interesting example that illustrate the importance of our results is also included. In the sequel, when we write a functional inequality we will assume that it holds for all sufficiently large values of t. Throughout this paper. we say that a function $H = H\left(t,s\right)$ belongs to a function class $\mathbb{X}$, denoted by $H \in \mathbb{X}$, if $H \in C\left(\mathbb{D}, \mathbb{R}^{+}\right)$, where

$$\mathbb{D} := \left\{\left(t,s\right) : t_0 \leq s \leq t < \infty\right\},$$

which satisfies

$$H\left(t,t\right) = 0, \quad \text{for} \quad t \geq t_0, \quad H\left(t,s\right) > 0, \quad \text{for} \quad t > s \geq t_0, \tag{1.16}$$

and has partial derivatives $\dfrac{\partial H}{\partial t}$ and $\dfrac{\partial H}{\partial s}$ on $\mathbb{D}$ such that

$$\frac{\partial H}{\partial t} = h_1\left(t,s\right)\sqrt{H\left(t,s\right)}, \quad \frac{\partial H}{\partial s} = -h_2\left(t,s\right)\sqrt{H\left(t,s\right)}, \tag{1.17}$$

for some locally integrable functions h_1 and h_2. Also, given a differentiable function $\phi(t)$ and a positive differentiable function $\Phi(t)$, we let, for some a positive constant N

$$P\left(t\right) := \frac{-1}{\Phi\left(t\right)}\left(p\left(t\right)\Phi\left(t\right) - a\left(t\right)\Phi'\left(t\right)\right),$$

$$A\left(t\right) := \frac{-1}{a\left(t\right)\sigma\left(t\right)}\left(P\left(t\right)\sigma(t) + 2Na\left(t\right)\phi\left(t\right)\right), \quad B\left(t\right) := \frac{N}{a\left(t\right)\sigma\left(t\right)\Phi\left(t\right)},$$

$$\psi\left(t\right) := \Phi\left(t\right)\left[q\left(t\right) + P\left(t\right)\phi\left(t\right) + \frac{Na\left(t\right)\phi^2\left(t\right)}{\sigma\left(t\right)} - \frac{a\left(t\right)\phi\left(t\right)\Phi'\left(t\right)}{\Phi\left(t\right)} - \left(a\left(t\right)\phi\left(t\right)\right)'\right],$$

$$\sigma(t) := \int_{t_0}^{t} \frac{ds}{a(s)\Phi(s)},$$

and

$$G_1(t,v) := \frac{1}{4B(t)}\left[h_1(t,v) - \sqrt{H(t,v)}A(t)\right]^2,$$
$$G_2(u,t) := \frac{1}{4B(t)}\left[h_2(u,t) + \sqrt{H(u,t)}A(t)\right]^2.$$

Main results. In this section, we will establish oscillation criteria of equation (1.1) when $\alpha > 1$.

Theorem 2.1. *Suppose that there exist a differentiable function $\phi(t)$, a positive differentiable function $\Phi(t)$ such that*

$$P(t) \geq 0, \quad (P(t)\Phi(t))' \leq 0, \tag{2.1}$$

and

$$\lim_{t\to\infty} \sigma(t) = \infty, \quad \text{and} \quad \liminf_{t\to\infty} \int_{t_0}^{t} \Phi(s) q(s)\, ds > -\infty. \tag{2.2}$$

If, for a sufficiently large $T > t_0$, there exist $H \in \mathbb{X}$ and $a, b, c \in \mathbb{R}$ such that $T \leq a < b < c$ and

$$\frac{1}{H(b,a)} \int_a^b \left[H(s,a)\psi(s) - G_1(s,a)\right] ds + \frac{1}{H(c,b)} \int_b^c \left[H(c,s)\psi(s) - G_2(c,s)\right] ds > 0, \tag{2.3}$$

then every solution of equation (1.1) is oscillatory.

Proof. Assume (1.1) has a nonoscillatory solution on $[t_0, \infty)$. Then without loss of generality, $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)$. Consider the generalized Riccati substitution

$$w(t) := a(t)\Phi(t)\left(\frac{y'(t)}{y^{\alpha}(t)} + \phi(t)\right). \tag{2.4}$$

From (1.1) and (2.4), we find, for $t \geq t_0$

$$w'(t) = -\Phi(t) q(t) - \left(p(t)\Phi(t) - a(t)\Phi'(t)\right)\frac{y'(t)}{y^{\alpha}(t)} - \alpha a(t)\Phi(t)\frac{(y'(t))^2}{y^{\alpha+1}(t)} + (a(t)\phi(t)\Phi(t))'. \tag{2.5}$$

It follows from the definition of $P(t)$ that

$$w'(t) = -\Phi(t)\,q(t) + P(t)\,\Phi(t)\,\frac{y'(t)}{y^{\alpha}(t)} - \alpha a(t)\,\Phi(t)\left(\frac{y'(t)}{y^{\gamma}(t)}\right)^2 + \left(a(t)\,\phi(t)\,\Phi(t)\right)', \tag{2.6}$$

where $\gamma := \dfrac{\alpha+1}{2}$. We consider the following two cases.

Case 1. The integral $\int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\dfrac{y'(s)}{y^{\gamma}(s)}\right)^2 ds$ converges as $t \to \infty$. Then there exists a positive constant N_1 such that

$$\int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds \le N_1, \quad \text{for all } t \ge t_0.$$

By Schwarz's inequality, we get

$$\begin{aligned}\left|\int_{t_0}^{t} \frac{y'(s)}{y^{\gamma}(s)}ds\right|^2 &= \left|\int_{t_0}^{t} \sqrt{\frac{1}{\alpha a(s)\,\Phi(s)}}\sqrt{\alpha a(s)\,\Phi(s)}\,\frac{y'(s)}{y^{\gamma}(s)}ds\right|^2 \\ &\le \int_{t_0}^{t} \frac{1}{\alpha a(s)\,\Phi(s)}ds\left[\int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds\right] \le \frac{N_1}{\alpha}\sigma(t).\end{aligned}$$

Hence, for $t \ge t_0$

$$\left|y^{1-\gamma}(t) - y^{1-\gamma}(t_1)\right| \le (1-\gamma)\sqrt{\frac{N_1}{\alpha}}\sqrt{\sigma(t)}.$$

Therefore, there exists a constant N and $T > t_0$ such that

$$y^{\gamma}(t) \le y^{\alpha}(t)\sqrt{\frac{\alpha}{N}}\sqrt{\sigma(t)}, \quad \text{for} \quad t \ge T. \tag{2.7}$$

Using (2.7) in (2.6), we get, for $t \ge T$

$$w'(t) \le -\Phi(t)\,q(t) + P(t)\,\Phi(t)\,\frac{y'(t)}{y^{\alpha}(t)} - \frac{Na(t)\,\Phi(t)}{\sigma(t)}\left(\frac{y'(t)}{y^{\alpha}(t)}\right)^2 + \left(a(t)\,\phi(t)\,\Phi(t)\right)'.$$

By the definitions of $w(t)$, $\psi(t)$, $A(t)$ and $B(t)$, we have

$$w'(t) \le -\psi(t) - A(t)\,w(t) - B(t)\,w^2(t), \quad \text{for } t \ge T > t_0.$$

Case 2. The integral $\int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds$ diverges as $t \to \infty$. Integrating (2.6) from t_0 to t, we get

$$\Phi(t)\frac{a(t)\,y'(t)}{y^{\alpha}(t)} = C - \int_{t_0}^{t} \Phi(s) q(s)\,ds + \int_{t_0}^{t} P(s)\,\Phi(s)\frac{y'(s)}{y^{\alpha}(s)}ds - \int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds, \tag{2.8}$$

where $C := w(t_0) - a(t_0)\,\phi(t_0)\,\Phi(t_0)$. Then, by Bonnet's Theorem (see [1]), since $P(t)\,\Phi(t)$ is nonnegative and nonincreasing, for a fixed $t \geq t_0$, there exists $\xi \in [t_0, t]$ such that

$$\int_{t_0}^{t} P(s)\,\Phi(s)\frac{y'(s)}{y^{\alpha}(s)}ds = P(t_0)\,\Phi(t_0)\int_{t_0}^{\xi}\frac{y'(s)}{y^{\alpha}(s)}ds = P(t_0)\,\Phi(t_0)\int_{y(t_0)}^{y(\xi)} u^{-\alpha}du$$

$$= \frac{P(t_0)\,\Phi(t_0)}{1-\alpha}\left(y^{1-\alpha}(\xi) - y^{1-\alpha}(t_0)\right) < \frac{P(t_0)\,\Phi(t_0)}{\alpha - 1}y^{1-\alpha}(t_0) =: M.$$

Therefore, for $t \geq t_0$, we find from (2.8) that

$$\Phi(t)\frac{a(t)\,y'(t)}{y^{\alpha}(t)} \leq L - \int_{t_0}^{t} q(s)\,\Phi(s)\,ds - \int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds,$$

where $L := C + M$. Also, from (2.2), we can find a constant E such that

$$-\Phi(t)\frac{a(t)\,y'(t)}{y^{\alpha}(t)} \geq E + \int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds, \quad \text{for } t \geq t_0. \tag{2.9}$$

Now, we can choose $t_1 \geq t_0$ so that

$$V := E + \int_{t_0}^{t_1} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds > 1,$$

which means that $y'(t) < 0$ for $t \geq t_1$. Therefore, (2.9) yields

$$-\frac{\alpha y'(t)}{y(t)} \leq \frac{\alpha a(t)\,\Phi(t)\left(\frac{y'(t)}{y^{\gamma}(t)}\right)^2}{E + \int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds}, \quad \text{for } t \geq t_1 \tag{2.10}$$

Integrating (2.10) from t_1 to t, $t \geq t_1$, we get

$$\log\frac{E + \int_{t_0}^{t} \alpha a(s)\,\Phi(s)\left(\frac{y'(s)}{y^{\gamma}(s)}\right)^2 ds}{V} \geq \alpha \log\frac{y(t_1)}{y(t)}.$$

Hence

$$E+\int_{t_0}^{t}\alpha a\left(s\right)\Phi\left(s\right)\left(\frac{y'\left(s\right)}{y^{\gamma}\left(s\right)}\right)^{2}ds\geq V\left(\frac{y\left(t_1\right)}{y\left(t\right)}\right)^{\alpha}\geq\left(\frac{y\left(t_1\right)}{y\left(t\right)}\right)^{\alpha},\quad\text{for } t\geq t_1.$$

So, (2.9) yields

$$y'(t)\leq\frac{-y^{\alpha}\left(t_1\right)}{a\left(t\right)\Phi\left(t\right)},\quad\text{for } t\geq t_1,$$

which implies that

$$y(t)\leq y\left(t_1\right)-y^{\alpha}\left(t_1\right)\int_{t_1}^{t}\frac{ds}{a\left(s\right)\Phi\left(s\right)}\to-\infty\quad\text{as } t\to\infty,$$

since $\lim_{t\to\infty}\sigma\left(t\right)=\infty$, which contradicts the fact that y is a positive solution of (1.1). Therefore, we can choose a sequence $\{T_i\}\subset[T,\infty)$ such that $T_i\to\infty$ as $i\to\infty$. By the assumption, for each $i\in\mathbb{N}$, there exist $a_i,b_i,c_i\in\mathbb{R}$ such that $T_i\leq a_i<b_i<c_i$, $i\in\mathbb{N}$. Then, we get for $s\in(a_i,b_i]$, for $i\in\mathbb{N}$

$$\psi\left(s\right)\leq-w'\left(s\right)-A\left(s\right)w\left(s\right)-B\left(s\right)w^{2}\left(s\right).\tag{2.11}$$

Multiplying both sides of (2.11) by $H\left(s,t\right)$, integrating with respect to s from t to b for $t\in(a_i,b_i]$,

$$\begin{aligned}\int_{t}^{b_i}H\left(s,t\right)\psi\left(s\right)ds\quad\leq\quad&-\int_{t}^{b_i}H\left(s,t\right)w'\left(s\right)ds-\int_{t}^{b_i}H\left(s,t\right)A\left(s\right)w\left(s\right)ds\\&-\int_{t}^{b_i}H\left(s,t\right)B\left(s\right)w^{2}\left(s\right)ds.\end{aligned}$$

Integrating by parts and using (1.16) and then (1.17), we obtain

$$\int_{t}^{b_i}H\left(s,t\right)\psi\left(s\right)ds\leq-H\left(b_i,t\right)w\left(b_i\right)$$

$$-\int_{t}^{b_i}\left\{H\left(s,t\right)B\left(s\right)w^{2}\left(s\right)-\left[h_1\left(s,t\right)\sqrt{H\left(s,t\right)}-H\left(s,t\right)A\left(s\right)\right]w\left(s\right)\right\}ds$$

$$=-H\left(b_i,t\right)w\left(b_i\right)$$

$$-\int_{t}^{b_i}\left\{\sqrt{H\left(s,t\right)B\left(s\right)}w\left(s\right)-\frac{1}{2\sqrt{B\left(s\right)}}\left[h_1\left(s,t\right)-\sqrt{H\left(s,t\right)}A\left(s\right)\right]\right\}^{2}ds$$

$$+\frac{1}{4}\int_{t}^{b_i}\frac{1}{B\left(s\right)}\left[h_1\left(s,t\right)-A\left(s\right)\sqrt{H\left(s,t\right)}\right]^{2}ds$$

$$\leq -H(b_i,t)\,w(b)+\frac{1}{4}\int_t^{b_i}\frac{1}{B(s)}\left[h_1(s,t)-A(s)\sqrt{H(s,t)}\right]^2 ds.$$

Letting $t\to a_i^+$ in the above, we have

$$\frac{1}{H(b_i,a_i)}\int_{a_i}^{b_i}\left[H(s,a_i)\,\psi(s)-G_1(s,a_i)\right]ds\leq -w(b_i). \tag{2.12}$$

Similarly, Multiplying both sides of (2.11) by $H(t,s)$, integrate with respect to s from b_i to t for $t\in[b_i,c_i)$,

$$\begin{aligned}\int_{b_i}^{t}H(t,s)\,\psi(s)\,ds \quad\leq\quad & -\int_{b_i}^{t}H(t,s)\,w'(s)\,ds-\int_{b_i}^{t}H(t,s)\,A(s)\,w(s)\,ds\\ & -\int_{b_i}^{t}H(t,s)\,B(s)\,w^2(s)\,ds.\end{aligned}$$

Integrating by parts and using (1.16) and then (1.17), we obtain

$$\int_{b_i}^{t}H(t,s)\,\psi(s)\,ds\leq H(t,b_i)\,w(b_i)$$

$$-\int_{b_i}^{t}\left\{H(t,s)\,B(s)\,w^2(s)+\left[h_2(t,s)\sqrt{H(t,s)}+H(t,s)\,A(s)\right]w(s)\right\}ds$$

$$=H(t,b_i)\,w(b_i)$$

$$-\int_{b_i}^{t}\left\{\sqrt{H(t,s)\,B(s)}w(s)+\frac{1}{2\sqrt{B(s)}}\left[h_2(t,s)+A(s)\sqrt{H(t,s)}\right]\right\}^2 ds$$

$$+\frac{1}{4}\int_{b_i}^{t}\frac{1}{B(s)}\left[h_2(t,s)+A(s)\sqrt{H(t,s)}\right]^2 ds$$

$$\leq H(t,b_i)\,w(b_i)+\frac{1}{4}\int_{b_i}^{t}\frac{1}{B(s)}\left[h_2(t,s)+A(s)\sqrt{H(t,s)}\right]^2 ds.$$

Letting $t\to c_i^-$ in the above, we get

$$\frac{1}{H(c_i,b_i)}\int_{b_i}^{c_i}\left[H(c_i,s)\,\psi(s)-G_2(c_i,s)\right]ds\leq w(b_i). \tag{2.13}$$

From (2.12) and (2.13), we obtain

$$\frac{1}{H(b_i,a_i)}\int_{a_i}^{b_i}\left[H(s,a_i)\,\psi(s)-G_1(s,a_i)\right]ds \tag{2.14}$$

$$+\frac{1}{H(c_i,b_i)}\int_{b_i}^{c_i}\left[H(c_i,s)\psi(s)-G_2(c_i,s)\right]ds\leq 0,$$

for any one interval (a_i,c_i), $b_i\in(a_i,c_i)$, $i\in\mathbb{N}$, which contradicts the assumption (2.3). Then every solution $y(t)$ has at least one zero $\upsilon_i\in(a_i,c_i)$, $i\in\mathbb{N}$. Thus, *every solution of equation* (1.1) *is oscillatory* and the proof is complete. □

Example 2.2. Consider the nonlinear differential equation

$$y''(t)-y'(t)+q(t)|y(t)|^{\alpha-1}y(t)=0,\quad t\geq 1, \tag{2.15}$$

where

$$q(t)=\begin{cases}\alpha_1, & 3n\leq t\leq 3n+2,\\ \\ \alpha_2, & 3n+2<t\leq 3n+3\end{cases},$$

and

$$\alpha_1=\frac{e\left(2e^{3n+1}-e^2+2\right)}{2N\left(e^2-2e-1\right)}+\epsilon,\quad \alpha_2=-2\left(\frac{e^{3n+2}}{N\left(e^2-2e-1\right)}+\epsilon\right),$$

for $n\in\mathbb{N}_0=\{0,1,2,\dots\}$ and $\epsilon\in(0,1)$. For any $T>1$ there exists $n\in\mathbb{N}_0$ such that $3n\geq T$. Let $\phi(t)=\dfrac{1}{2}$ and $\Phi(t)=e^{-t}$, then

$$P(t)=0,\quad \sigma(t)=e^t-e,\quad \psi(t)=\frac{q(t)}{e^t}+\frac{2\left(e^t-e\right)+N}{4e^t\left(e^t-e\right)},$$

and

$$A(t)=\frac{N}{e-e^t},\quad B(t)=\frac{N}{1-e^{1-t}}.$$

It is clear that conditions (2.1) and (2.2) are satisfied. By taking $a=3n$, $b=3n+1$, $c=3n+2$ and $H(t,s)=(t-s)^2$. Note that $h_1(t,s)=h_2(t,s)=2$. It is easy to see that

$$\begin{aligned}G_1(t,a) &= \frac{1-e^{1-t}}{4N}\left(2+\frac{N(t-3n)}{e^t-e}\right)^2,\\ G_2(c,t) &= \frac{1-e^{1-t}}{4N}\left(2-\frac{N(3n+2-t)}{e^t-e}\right)^2,\end{aligned}$$

and

$$\frac{1}{H(b,a)}\int_a^b\left[H(s,a)\psi(s)-G_1(s,a)\right]ds$$

$$+\frac{1}{H(c,b)}\int_b^c\left[H(c,s)\psi(s)-G_2(c,s)\right]ds=2\epsilon\left(e^2-2e-1\right)e^{-2-3n}>0.$$

Hence, every solution of equation (2.15) is oscillatory by Theorem 2.1. Note that in this equation, we have $\int_1^\infty q(t)\,dt = -\infty$.

Corollary 2.3. *If*

$$\limsup_{t\to\infty} \int_l^t [\psi(s) H(s,l) - G_1(s,l)]\,ds > 0, \tag{2.16}$$

and

$$\limsup_{t\to\infty} \int_l^t [\psi(s) H(t,s) - G_2(t,s)]\,ds > 0, \tag{2.17}$$

for each $l \geq T > t_0$, then every solution of equation (1.1) *is oscillatory.*

P r o o f. For any $T_1 \geq T$, let $a = T_1$. In (2.16), we choose $l = a$. Then there exists $b > a$ such that

$$\int_a^b [H(s,a)\psi(s) - G_1(s,a)]\,ds > 0. \tag{2.18}$$

In (2.17), we choose $l = b$. Then there exists $c > b$ such that

$$\int_b^c [H(c,s)\psi(s) - G_2(c,s)]\,ds > 0. \tag{2.19}$$

Combining (2.18) and (2.19), we get (2.3). The conclusion thus comes from Theorem 2.1. The proof is complete. □

Next define

$$H(t,s) = [\sigma(t) - \sigma(s)]^\lambda, \quad t \geq s \geq T > t_0,$$

where $\lambda > 1$ is a constant.

Corollary 2.4. *If*

$$\limsup_{t\to\infty} \frac{1}{\sigma^{\lambda-1}(t)} \int_l^t \left\{ \left(\psi(s) - \frac{A^2(s)}{4B(s)} \right) [\sigma(s) - \sigma(l)]^\lambda \right. \tag{2.20}$$

$$\left. + \frac{\lambda A(s)[\sigma(s) - \sigma(l)]^{\lambda-1}}{2B(s)a(s)\Phi(s)} - \frac{\lambda^2(\sigma(s)-1)}{4Na(s)\Phi(s)} [\sigma(s) - \sigma(l)]^{\lambda-2} \right\} ds$$

$$> \frac{\lambda^2}{4N(\lambda-1)},$$

and

$$\limsup_{t\to\infty} \frac{1}{\sigma^{\lambda-1}(t)} \int_l^t \left\{ \left(\psi(s) - \frac{A^2(s)}{4B(s)} \right) [\sigma(t) - \sigma(s)]^\lambda \right. \tag{2.21}$$

$$-\frac{\lambda A(s)\left[\sigma(t)-\sigma(s)\right]^{\lambda-1}}{2B(s)\,a(s)\,\Phi(s)}-\frac{\lambda^2(\sigma(s)-1)}{4Na(s)\,\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-2}\Bigg\}ds$$

$$>\frac{\lambda^2}{4N(\lambda-1)},$$

for each $l\geq T>t_0$, then every solution of equation (1.1) *is oscillatory.*

P r o o f. From (1.17), we can find

$$h_1(t,s)=\lambda\left[\sigma(t)-\sigma(s)\right]^{(\lambda-2)/2}\frac{1}{a(t)\,\Phi(t)},$$

and

$$h_2(t,s)=\lambda\left[\sigma(t)-\sigma(s)\right]^{(\lambda-2)/2}\frac{1}{a(s)\,\Phi(s)}.$$

Note that

$$\int_l^t\frac{1}{4B(s)}h_1^2(s,l)\,ds=\frac{\lambda^2}{4N}\int_l^t\frac{\sigma(s)}{a(s)\,\Phi(s)}\left[\sigma(s)-\sigma(l)\right]^{\lambda-2}ds$$

$$=\frac{\lambda^2}{4N}\int_l^t\{\frac{(\sigma(s)-1)}{a(s)\,\Phi(s)}\left[\sigma(s)-\sigma(l)\right]^{\lambda-2}+\frac{1}{a(s)\,\Phi(s)}\left[\sigma(s)-\sigma(l)\right]^{\lambda-2}\}ds$$

$$=\frac{\lambda^2}{4N}\int_l^t\frac{(\sigma(s)-1)}{a(s)\,\Phi(s)}\left[\sigma(s)-\sigma(l)\right]^{\lambda-2}ds+\frac{\lambda^2}{4N(\lambda-1)}\left[\sigma(t)-\sigma(l)\right]^{\lambda-1},$$

and

$$\int_l^t\frac{1}{4B(s)}h_2^2(t,s)\,ds=\frac{\lambda^2}{4N}\int_l^t\frac{\sigma(s)}{a(s)\,\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-2}ds$$

$$=\frac{\lambda^2}{4N}\int_l^t\{\frac{(\sigma(s)-1)}{a(s)\,\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-2}+\frac{1}{a(s)\,\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-2}\}ds$$

$$=\frac{\lambda^2}{4N}\int_l^t\frac{(\sigma(s)-1)}{a(s)\,\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-2}ds+\frac{\lambda^2}{4N(\lambda-1)}\left[\sigma(t)-\sigma(l)\right]^{\lambda-1}.$$

Since $\lim_{t\to\infty}\sigma(t)=\infty$, we get

$$\lim_{t\to\infty}\frac{1}{\sigma^{\lambda-1}(t)}\int_l^t\frac{1}{4B(s)}h_1^2(s,l)\,ds \tag{2.22}$$

$$=\lim_{t\to\infty}\frac{\lambda^2}{4N\sigma^{\lambda-1}(t)}\int_l^t\frac{(\sigma(s)-1)}{a(s)\,\Phi(s)}\left[\sigma(s)-\sigma(l)\right]^{\lambda-2}ds+\frac{\lambda^2}{4N(\lambda-1)},$$

and

$$\lim_{t\to\infty}\frac{1}{\sigma^{\lambda-1}(t)}\int_l^t\frac{1}{4B(s)}h_2^2(t,s)\,ds \tag{2.23}$$

$$=\lim_{t\to\infty}\frac{\lambda^2}{4N\sigma^{\lambda-1}(t)}\int_l^t\frac{(\sigma(s)-1)}{a(s)\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-2}ds+\frac{\lambda^2}{4N(\lambda-1)}.$$

From (2.20) and (2.22), we have

$$\limsup_{t\to\infty}\frac{1}{\sigma^{\lambda-1}(t)}\int_l^t\{(\psi(s)-\frac{A^2(s)}{4B(s)})\left[\sigma(s)-\sigma(l)\right]^{\lambda}$$
$$+\frac{\lambda A(s)}{2B(s)a(s)\Phi(s)}\left[\sigma(s)-\sigma(l)\right]^{\lambda-1}-\frac{1}{4B(s)}h_1^2(s,l)\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{\sigma^{\lambda-1}(t)}\int_l^t\{(\psi(s)-\frac{A^2(s)}{4B(s)})\left[\sigma(s)-\sigma(l)\right]^{\lambda}$$

$$+\frac{\lambda A(s)}{2B(s)a(s)\Phi(s)}\left[\sigma(s)-\sigma(l)\right]^{\lambda-1}-\frac{\lambda^2(\sigma(s)-1)}{4Na(s)\Phi(s)}\left[\sigma(s)-\sigma(l)\right]^{\lambda-2}\}ds$$

$$-\frac{\lambda^2}{4N(\lambda-1)}>0,$$

which implies that (2.16) holds. Similarly, from (2.21) and (2.23), we have

$$\limsup_{t\to\infty}\frac{1}{\sigma^{\lambda-1}(t)}\int_l^t\{(\psi(s)-\frac{A^2(s)}{4B(s)})\left[\sigma(t)-\sigma(s)\right]^{\lambda}$$

$$-\frac{\lambda A(s)}{2B(s)a(s)\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-1}-\frac{1}{4B(s)}h_2^2(s,l)\}ds$$

$$=\limsup_{t\to\infty}\frac{1}{\sigma^{\lambda-1}(t)}\int_l^t\{(\psi(s)-\frac{A^2(s)}{4B(s)})\left[\sigma(t)-\sigma(s)\right]^{\lambda}$$

$$-\frac{\lambda A(s)}{2B(s)a(s)\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-1}-\frac{\lambda^2(\sigma(s)-1)}{4Na(s)\Phi(s)}\left[\sigma(t)-\sigma(s)\right]^{\lambda-2}\}ds$$

$$-\frac{\lambda^2}{4N(\lambda-1)}>0,$$

which implies that (2.17) holds. By Corollary 2.3, (1.1) is oscillatory. □

Remark 2.5. These results are extendable to equation (1.1) when $\alpha = 1$. In this case The details are left to the reader.

Remark 2.6. The results in this paper are in a form with a high degree of generality, and so with an appropriate choice of the functions $H(t,s)$, $\phi(t)$ and $\Phi(t)$, we can get other interval oscillation criteria for equation (1.1). For instance, if we choose, for some a constant $\lambda > 1$,

$$\begin{aligned} H(t,s) &= (t-s)^{\lambda}, \\ H(t,s) &= \left(\log \frac{t}{s}\right)^{\lambda}, \\ H(t,s) &= \left(\int_s^t \frac{dx}{\beta(x)}\right)^{\lambda}, \end{aligned}$$

for $(t,s) \in \mathbb{D}$, where β is a positive continuous function on $[t_0,\infty)$ such that $\int_{t_0}^{\infty} \frac{dx}{\beta(x)} = \infty$, then $\phi(t)$ and $\Phi(t)$ may be chosen 1 or t, etc. Also, if we replace the condition (1.17) by

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)} + A(t)H(t,s),$$

and

$$\frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)} + A(t)H(t,s),$$

or

$$\frac{\partial H}{\partial t} = h_1(t,s), \ \frac{\partial H}{\partial s} = -h_2(t,s),$$

we get other oscillation criteria for equation (1.1).

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