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STRUCTURE OF THE UNIT GROUP OF FD_{10}*

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Abstract. The structure of the unit group of the group algebra FD_{10} of the dihedral group D_{10} of order 10 over a finite field F has been obtained.

1. Introduction. Let FG be the group algebra of a group G over a field F. For a normal subgroup H of G, the natural homomorphism from G to G/H can be extended to an F-algebra homomorphism from FG onto $F[G/H]$ defined by $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g gH$. The kernel of this homomorphism, denoted by $\omega(H)$, is the ideal of FG generated by $\{h-1 \mid h\in H\}$. It is clear that $FG/\omega(H) \cong F[G/H]$. The augmentation ideal $\omega(FG)$ of the group algebra FG is defined by

$$\omega(FG) = \left\{ \sum_{g\in G} a_g g \;\middle|\; a_g \in F, \sum_{g\in G} a_g = 0 \right\}.$$

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Note that $\omega(G) = \omega(FG)$ and $\omega(H) = \omega(FH)FG = FG\omega(FH)$. Also $FG/\omega(G) \cong F$ showing that the Jacobson radical $J(FG)$ is contained in $\omega(FG)$. The equality occurs if G is a finite p-group and the characteristic of F is p. For an ideal $I \subseteq J(FG)$, the natural homomorphism from FG to FG/I induces an epimorphism from the unit group $\mathscr{U}(FG)$ of FG, to $\mathscr{U}(FG/I)$ with kernel $1+I$ and so that $\mathscr{U}(FG)/(1+I) \cong \mathscr{U}(FG/I)$.

For $g_1, g_2 \in G$, the commutator is $(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$. The lower central chain of G is given by

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \cdots \supseteq \gamma_m(G) \supseteq \cdots$$

where $\gamma_{c+1}(G) = (\gamma_c(G), G)$ is the group generated by (g, h) with $g \in \gamma_c(G), h \in G$, for $c \geq 1$. The group G is said to be nilpotent of class c if $\gamma_{c+1}(G) = \{1\}$ but $\gamma_c(G) \neq \{1\}$.

Passman and Smith [4] studied the structure of the unit group of the integral group ring $\mathbb{Z}D_{2p}$. The number of conjugacy classes of elements of finite order in the normalized unit group of the integral group ring $\mathbb{Z}D_{2p}$ has been determined by Bhandari and Luther [1]. However, the structure of the unit group $\mathscr{U}(FD_{2p})$ over a field F of positive characteristic is not known.

Recently the author, Sharma and Srivastava [3, 6, 7] have determined the structure of the unit group of FG for $G = S_3,\ S_4,\ A_4$. The work in this paper is on the unit group $\mathscr{U}(FD_{10})$ of the group algebra FD_{10} of the dihedral group D_{10} of order 10 over a finite field F. The presentation of D_{10} is given by

$$D_{10} = \langle a, b \mid a^5 = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

Consequently the commutator subgroup of D_{10} is $A = \langle a \rangle$. The distinct conjugacy classes of D_{10} are $\mathscr{C}_0 = \{1\}, \mathscr{C}_1 = \{a, a^{-1}\}, \mathscr{C}_2 = \{a^2, a^{-2}\}$ and $\mathscr{C}_3 = \{b, ab, a^2b, a^3b, a^4b\}$. Hence $\{\widehat{\mathscr{C}_0}, \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3}\}$ form an F-basis for the center $\mathscr{Z}(FD_{10})$ of the group algebra FD_{10}, where $\widehat{\mathscr{C}_i}$ denotes the sum of all elements in the conjugacy class $\mathscr{C}_i$ (cf. Lemma 4.1.1 [5]).

2. Unit group of FD_{10}.

Theorem 2.1. *Let $\mathscr{U}(FD_{10})$ be the unit group of the group algebra FD_{10} of the dihedral group D_{10} of order* 10 *over a finite field F. Let $V = 1 + J(FD_{10})$ where $J(FD_{10})$ denotes the Jacobson radical of the group algebra FD_{10}.*

(1) *If $|F| = 5^n$, then $\mathscr{U}(FD_{10})/V \cong F^* \times F^*$ and V is a nilpotent group of class* 4. *Moreover, the center $\mathscr{Z}(V)$ of V is an elementary abelian* 5*-group of order 5^{3n}.*

(2) *Let $|F| = 2^n$. If the extension field F of F_2 contains a primitive 5th root of unity, then $\mathscr{U}(FD_{10})/V \cong F^* \times GL(2,F) \times GL(2,F)$ and V is an elementary abelian 2-group of order 2^n.*

(3) *If $|F| = r^n$, where r is prime and $r \neq 2, 5$, then*

$$\mathscr{U}(FD_{10}) \cong \begin{cases} GL(2,F) \times GL(2,F) \times F^* \times F^*, & \text{if } r \equiv \pm 1 \pmod 5; \\ GL(2,F) \times GL(2,F) \times F^* \times F^*, & \text{if } r \equiv \pm 2 \pmod 5 \\ & \quad \text{and } n \text{ is even}; \\ GL(2,\widetilde{F}) \times F^* \times F^*, & \text{if } r \equiv \pm 2 \pmod 5 \\ & \quad \text{and } n \text{ is odd}. \end{cases}$$

Here $F^* = F \backslash \{0\}$, $GL(2,F)$ is the general linear group of degree 2 over F and $\widetilde{F}$ is the quadratic extension of F.

Proof. (1) Since A is a normal subgroup of D_{10} of index 2, we have $J(FD_{10}) = J(FA)(FD_{10})$ (cf. Theorem 7.2.7 of [5]). The group A is of order 5 and $\operatorname{char} F = 5$. This implies that $J(FA) = \omega(FA)$ and so $J(FD_{10}) = \omega(A)$. Note that $\omega(A)$ is a nilpotent ideal with nilpotency index 5. Therefore, the natural homomorphism from FD_{10} onto $FD_{10}/\omega(A)$ induces an epimorphism from $\mathscr{U}(FD_{10})$ to $\mathscr{U}(FD_{10}/\omega(A))$ with kernel $V = 1 + \omega(A)$ and so

$$\mathscr{U}(FD_{10})/V \cong \mathscr{U}(F[D_{10}/A]) \cong F^* \times F^*.$$

Further, as $\omega(A)^5 = 0$ and $\operatorname{char} F = 5$, the order of any nontrivial element of V is 5. Clearly V is a nilpotent group. One can observe that $\gamma_2(V) \subseteq 1 + \omega(A)^2, \gamma_3(V) \subseteq 1 + \omega(A)^3, \gamma_4(V) \subseteq 1 + \omega(A)^4$ and so the nilpotency class of V is at most 4.

The element $x = \alpha_0 + \alpha_1\widehat{\mathscr{C}_1} + \alpha_2\widehat{\mathscr{C}_2} + \alpha_3\widehat{\mathscr{C}_3}$ belongs to V of $\mathscr{Z}(FD_{10})$, if and only if $\alpha_0 + 2\alpha_1 + 2\alpha_2 = 1$. If $H = \mathscr{Z}(FD_{10}) \cap V$ then

$$H = \{1 + \alpha_1(\widehat{\mathscr{C}_1} - 2) + \alpha_2(\widehat{\mathscr{C}_2} - 2) + \alpha_3\widehat{\mathscr{C}_3} \mid \alpha_1, \alpha_2, \alpha_3 \in F\}$$

is a central subgroup of V of order 5^{3n}. Let

$$\begin{aligned} \omega_1 &= (a - a^{-1})(1 + b), & \omega_2 &= (a - a^{-1})(1 - b), \\ \omega_3 &= (a^2 - a^{-2})(1 + b), & \omega_4 &= (a^2 - a^{-2})(1 - b). \end{aligned}$$

Note that $\omega_i^2 = 0$, $1 \leq i \leq 4$, and $\omega_1\omega_3 = \omega_3\omega_1 = 0$, $\omega_2\omega_4 = \omega_4\omega_2 = 0$. Also observe that

$$\begin{aligned} \omega_1\omega_2 &= (a^2 + a^3 - 2)(2 - 2b), & \omega_2\omega_1 &= (a^2 + a^3 - 2)(2 + 2b), \\ \omega_3\omega_4 &= (a + a^4 - 2)(2 - 2b), & \omega_4\omega_3 &= (a + a^4 - 2)(2 + 2b). \end{aligned}$$

It is known that $\{(a^i-1),(a^i-1)b \mid 1\le i\le 4\}$ forms a basis of $\omega(A)$ as an F-vector space. Since, for $\omega_i\in\omega(A)$, $1\le i\le 4$ and

$$\begin{aligned}
(a-1) &= 2(\omega_3\omega_4+\omega_4\omega_3)-(\omega_1+\omega_2),\\
(a^2-1) &= 2(\omega_1\omega_2+\omega_2\omega_1)-(\omega_3+\omega_4),\\
(a^3-1) &= 2(\omega_1\omega_2+\omega_2\omega_1)+(\omega_3+\omega_4),\\
(a^4-1) &= 2(\omega_3\omega_4+\omega_4\omega_3)+(\omega_1+\omega_2),\\
(a-1)b &= 3(\omega_3\omega_4-\omega_4\omega_3)+4(\omega_1-\omega_2),\\
(a^2-1)b &= 3(\omega_1\omega_2-\omega_2\omega_1)+4(\omega_3-\omega_4),\\
(a^3-1)b &= 3(\omega_1\omega_2-\omega_2\omega_1)+(\omega_3-\omega_4),\\
(a^4-1)b &= 3(\omega_3\omega_4-\omega_4\omega_3)+(\omega_1-\omega_2),
\end{aligned}$$

we have

$$\omega(A)=F\omega_1+F\omega_2+F\omega_3+F\omega_4+F\omega_1\omega_2+F\omega_2\omega_1+F\omega_3\omega_4+F\omega_4\omega_3.$$

In fact this sum is a direct sum.

For $1\le i\le 3$, let $u_i=1+\omega_i$. Then $(u_1,u_2)\equiv 1+y\,(\mathrm{mod}\,\mathscr{Z}(V))$, where $y=\omega_1\omega_2-\omega_2\omega_1+\omega_1\omega_2\omega_1-\omega_2\omega_1\omega_2$. Since $y\in\omega(A)^2$, we have $(1+y)^{-1}\equiv 1-y\,(\mathrm{mod}\,\mathscr{Z}(V))$ and so $(u_1,u_2,u_3)\equiv 1+y\omega_3-\omega_3 y\,(\mathrm{mod}\,\mathscr{Z}(V))$. Hence V is a nilpotent group of class 4.

Assume $x\in\omega(A)$ with

$$x=\alpha_1\omega_1+\alpha_2\omega_2+\alpha_3\omega_3+\alpha_4\omega_4+\beta_1\omega_1\omega_2+\beta_2\omega_2\omega_1+\beta_3\omega_3\omega_4+\beta_4\omega_4\omega_3,\quad \alpha_i,\beta_i\in F.$$

If $1+x\in\mathscr{Z}(V)$ then $\omega_1 x=x\omega_1$ and hence $\alpha_2=\alpha_4=0$ and $\beta_1=\beta_2,\beta_3=\beta_4$. Thus

$$x=\alpha_1\omega_1+\alpha_3\omega_3+\beta_1(\omega_1\omega_2+\omega_2\omega_1)+\beta_2(\omega_3\omega_4+\omega_4\omega_3).$$

Since x commute with ω_2, we have $\alpha_1=\alpha_3=0$ and therefore $x=\beta_1(\omega_1\omega_2+\omega_2\omega_1)+\beta_2(\omega_3\omega_4+\omega_4\omega_3)$, where $(\omega_1\omega_2+\omega_2\omega_1)=4(\widehat{\mathscr{C}_2}-2)$ and $\omega_3\omega_4+\omega_4\omega_3=4(\widehat{\mathscr{C}_1}-2)$. Thus for any $\beta_1,\beta_2\in F$, $1+x\in H$. Hence $H=\mathscr{Z}(V)$ and so $\mathscr{Z}(V)$ is an elementary abelian 5-group of order 5^{3n}. Note that $\mathscr{Z}(V)=\mathscr{V}_1\times\mathscr{V}_2$, where

$$\begin{aligned}
\mathscr{V}_1 &= \{1+\alpha_1(\widehat{\mathscr{C}_1}-2)+\alpha_2(\widehat{\mathscr{C}_2}-2)\mid \alpha_1,\alpha_2\in F\},\\
\mathscr{V}_2 &= \{1+\alpha\widehat{\mathscr{C}_3}\mid \alpha\in F\}.
\end{aligned}$$

Let $f(x)$ be a monic irreducible polynomial of degree n over the prime field F_5 such that $F_5[x]/\langle f(x)\rangle \cong F$. Assume α is the residue class of x modulo $\langle f(x)\rangle$. We claim that

$$\mathscr{V}_1 = \prod_{i=0}^{n-1}\langle 1+\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle \times \prod_{i=0}^{n-1}\langle 1+2\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle.$$

For that take $u_{\alpha^i} = 1+\alpha^i(\widehat{\mathscr{C}_1}-2)$. Note that

$$(\widehat{\mathscr{C}_1}-2)^2 = (\widehat{\mathscr{C}_2}-2)^2 = (\widehat{\mathscr{C}_1}+\widehat{\mathscr{C}_2}-4),$$

$$(\widehat{\mathscr{C}_1}-2)(\widehat{\mathscr{C}_2}-2) = -(\widehat{\mathscr{C}_1}+\widehat{\mathscr{C}_2}-4)$$

and so

$$\begin{aligned} u_{\alpha^i}u_{\alpha^j} &= (1+\alpha^i(\widehat{\mathscr{C}_1}-2))(1+\alpha^j(\widehat{\mathscr{C}_1}-2)) \\ &= 1+(\alpha^i+\alpha^j+\alpha^{i+j})(\widehat{\mathscr{C}_1}-2)+\alpha^{i+j}(\widehat{\mathscr{C}_2}-2). \end{aligned}$$

By induction one can prove that

$$u_{\alpha^{i_1}}u_{\alpha^{i_2}}\dots u_{\alpha^{i_l}} = 1+(\delta_1+\delta_2)(\widehat{\mathscr{C}_1}-2)+\delta_2(\widehat{\mathscr{C}_2}-2),$$

where $\delta_1 = \sum_{j=1}^{l}\alpha^{i_j}$ and $\delta_2 = \sum_{\substack{j,k=1\\ j\neq k}}^{l}\alpha^{i_j}\alpha^{i_k}$. We claim that for any $0 \le l \le (n-1)$,

$$\langle 1+\alpha^l(\widehat{\mathscr{C}_1}-2)\rangle \bigcap \prod_{\substack{i=0\\ i\neq l}}^{n-1}\langle 1+\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle = \{1\}.$$

Let, if possible, $u_{\alpha^l} = u_{\alpha^{i_1}}u_{\alpha^{i_2}}\dots u_{\alpha^{i_k}}$ so that $\alpha^l = \delta_1+\delta_2$ and $\delta_2 = 0$. Thus $\alpha_l = \delta_1$. Since $0 \le i_1, i_2, \dots, i_k, l \le n-1$ and $\{1, \alpha, \dots, \alpha^{n-1}\}$ is a linearly independent set, we reach a contradiction. Hence $\prod_{i=0}^{n-1}\langle 1+\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle$ is a direct product of cyclic groups of order 5. Similarly one can show that $\prod_{i=0}^{n-1}\langle 1+2\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle$ is also a direct product of cyclic groups of order 5. As $\prod_{i=0}^{n-1}\langle 1+2\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle$ and $\prod_{i=0}^{n-1}\langle 1+\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle$

do not have any common element, we have $\prod_{i=0}^{n-1}\langle 1+\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle\times\prod_{i=0}^{n-1}\langle 1+2\alpha^i(\widehat{\mathscr{C}_1}-2)\rangle$ is a direct product of cyclic groups of order 5^{2n}. Note that this is a subgroup of $\mathscr{V}_1$ with $|\mathscr{V}_1| = 5^{2n}$. Hence the result follows. Further, the structure of $\mathscr{V}_2$ is given as follows:

$$\mathscr{V}_2 = \prod_{i=0}^{n-1}\langle 1+\alpha^i(1+a+a^2+a^3+a^4)b\rangle.$$

(2) Assume the field F contains a primitive 5-th root of unity, say ε. We define a matrix representation of D_{10},

$$\theta : D_{10} \longrightarrow \mathscr{U}(F\oplus \mathbb{M}(2,F)\oplus \mathbb{M}(2,F))$$

by the assignment

$$a\mapsto \left(1, \begin{pmatrix} \varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix}, \begin{pmatrix} \varepsilon^2 & 0\\ 0 & \varepsilon^{-2}\end{pmatrix}\right) \quad\text{and}\quad b\mapsto \left(1, \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\right)$$

and extend it to an algebra homomorphism

$$\theta^* : FD_{10} \longrightarrow F\oplus \mathbb{M}(2,F)\oplus \mathbb{M}(2,F)$$

where $\mathbb{M}(2,F)$ is the algebra of 2×2 matrices over the field F. Let $x = \sum_{i=0}^{4}\alpha_i a^i + \sum_{i=0}^{4}\beta_i a^i b \in \operatorname{Ker}\theta^*$, where $\alpha_i, \beta_i \in F$. Thus $\theta^*(x) = 0$ gives the following system of equations:

$$\sum_{i=0}^{4}\alpha_i + \sum_{i=0}^{4}\beta_i = 0 \tag{1}$$

$$\alpha_0+\alpha_1\varepsilon+\alpha_2\varepsilon^2+\alpha_3\varepsilon^3+\alpha_4\varepsilon^4 = 0 \tag{2}$$

$$\alpha_0+\alpha_1\varepsilon^4+\alpha_2\varepsilon^3+\alpha_3\varepsilon^2+\alpha_4\varepsilon = 0 \tag{3}$$

$$\begin{aligned}
(4)\qquad \alpha_0 + \alpha_1\varepsilon^2 + \alpha_2\varepsilon^4 + \alpha_3\varepsilon + \alpha_4\varepsilon^3 &= 0\\
(5)\qquad \alpha_0 + \alpha_1\varepsilon^3 + \alpha_2\varepsilon + \alpha_3\varepsilon^4 + \alpha_4\varepsilon^2 &= 0\\
(6)\qquad \beta_0 + \beta_1\varepsilon + \beta_2\varepsilon^2 + \beta_3\varepsilon^3 + \beta_4\varepsilon^4 &= 0\\
(7)\qquad \beta_0 + \beta_1\varepsilon^4 + \beta_2\varepsilon^3 + \beta_3\varepsilon^2 + \beta_4\varepsilon &= 0\\
(8)\qquad \beta_0 + \beta_1\varepsilon^2 + \beta_2\varepsilon^4 + \beta_3\varepsilon + \beta_4\varepsilon^3 &= 0\\
(9)\qquad \beta_0 + \beta_1\varepsilon^3 + \beta_2\varepsilon + \beta_3\varepsilon^4 + \beta_4\varepsilon^2 &= 0.
\end{aligned}$$

Since ε is a primitive 5-th root of unity, we have ε is a root of the equation $x^4 + x^3 + x^2 + x + 1 \in F_2[x]$. From equations $(2), (3), (4)$ and (5) and using $\operatorname{char} F = 2$ we get $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$. Also multiplying equation (2) by ε^4, (3) by ε, (4) by ε^3, and (5) by ε^2 and after adding we get $\alpha_0 + \alpha_2 + \alpha_3 + \alpha_4 = 0$. Thus $\alpha_0 = \alpha_1$. Similarly we get $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. By using the same arguments in equations (6), (7), (8) and (9), we get $\beta_0 = \beta_1 = \beta_2 = \beta_3 = \beta_4$. Hence from equation (1) we get all coefficients of x are the same and therefore $\operatorname{Ker}\theta^* = F\widehat{D}_{10}$, where $\widehat{D}_{10}$ is the sum of all elements in D_{10}. Since $\dim_F(\operatorname{Ker}\theta^*) = 1$, we have

$$FD_{10}/\operatorname{Ker}\theta^* \cong F \oplus \mathbb{M}(2, F) \oplus \mathbb{M}(2, F).$$

As θ^* is onto, $\theta^*(J(FD_{10})) \subseteq J(F \oplus \mathbb{M}(2, F) \oplus \mathbb{M}(2, F)) = 0$ implies that $J(FD_{10}) \subseteq \operatorname{Ker}\theta^*$. Further, since $\widehat{D}_{10}^2 = 0$, we have $\operatorname{Ker}\theta^* \subseteq J(FD_{10})$ and therefore $J(FD_{10}) = F\widehat{D}_{10}$.

Since $J(FD_{10})$ is a nilpotent ideal, we have

$$\mathscr{U}(FD_{10})/V \cong F^* \times GL(2, F) \times GL(2, F)$$

where $V = 1 + J(FD_{10})$. Here V is an elementary abelian 2-group of order 2^n whose structure is given as

$$V = \prod_{i=0}^{n-1} \langle 1 + \alpha^i \widehat{D}_{10} \rangle,$$

where α is the residue class of $x \mod \langle f(x) \rangle$. Here $f(x)$ is a monic irreducible polynomial of degree n over F_2.

(3) Since the group algebra FD_{10} is a semi-simple Artinian ring, by Wedderburn structure theorem we get

$$FD_{10} \cong \mathbb{M}(n_1, D_1) \oplus \mathbb{M}(n_2, D_2) \oplus \cdots \oplus \mathbb{M}(n_j, D_j)$$

where the D_i's are finite dimensional division algebras over F. Since F is a finite field, we have the D_i's are finite division rings and so the D_i's are finite field extensions of F.

Further, we can observe that $r \equiv \pm 1$ or $\pm 2 \pmod 5$. If $r \equiv \pm 1 \pmod 5$, then $(a^i + a^{-i})^r = (a^i + a^{-i})$ for $i = 1, 2$. Hence for any element $x \in \mathscr{Z}(FD_{10})$, $x^{r^n} = x$ and so

$$FD_{10} \cong \mathbb{M}(2, F) \oplus \mathbb{M}(2, F) \oplus F \oplus F.$$

Now if $r \equiv \pm 2 \pmod 5$ then $r^2 \equiv \pm 1 \pmod 5$. Now if n is even then $r^n \equiv \pm 1 \pmod p$ which implies that $x^{r^n} = x$ for all $x \in \mathscr{Z}(F_r D_{2p})$ and so $FD_{10} \cong \mathbb{M}(2, F) \oplus \mathbb{M}(2, F) \oplus F \oplus F$. If n is odd then $r^{2n} \equiv 1 \pmod p$ and so $x^{r^{2n}} = x$ for any element in the center of FD_{10}. Thus

$$\begin{aligned} FD_{10} &\cong \mathbb{M}(2, \widetilde{F}) \oplus \widetilde{F} \\ \text{or} &\cong \mathbb{M}(2, \widetilde{F}) \oplus F \oplus F. \end{aligned}$$

Since A is a derived subgroup of D_{10}, we have $FD_{10} \cong F(D_{10}/A) \oplus \omega(A)$. Further, $FD_{10}/\omega(A) \cong F(D_{10}/A) \cong FC_2 \cong F \oplus F$. So finally we have $FD_{10} \cong \omega(A) \oplus F \oplus F$. As $\omega(A)$ is a two-sided ideal of the group algebra FD_{10} then it will direct sum of simple module and each simple module is isomorphic to a matrix ring over F. Thus the group algebra $FD_{10} \cong \mathbb{M}(2, \widetilde{F}) \oplus F \oplus F$. Hence

$$\mathscr{U}(FD_{10}) \cong \begin{cases} GL(2,F) \times GL(2,F) \times F^* \times F^*, & \text{if } r \equiv \pm 1 \pmod 5; \\ GL(2,F) \times GL(2,F) \times F^* \times F^*, & \text{if } r \equiv \pm 2 \pmod 5 \\ & \quad \text{and } n \text{ is even}; \\ GL(2,\widetilde{F}) \times F^* \times F^*, & \text{if } r \equiv \pm 2 \pmod 5 \\ & \quad \text{and } n \text{ is odd}. \end{cases} \quad \square$$

Remark 1. Although our methods were theoretical, the use of the GAP package LAGUNA [2] helped us to verify certain long and involved computations.

Remark 2. We have not handled the case when the extension field F of F_2 does not have a primitive 5-th root of unity. However, we have the following proposition in the case of F_2.

Proposition 2.2. *$\mathscr{U}(F_2D_{10}) \cong V'(A) \rtimes \langle b\rangle$, the semi-direct product of $V'(A)$ with $\langle b\rangle$ where $V'(A) = (1+\omega(A)) \cap \mathscr{U}(F_2D_{10})$.*

Proof. Since A is a normal subgroup of D_{10}, the natural homomorphism $D_{10} \hookrightarrow D_{10}/A$ induces an algebra homomorphism, say θ, from F_2D_{10} onto $F_2[D_{10}/A]$. The kernel of this map is $\omega(A)$ and so $F_2D_{10}/\omega(A) \cong F_2C_2$. Assume $\theta^* = \theta|_{V'(F_2D_{10})}$, the restriction of θ on $V'(F_2D_{10})$, where

$$V'(F_2D_{10}) = \left\{ \sum_{g\in G} a_g g \in \mathscr{U}(F_2D_{10}) \mid \Sigma a_g = 1 \right\}.$$

Note that if $u \in V'(F_2D_{10})$ then $\theta^*(u) \in V'(F_2[D_{10}/A])$ and therefore $\theta^* : V'(F_2D_{10}) \longrightarrow V'(F_2[D_{10}/A])$ is a group homomorphism with $\operatorname{Ker}\theta^* = V'(A) = (1+\omega(A)) \cap V'(F_2D_{10})$. Further, assume

$$\theta' = \theta|_{\mathscr{U}(FD_{10})} : \mathscr{U}(FD_{10}) \longrightarrow \mathscr{U}(F[D_{10}/A])$$

is a group homomorphism. It is easy to observe that the kernel of θ' is $V'(A)$ and so $\mathscr{U}(F_2D_{10})/V'(A) \cong \operatorname{Im}\theta' \subseteq \mathscr{U}(F_2\langle b\rangle) = \langle b\rangle$. Hence

$$\mathscr{U}(F_2D_{10}) \cong V'(A) \rtimes \langle b\rangle. \qquad \square$$

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