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Hermite Series with Polar Singularities

Georgi S. Boychev

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Series in Hermite polynomials with poles on the boundaries of their regions of convergence are considered.

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1. Introduction

Definition 1. The polynomials $\{H_n(z)\}_{n=0}^{+\infty}$ defined by equalities

$$H_n(z) = (-1)^n \exp(z^2) \frac{d^n}{dz^n}\{\exp(-z^2)\},\ n = 0,\ 1,\ 2,\ \dots;\ z \in \mathbb{C},$$

where $\mathbb{C}$ is the complex plane, are called *Hermite polynomials.*

Let $\lambda(z) = \sqrt{2}\exp(z^2/2)$ and $c_n(z) = (2n/e)^{n/2}\cos[(2n+1)^{1/2}z - n\pi/2]$. Then the asymptotic formula for the Hermite polynomials ($n \geq 1$) [1, Chapter III, (2.2)] yield that

$$H_n(z) = \lambda(z)c_n(z)\{1 + h_n(z)\}, \tag{1.1}$$

where $\{h_n(z)\}_{n=1}^{+\infty}$ are holomorphic functions in the open set $G = \mathbb{C}\backslash(-\infty,\ +\infty)$ and

$$h_n(z) = O(n^{-1/2}) \qquad (n \to +\infty)$$

uniformly on every compact subset of G.

Definition 2. The series of the kind

$$\sum_{n=0}^{+\infty} a_n H_n(z), \tag{1.2}$$

we shall call *Hermite series.*

Let $0 < \tau < +\infty$ and define $S(\tau) = \{z \in \mathbb{C} :\ |\Im z| < \tau\}$ and $S^*(\tau) = \mathbb{C}\backslash\overline{S(\tau)}$. We assume $S(0) = \emptyset, S(\infty) = \mathbb{C}, S^*(0) = \mathbb{C}\backslash\mathbb{R}$ and $S^*(\infty) = \emptyset$. Then:

Theorem 1. *If*

$$\tau = \max\{0, -\lim_{n\to+\infty}\sup\ (2n+1)^{-1/2}\log|\,(2n/e)^{n/2}a_n|\} \tag{1.3}$$

then the series (1.2) is absolutely uniformly convergent on every compact subset of the set $S(\tau)$ *and diverges in* $S^*(\tau)$ ([1], Theorem IV.3.1, b)).

Remark 1. The equality (1.3) can be regarded as a formula of Cauchy-Hadamard type for the series of kind (1.2).

Remark 2. In the proof of Theorem 1 it is used the asymptotic formula (1.1).

In [2] we prove the following

Theorem 2. *Let* $z_0 \in G$ *and* $a_nH_n(z_0) = O(n^p)(n \to +\infty)$, *where* $p \geq -1$. *Then the series (1.2) is absolutely convergent in the strip* $S(|\Im z_0|)$.

Remark 3. If the conditions of Theorem 2 are satisfied, then the series (1.2) is absolutely uniformly convergent on every compact subset of the strip $S(|\Im z_0|)$ and the sum of this series is a complex function holomorphic in $S(|\Im z_0|)$.

2. The main result

The basic result is given by the following

Theorem 3. *Let* $z_0 \in G$ *and* $a_nH_n(z_0) = o(n^p)(n \to +\infty)$, *where* p *is a nonnegative integer. If* $f(z)$ *is the sum of the series* (1.2) *in the strip* $S(\tau_0)$ *with* $\tau_0 = |\Im z_0|$ *and* $f(z)$ *has a pole of order* m *on* $\partial S(\tau_0)$, *then* $m \leq 2p+1$.

P r o o f. Suppose that there is a point $\zeta \in \partial S(\tau_0)$ such that the function $f(z)$ has a pole at ζ of order m and $m > 2p+1$. Then

$$m \geq 2p+2. \tag{2.1}$$

Since the Hermite polynomials have no zeros outside real line, we can write that

$$g(z) = (z-\zeta)^m f(z) = (z-\zeta)^m \sum_{n=0}^{+\infty} a_nH_n(z_0)\frac{H_n(z)}{H_n(z_0)},$$

where $z \in S(\tau_0)\backslash(-\infty,\ +\infty)$. Hence,

$$|g(z)| \leq |(z-\zeta)^m \sum_{n=0}^{N} a_nH_n(z)| + |(z-\zeta)^m \sum_{n=N+1}^{+\infty} a_nH_n(\zeta)\frac{H_n(z)}{H_n(\zeta)}| = g_{N,1}(z) + g_{N,2}(z)$$

whatever the non-negative integer N can be.

Let $\varepsilon > 0$. Then it follows that exists a positive N_0 such that

$$|a_nH_n(z_0)| < \varepsilon n^p$$

for $n > N_0$. Then for $N > N_0$ we obtain that

$$g_{N,2}(z) \le \varepsilon |z-\zeta|^m \sum_{n=N+1}^{+\infty} n^p \left|\frac{H_n(z)}{H_n(z_0)}\right|. \tag{2.2}$$

Let $\max(0, \tau_0 - 1) < \delta < \tau_0$ and $\overline{D(\zeta;\delta)} = \{z \in \mathbb{C} : \Re z = \Re\zeta\} \cap \{z \in S(\tau_0) - (-\infty,\ +\infty) : |\Im z| \ge \delta\}$. Since $\overline{D(\zeta;\delta)}$ is a compact subset of G, the asymptotic formula (1.1) yields that

$$\frac{H_n(z)}{H_n(z_0)} = O\{\exp(-\sqrt{2n+1}(\tau_0 - |\Im z|)\}, \quad n \to +\infty,$$

uniformly on $D(\zeta;\delta)$. Then,

$$\sum_{n=N+1}^{+\infty} n^p \left|\frac{H_n(z)}{H_n(z_0)}\right| = O\{\sum_{n=N+1}^{+\infty} n^p \exp[-\sqrt{2n+1}(\tau_0 - |\Im z|)]\}$$
$$= O(\int_1^{+\infty} t^p \exp[-(\tau_0 - |\Im z|)\sqrt{2t+1}]\mathrm{d}t).$$

It is not difficult to prove that

$$\int_1^{+\infty} t^p \exp[-(\tau_0 - |\Im z|)\sqrt{2t+1}]\mathrm{d}t \le M(\tau_o - |\Im z|)^{-2p-2},$$

where M is a constant not depending of N. Hence,

$$\sum_{n=N+1}^{+\infty} n^p \left|\frac{H_n(z)}{H_n(z_0)}\right| \le K(\tau_o - |\Im z|)^{-2p-2}, \;\; z \in D(\zeta;\delta),$$

where K is a constant not depending of N. Then from (2.2) it follows that

$$g_{N,2}(z) \le \varepsilon K|z-\zeta|^m(\tau_0 - |\Im z|)^{-2p-2}.$$

Obviously, $|z-\zeta|^m = (\tau_0 - |\Im z|)^m$ for $z \in D(\zeta;\delta)$. Using (2.1) and the inequality $\tau_0 - |\Im z| < 1$, we obtain that

$$g_{N,2}(z) \le \varepsilon K \tag{2.3}$$

for each $N > N_0$ and $z \in D(\zeta;\delta)$. Let such N be fixed, then there exists a positive constant L such that

$$g_{N,1}(z) \le L(\tau_0 - |\Im z|)^m,$$

for $z \in D(\zeta;\delta)$. Moreover let $(\tau_0 - |\Im z|)^m < \varepsilon$. Then

$$g_{N,1}(z) \le \varepsilon L. \tag{2.4}$$

From the inequalities (2.3) and (2.4) it follows that

$$|g(z)| \leq \varepsilon(K+L)$$

for $z \in D(\zeta;\delta)$ and sufficiently closed to ζ. This means that

$$\lim_{z\to\zeta}(z-\zeta)^m f(z) = 0, \qquad z \in D(\zeta;\delta).$$

However, this contradicts the assumption that the function $f(z)$ has a pole of order m at the point ζ. This completes the proof of Theorem 3. ■

Corollary. *Let* $z_0 \in G$, $\lim_{n\to+\infty} a_n H_n(z_0) = 0$ *and* $f(z)$ *is the sum of* (1.2) *in the strip* $S(\tau_0)$, *where* $\tau_0 = |\Im z_0|$. *If the function* $f(z)$ *has a pole on* $\partial S(\tau_0)$, *then it is a simple pole.*

Finally we shall note that the following assertion holds:

Theorem 4. *Let* $z_0 \in G$ *and* $a_n H_n(z_0) = O(n^p)(n \to +\infty)$, *where* p *is a nonnegative integer. If* $f(z)$ *is the sum of the series (1.2) in the strip* $S(\tau_0)$ *with* $\tau_0 = |\Im z_0|$ *and* $f(z)$ *has a pole of order* m *on* $\partial S(\tau_0)$, *then* $m \leq 2p+2$.

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Trakia University
Stara Zagora – 6000, BULGARIA
e-mail: GBoychev@hotmail.com

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