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A NEW HEREDITARILY ℓ^2 BANACH SPACE

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Abstract. We construct a non-reflexive, ℓ^2 saturated Banach space such that every non-reflexive subspace has non-separable dual.

1. Introduction. The aim of the present paper is to provide a new Banach space denoted by $\mathfrak{X}_{nqr}$ which answers a question posed by H. P. Rosenthal. More precisely H. P. Rosenthal had asked if every non-reflexive Banach space X, which is reflexively saturated must contain a proper quasi-reflexive subspace (i.e. a subspace Y such that $0 < \dim Y^{**}/Y < \infty$). We answer this question in negative. Namely the space $\mathfrak{X}_{nqr}$ is ℓ^2 saturated and every non-reflexive subspace has non-separable dual.

In the following paragraphs we present a historical overview of Rosenthal's problem and we analyze the main features of the space $\mathfrak{X}_{nqr}$ and its basic properties.

The class of quasi-reflexive Banach spaces is established with the famous James space J constructed in the early 50's, by R. C. James [11] and is the class of non-reflexive Banach spaces which are nearest to reflexive ones.

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As P. Civin and B. Yood [7] have proved every quasi-reflexive Banach space is reflexively saturated, this result has been generalized by W. Johnson and H. P. Rosenthal [13] to the class of separable Banach spaces X with separable second dual X^{**}.

We recall the definition of two well known classes of Banach spaces.

Definition I. *A Banach space X has the RNP (Radon-Nikodym property) if every closed and bounded subset of X is dentable. A non empty subset F of X is dentable, if for every $\epsilon > 0$ there exists $x_\epsilon \in F$, which does not belong to $\overline{\text{conv}}(F \backslash S(x_\epsilon, \epsilon))$.*

Also

Definition II. *A Banach space $(X, \|\cdot\|)$ has the PCP (Point Continuity Property), if for every non-empty and closed subset F of X, the identity operator $id : (F, w) \longrightarrow (F, \|\cdot\|)$ has at least one point of continuity.*

It is known that if X is a Banach space with separable X^* then X^* has the RNP and if X has the RNP then has the PCP. It is obvious that if X is a Banach space with separable X^{**}, then X has the PCP.

S. F. Bellenot [6] and C. Finet [8] proved independently, in 1987, the following theorem:

Theorem I. *If X is a non-reflexive Banach space which has the PCP and X^* is separable, then every non-trivial w-cauchy sequence $(x_n)_{n\in\mathbb{N}}$ contains a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ which is boundedly complete and $\dim Y^{**}/Y = 1$, where $Y = \overline{\langle x_{k_n}, n \in \mathbb{N}\rangle}$.*

The aforementioned problem posed by Rosenthal is restated as follows.

Problem. *Does every non-reflexive and reflexively saturated Banach space X with the PCP, contains a strictly quasi-reflexive subspace?*

In the case of a Banach space X which has also separable dual the answer is affirmative according to the theorem of Bellenot and Finet. The main goal of this paper is to give negative answer to the problem of H. P. Rosenthal. This is done with the construction of the space $\mathfrak{X}_{nqr}$ which has the following properties.

Theorem. *There exists a separable Banach space $\mathfrak{X}_{nqr}$ with the following properties:*

(1) *The space has a boundedly complete Schauder basis $(e_n)_{n\in\mathbb{N}}$, hence the space has the RNP.*

(2) *The space is $\ell^2(\mathbb{N})$ saturated, namely every closed and infinite dimensional subspace contains an isomorphic copy of $\ell^2(\mathbb{N})$.*

(3) *The space $\mathfrak{X}_{nqr}$ is non-reflexive.*

(4) *Every closed, infinite dimensional and non-reflexive subspace has non-separable dual.*

The norm in $\mathfrak{X}_{nqr}$ is defined to be the completion of a norm on $c_{00}(\mathbb{N})$, which is defined using a norming set G. The norming set G of the space $\mathfrak{X}_{nqr}$ is defined inductively as a subset of $c_{00}(\mathbb{N})$. Its definition is mainly divided in two parts.

In the first part using induction we construct a sequence $(T_r)_{r\in\mathbb{N}}$ of infinitely branching trees of height ω with each branch of T_r consisting of a block sequence $(\phi_i)_{i\in N}$ in $c_{00}(\mathbb{N})$. The T_r special functionals are of the form

$$E\left(\frac{1}{2}\sum_{i=1}^{n}\phi_i\right)$$

where $(\phi_1,\dots,\phi_n)$ is initial segment of T_r and E is an interval of $\mathbb{N}$. The nodes ϕ_i of the tree T_r are built using special segments of the previous trees and are of the form

$$\frac{1}{m_j}\sum_{i=1}^{d}\psi_i$$

where $d\in\mathbb{N}$ with $d\le n_j$ and ψ_i successive elements of $c_{00}(\mathbb{N})$. For each ϕ as above we denote by $w(\phi)$ the weight of ϕ which is equal to m_j. To each T_r special functional x^* we associate the $\mathrm{ind}(x^*)$ be the set of the weights of ϕ_i that involves in the definition of x^*.

In the second stage of the definition of the norming set G we define the functionals x^*, which are of the form

$$x^*=\sum_{i=1}^{d}\lambda_i x_i^*$$

where $\sum_{i=1}^{d}\lambda_i^2\le 1$, x_i^* T_{r_i} special functional and the sets $\mathrm{ind}(x_i^*)$ pairwise disjoint. The fact that the norming set G consists of ℓ^2 convex combinations of structures resulting of trees, a property reminding the classical James tree space [10], yields that the space $\mathfrak{X}_{nqr}$ is ℓ^2 saturated and this is shown in Section 5. In Section 4 we also show that the space $\mathfrak{X}_{nqr}$ has a boundedly complete basis and hence has the RN property. The latter yields that $\mathfrak{X}_{nqr}$ has also the PC property. The most delicated part of the proof is the main property of the space, namely that every non-reflexive subspace Y of $\mathfrak{X}_{nqr}$ has non-separable dual. To prove this we use the sequence of trees $(T_r)_{r\in\mathbb{N}}$ described above. In particular we use the fact that using only one branch of the tree T_r we can produce a dyadic subtree of tree T_{r+1} with each node using exclusively parts of that branch for its definition.

The proof of the property that every closed, non-reflexive subspace of $\mathfrak{X}_{nqr}$ has non-separable dual, uses techniques of the theory of Hereditarily Indecomposable Banach spaces combining with Ramsey type results, which yield the following inequality.

Proposition. *Let $j_0 \in \mathbb{N}$ and $(y_k)_{k\in\mathbb{N}}$ be a block sequence of averages with increasing lengths (as in Remark 7.1).*

Then there exists an $L \in [\mathbb{N}]$ such that for every $f \in (G\backslash F_0)$ with $\mathrm{ind}(f) \subset \{j_0+1, \dots\}$ we have that

$$\left|\left\{n \in L:\ |f(y_k)| \geq \frac{2}{m_{j_0}^2}\right\}\right| \leq 257 m_{j_0}^4.$$

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2. Preliminaries. We make use of the following standard notation throughout this article.

Notation

i. We denote by $c_{00}(\mathbb{N})$ the set $c_{00}(\mathbb{N}) = \{f : \mathbb{N} \to \mathbb{R} : f(n) \neq 0$ for finitely many $n \in \mathbb{N}\}$. For every $x \in c_{00}(\mathbb{N})$ we denote by $\operatorname{supp} x$ the set $\operatorname{supp} x = \{n \in \mathbb{N} : x(n) \neq 0\}$ and by $\operatorname{ran} x$ the minimal interval of $\mathbb{N}$ that contains $\operatorname{supp} x$.

ii. We denote by $(e_n)_n$ the standard Hamel basis of $c_{00}(\mathbb{N})$.

iii. Let E_1, E_2 be two nonempty finite subsets of $\mathbb{N}$. We write $E_1 < E_2$ if $\max E_1 < \min E_2$. If $x_1, x_2 \in c_{00}(\mathbb{N})$ we write $x_1 < x_2$ whenever $\operatorname{ran} x_1 < \operatorname{ran} x_2$. In addition for a sequence $f : \mathbb{N} \to \mathbb{R}$ and E an interval of $\mathbb{N}$ we denote by Ef the sequence $f \cdot X_E$, where X_E is the characteristic function of E.

iv. We fix two sequences of natural numbers $(m_j)_j$ and $(n_j)_j$ defined recursively as follows. We set $m_1 = 2^4$ and $m_{j+1} = m_j^5$ and $n_1 = 2^7$ and $n_{j+1} = (2n_j)^{s_{j+1}}$ where $s_{j+1} = \log_2(m_{j+1}^3)$, $j \geq 1$.

v. For a set A we denote by $|A|$ the cardinality of A and by $[A]$ the set of its infinite subsets.

3. The norming set G of the space $\mathfrak{X}_{nqr}$. In this section we define the norming set of the space $\mathfrak{X}_{nqr}$.

Let $\mathbb{N} = \bigcup_{k\in\mathbb{N}} L_k$, $L_k \subset \mathbb{N}$, $k \in \mathbb{N}$, L_k infinite and pairwise disjoint subsets of $\mathbb{N}$, Ω_1, Ω_2 infinite subsets of $\mathbb{N}$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and $(m_j)_{j\in\mathbb{N}}$, $(n_j)_{j\in\mathbb{N}}$ the sequences defined before.

We set

$$F_0 = \{|q_n|e_n^* : \ |q_n| = 1, n \in \mathbb{N}\} \bigcup \{0\} \text{ and}$$

$$F_j = \left\{ \frac{1}{m_j} \sum_{i\in F} \epsilon_i e_i^* : \ F \text{ finite with } |F| \leq n_j, |\epsilon_i| = 1, i \in F \right\} \text{ where } j \in L_1.$$

Let

$$K_1 = \left(\bigcup_{j\in L_1} F_j \right) \bigcup F_0,$$

$$W_1 = \{(f_1, \ldots, f_d) : \ d \in \mathbb{N}, f_1 < \ldots < f_d, f_i \in (K_1 \backslash F_0)\} \text{ and}$$

$$L_1 = \{l_n^{(1)} : \ n \in \mathbb{N}\}$$

We observe that $\|f\|_\infty = \dfrac{1}{m_j}$, $f \in F_j$, $j \in L_1$.

Since W_1 is countable there exists an injective coding map $\sigma_1 : W_1 \longrightarrow \{l_n^{(1)} : \ n \in \Omega_2\}$ such that

$$\sigma_1(f_1, \ldots, f_d) > \max\{k \in L_1 : \text{ exists } i \in \{1, \ldots, d\} \text{ with } f_i \in F_k\}$$

for all $(f_1, \ldots, f_d) \in W_1$.

A finite or infinite sequence $(f_i)_i$ with $f_i \in (K_1 \backslash F_0)$ is said to be σ_1 special if

(1) $f_i < f_{i+1}$ for all i.

(2) $f_1 \in \bigcup_{n\in\Omega_1} F_{l_n^{(1)}}$ and $f_{i+1} \in F_{\sigma_1(f_1,\ldots,f_i)}$ for all i.

If $(f_i)_i$ is σ_1 special sequence, then we define the sequence of indices $(\mathrm{ind}(f_i))_i$ as follows:

(1) $\mathrm{ind}(f_i) \in L_1$ for all i.

(2) $f_1 \in F_{\mathrm{ind}(f_1)}$ and $\mathrm{ind}(f_1) \in \{l_n^1 : \ n \in \Omega_1\}$.

(3) $\mathrm{ind}(f_{i+1}) = \sigma_1(f_1, \ldots, f_i)$ for all i.

Hence in every σ_1 special sequence we correspond the sequence of indices.

The set W_1 with the relation

$$(f_1, \ldots, f_k) \leq_1 (g_1, \ldots, g_n) \text{ if and only if } k \leq n \text{ and } f_i = g_i \text{ for all } i = 1, \ldots, k$$

is a tree and the set of all finite σ_1 special sequences which is denoted with T_1 is a complete subtree of W_1. The infinite σ_1 branches of the tree T_1 are identified with the set of all infinite σ_1 special sequences and the set of finite σ_1 branches with the set of all finite σ_1 special sequences. The tree T_1 is called the tree of finite σ_1 special sequences.

A σ_1 special functional is a sequence of the form

$$x^* = \frac{1}{2}\left(E \sum_i f_i\right)$$

where $(f_i)_i$ is a σ_1 special sequence, E interval of $\mathbb{N}$ and $\sum\limits_i f_i$ is a finite or infinite sum.

$E\sum\limits_i f_i$ denotes the sequence $\left(\sum\limits_i f_i\right)\chi_E$, where χ_E is the characteristic function of E.

If E is infinite interval and $(f_i)_i$ is infinite sequence then the previous sum is considered in the topology of pointwise convergence.

If E is finite interval then x^* is said to be finite σ_1 special functional and the set of all these functionals is denoted with S_1. The set S_1 is said to be the set of finite σ_1 special functionals.

The set of indices of x^* is defined to be the set:

$$\mathrm{ind}(x^*) = \{\mathrm{ind}(f_i) : \ E \cap \mathrm{supp}(f_i) \neq \emptyset\}.$$

Therefore we have define the set $K_1 \subset c_{00}(\mathbb{N})$, the tree T_1 of finite σ_1 special sequences and the set S_1 of finite σ_1 special functionals.

We will define inductively

i. A sequence $(K_r)_{r\in\mathbb{N}}$ of subsets of $\mathbb{N}$.

ii. A sequence $(\sigma_r)_{r\in\mathbb{N}}$ of injective maps which are called coding maps.

iii. A sequence of trees $(T_r)_{r\in\mathbb{N}}$ (each tree T_r is called the tree of finite σ_r special sequences).

iv. A sequence of sets $(S_r)_{r\in\mathbb{N}}$ (each S_r is called the set of finite σ_r special functionals)

as follows:

Let $r\in\mathbb{N}$ and we assume that the following have been defined

i. The sets $K_1,\dots,K_r$.

ii. The coding maps $\sigma_1,\dots,\sigma_r$.

iii. The trees $T_1,\dots,T_r$.

iv. The sets $S_1,\dots,S_r$.

Then the set K_{r+1} is defined as follows:

$$K_{r+1}=\left(\bigcup_{j\in L_{r+1}} F_j\right)\bigcup F_0$$

where

$$F_j=\left\{\frac{1}{m_j}\sum_{i=1}^{d}\phi_i : d\in\mathbb{N},\ d\le n_j,\ \phi_1<\dots<\phi_d,\right.$$
$$\left.\phi_i\in\left(\bigcup_{i=1}^{r}K_i\right)\bigcup\left(\bigcup_{i=1}^{r}S_i\right),\ i=1,\dots,d\right\}\ \text{for } j\in L_{r+1}.$$

We observe that $\|f\|_\infty\le\dfrac{1}{m_j}$, $f\in F_j$, $j\in L_{r+1}$.

Let

$$L_{r+1}=\{l_n^{(r+1)}:\ n\in\mathbb{N}\}\ \text{and}$$
$$W_{r+1}=\{(f_1,\dots,f_d): d\in\mathbb{N},\ f_1<\dots<f_d,\ f_i\in(K_{r+1}\backslash F_0)\},$$

W_{r+1} is countable, so we may choose an injective coding map $\sigma_{r+1}: W_{r+1}\longrightarrow\{l_n^{(r+1)}: n\in\Omega_2\}$ such that

$$\sigma_{r+1}(f_1,\dots,f_d)>\max\{k\in L_{r+1}:\ \text{exists } i\in\{1,\dots,d\} \text{ with } f_i\in F_k\}$$

for every $(f_1, \dots, f_d) \in W_{r+1}$.

A finite or infinite sequence $(f_i)_i$ with $f_i \in (K_{r+1} \backslash F_0)$ is called σ_{r+1} special sequence if

(1) $f_i < f_{i+1}$ for all i.

(2) $f_1 \in \bigcup\limits_{n \in \Omega_1} F_{l_n^{(r+1)}}$ and $f_{i+1} \in F_{\sigma_{r+1}(f_1, \dots, f_d)}$ for all i.

The sequence of indices of a σ_{r+1} special sequence is defined as in the case of σ_1 special sequences.

The set W_{r+1} endowed with a relation $\leq_{r+1}$ which is analogous of that of W_1 is a tree and the set of finite σ_{r+1} special sequences, denoted by T_{r+1}, is a complete subtree of W_{r+1}. The set of infinite σ_{r+1} branches of the tree T_{r+1} is identified with the set of infinite σ_{r+1} special sequences and the set of finite σ_{r+1} branches with the set of finite σ_{r+1} special sequences.

The σ_{r+1} special functionals are defined in a similar way as the σ_1 and the set of finite σ_{r+1} special functionals is denoted with S_{r+1}.

Analogously we define the set of indices of a σ_{r+1} special functional.

We set

$$K = \left(\bigcup_{r \in \mathbb{N}} K_r \right) \bigcup \left(\bigcup_{r \in \mathbb{N}} S_r \right)$$

and

$$W = \{(f_1, \dots, f_d) : d \in \mathbb{N},\ f_1 < \cdots , < f_d,\ f_i \in (K \backslash F_0)\}.$$

A block finite or infinite sequence $(f_i)_i$ with $f_i \in (K \backslash F_0)$ is said to be σ special sequence if and only if there exists $r \in \mathbb{N}$ such that $(f_i)_i$ is a σ_r special sequence.

The set W endowed with the relation

$$(f_1, \dots, f_k) \leq (g_1, \dots, g_n) \text{ if and only if}$$

$$(f_1, \dots, f_k), (g_1, \dots, g_n) \text{ belong to some } W_i \text{ and } (f_1, \dots, f_k) \leq_i (g_1, \dots, g_n)$$

is a tree and the set of finite σ special sequences, denoted with T, is a complete subtree of W. The set of infinite σ branches of the tree T is identified with the set of infinite σ special sequences and the set of finite σ branches with the set of finite σ special sequences.

A sequence x^* is said to be σ (finite) special functional if and only if there exists $r \in \mathbb{N}$ such that x^* is a (finite) σ_r special functional.

We denote by S the set $\bigcup\limits_{r \in \mathbb{N}} S_r$.

We set

$$G = \left\{ \sum_{i=1}^{d} a_i x_i^* : \ d \in \mathbb{N}, a_i \in \mathbb{Q}, \right.$$

$$\left. \sum_{i=1}^{d} a_i^2 \leq 1, x_i^* \in (K \backslash F_0), \ \text{ind}(x_i^*) \text{ pairwise disjoint} \right\} \bigcup F_0.$$

The space $\mathfrak{X}_{nqr}$ is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_G)$ where $\|x\|_G = \sup\{|f(x)| : f \in G\}$.

Remarks 3.1.

(1) The sets F_j, $j \in \mathbb{N}$ are closed in restrictions to finite intervals of $\mathbb{N}$.

(2) G is closed in restrictions to finite intervals of $\mathbb{N}$.

(3) If $f \in G$ then $\|f\|_\infty \leq 1$.

(4) The basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{nqr}$ is bimonotone and $\|e_n\|_G = 1$, $n \in \mathbb{N}$.

(5) The sets F_j, $j \in L_1$ are compact in the topology of pointwise convergence.

4. The basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{nqr}$ is boundedly complete. In this section we prove that the basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{nqr}$ is boundedly complete. The proof is based on the definition of $\mathfrak{X}_{nqr}$.

Proposition 4.1. *The basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{nqr}$ is boundedly complete.*

Proof. Assume that the conclusion of the proposition fails. Then there exist $M > 0$, $\epsilon_0 > 0$, $(a_n)_{n\in\mathbb{N}}$ sequence of real numbers, $(m_n)_{n\in\mathbb{N}}$ strictly increasing sequence of natural numbers and $u_n = \sum_{i=m_n+1}^{m_{n+1}} a_i e_i$, $n \in \mathbb{N}$, block of $(e_n)_{n\in\mathbb{N}}$ such that

$$\|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_G \leq M, n \in \mathbb{N} \tag{4.1}$$

and

$$\epsilon_0 < \|u_n\|_G, n \in \mathbb{N}. \tag{4.2}$$

We distinguish the following cases.

1. Suppose that there exist $\epsilon_1 > 0$ and $L \subset \mathbb{N}$ infinite with $\epsilon_1 \leq \|u_n\|_\infty$, $n \in L$.

We assume without loss of generality that $\epsilon_1 \leq \|u_n\|_\infty$, $n \in \mathbb{N}$.
For each $n \in \mathbb{N}$ there exists $t_n \in \operatorname{supp}(u_n)$ such that

$$|e_{t_n}(u_n)| \geq \epsilon_1.$$

We set

$$f_j = \frac{1}{m_j} \sum_{i=1}^{n_j} \epsilon_i e^*_{t_i}, \quad |\epsilon_i| = 1, i = 1, \ldots, n_j, \quad v_j = \sum_{i=1}^{n_j} u_i$$

and

$$k_j = \max \operatorname{supp}(u_{n_j}).$$

Then $f_j \in G$, $j \in \mathbb{N}$ and

$$\frac{n_j}{m_j} \cdot \epsilon_1 \leq f_j(v_j) = f_j \left(\sum_{i=1}^{k_j} \alpha_i e_i \right) \leq \left\| \sum_{i=1}^{k_j} \alpha_i e_i \right\|_G \leq M, \quad j \in L_1$$

a contradiction, since $\lim_j \dfrac{n_j}{m_j} = \infty$.

2. Suppose that for every $\epsilon > 0$ and every $L \subset \mathbb{N}$ infinite, there exists $n \in L$ with $\|u_n\|_\infty < \epsilon$.

It is obvious that $\lim_n \|u_n\|_\infty = 0$. We will prove inductively that

$$\lim_n \sup \{|\phi(u_n)| : \ \phi \in (K_l \cup S_l)\} = 0, \quad l \in \mathbb{N}.$$

Let $j \in L_1$. We will show that

$$\lim_n \sup\{|f(u_n)| : \ f \in F_j\} = 0.$$

Let $f \in F_j$. Then $f = \dfrac{1}{m_j} \sum_{i \in F} \epsilon_i e^*_i$, $|\epsilon_i| = 1$, $i \in F$, $|F| \leq n_j$. Therefore

$$|f(u_n)| = \frac{1}{m_j} \left| \left(\sum_{i \in G} \epsilon_i e^*_i \right) (u_n) \right| \leq \frac{1}{m_j} n_j \|u_n\|_\infty = \frac{n_j}{m_j} \|u_n\|_\infty$$

and thus

$$\lim_n \sup\{|f(u_n)| : \ f \in F_j\} = 0, \quad j \in L_1.$$

Let $l \in \mathbb{N}$ and we assume that

$$\lim_n \sup\{|\phi(u_n)| : \phi \in (K_i \cup S_i)\} = 0, \quad i = 1, \dots, l. \tag{4.3}$$

We will show that if $j \in L_{l+1}$ then $\lim_n \sup\{|f(u_n)| : f \in F_j\} = 0$.

Let $f \in F_j$. Then

$$f = \frac{1}{m_j} \sum_{i=1}^{d} \phi_i, d \le n_j, \phi_1 < \cdots < \phi_d, \phi_i \in \left(\bigcup_{i=1}^{l} K_i\right) \bigcup \left(\bigcup_{i=1}^{l} S_i\right), \quad i = 1, \dots, d.$$

We have that

$$|f(u_n)| = \left| \left(\frac{1}{m_j} \sum_{i=1}^{d} \phi_i \right) (u_n) \right| \le \frac{n_j}{m_j} \sum_{i=1}^{l} \sup\{|\phi(u_n)| : \phi \in (K_i \cup S_i)\}$$

hence

$$\sup\{|f(u_n)| : f \in F_j\} \le \frac{n_j}{m_j} \sum_{i=1}^{l} \sup\{|\phi(u_n)| : \phi \in (K_i \cup S_i)\}.$$

Equation (4.3) yields

$$\lim_n \sup\{|f(u_n)| : f \in F_j\} = 0, \quad j \in L_{l+1}. \tag{4.4}$$

The next step is to show that

$$\lim_n \sup\{|\phi(u_n)| : \phi \in S_{l+1}\} = 0.$$

Assume the contrary. Then there exist $\epsilon_1 > 0$ and $M_1 \subset \mathbb{N}$ infinite such that

$$\epsilon_1 < \sup\{|\phi(u_n)| : \phi \in S_{l+1}\}, \quad n \in M_1.$$

Without loss of generality we may assume that $\epsilon_1 < \sup\{|\phi(u_n)| : \phi \in S_{l+1}\}$, $n \in \mathbb{N}$.

The last inequality yields that for every $n \in \mathbb{N}$ there exists $\phi_n \in S_{l+1}$ with $\text{ran}(\phi_n) \subset \text{ran}(u_n)$ and $\epsilon_1 < |\phi_n(u_n)|$.

We distinguish the following cases.

2A. The set $A = \{\text{ind}(\phi_n) : n \in \mathbb{N}\}$ is finite.

Let $\bigcup_{n\in\mathbb{N}} \text{ind}(\phi_n) = \{j_1, \dots, j_k\} \subset L_{l+1}$. We have that

$$|\phi_n(u_n)| \le \sup\{|f(u_n)| : f \in F_{j_1}\} + \cdots + \sup\{|f(u_n)| : f \in F_{j_k}\}, n \in \mathbb{N}.$$

From equation (4.4) we get that $\lim_n |\phi_n(u_n)| = 0$, a contradiction, since $\epsilon_1 < |\phi_n(u_n)|$, $n \in \mathbb{N}$.

2B. The set $A = \{\mathrm{ind}(\phi_n) : n \in \mathbb{N}\}$ is infinite.

We may assume without loss of generality that $\mathrm{ind}(\phi_n) \neq \mathrm{ind}(\phi_m)$ for every $n \neq m$. We choose $q \in \mathbb{N}$ such that

$$q > \frac{2}{\epsilon_1} \cdot M. \tag{4.5}$$

We have that $\epsilon_1 < |\phi_1(u_1)|$. Let

$$\phi_n = z_n^2 + \psi_n^2, n \geq 2$$

where $\mathrm{ind}(z_n^2) \subset \mathrm{ind}(\phi_1)$, $n \geq 2$ and $\min \mathrm{ind}(\psi_n^2) > \max \mathrm{ind}(\phi_1)$, $n \geq 2$.

The set A is infinite, so there exist $i_2 \geq 2$ with $\mathrm{ind}(\psi_n^2) \neq \emptyset, n \geq i_2$.

Since $\epsilon_1 < |\phi_n(u_n)|$, $n \geq i_2$ it follows that for every $n \geq i_2$ we get

$$|z_n^2(u_n)| > \frac{\epsilon_1}{2} \text{ or } |\psi_n^2(u_n)| > \frac{\epsilon_1}{2}.$$

If the set $\left\{n \in \mathbb{N} : n \geq i_2 \text{ and } |z_n^2(u_n)| > \frac{\epsilon_1}{2}\right\}$ is infinite, then following the steps of case (1) we come to a contradiction.

If the set $\left\{n \in \mathbb{N} : \ n \geq i_2 \ \text{ and } \ |z_n^2(u_n)| > \frac{\epsilon_1}{2}\right\}$ is finite then there exist $j_2 > i_2$ such that

$$\min \mathrm{ind}(\psi_{j_2}^2) > \max \mathrm{ind}(\phi_1) \text{ and } |\psi_{j_2}^2(u_{j_2})| > \frac{\epsilon_1}{2}.$$

Let

$$\phi_n = z_n^3 + \psi_n^3, \quad n \geq j_2 + 1$$

where $\mathrm{ind}(z_n^3) \subset \mathrm{ind}(\phi_1) \cup \mathrm{ind}(\psi_{j_2}^2)$, $n \geq j_2 + 1$ and $\min \mathrm{ind}(\psi_n^3) > \max(\mathrm{ind}(\phi_1) \cup \mathrm{ind}(\psi_{j_2}^2))$, $n \geq j_2 + 1$.

Without loss of generality we assume that $\mathrm{ind}(\psi_n^3) \neq \emptyset$, $n \geq j_2 + 1$.

Since $\epsilon_1 < |\phi_n(u_n)|$, $n \geq j_2 + 1$ it follows that for every $n \geq j_2 + 1$ we get that

$$|z_n^3(u_n)| > \frac{\epsilon_1}{2} \quad \text{or} \quad |\psi_n^3(u_n)| > \frac{\epsilon_1}{2}.$$

If the set $\left\{n \in \mathbb{N} : \ n \geq j_2 + 1 \text{ and } |z_n^3(u_n)| > \frac{\epsilon_1}{2}\right\}$ is infinite, then following the steps of case (1) we come to a contradiction.

If the set $\left\{n \in \mathbb{N} : n \geq j_2 + 1 \text{ and } |z_n^3(u_n)| > \frac{\epsilon_1}{2}\right\}$ is finite there exists $j_3 > j_2$ with

$$\min \operatorname{ind}(\psi_{j_3}^3) > \max(\operatorname{ind}(\phi_1) \cup \operatorname{ind}(\psi_{j_2}^2))$$

and

$$|\psi_{j_3}^3(u_{j_3})| > \frac{\epsilon_1}{2}.$$

Hence we come to a contradiction or we construct functionals $\psi_{j_2}^2, \ldots, \psi_{j_{q^2}}^{q^2}$ with disjoint indices, $\operatorname{ran}(\psi_{j_i}) \subset \operatorname{ran}(u_{j_i})$, $i = 2, \ldots, q^2$ and $|\psi_{j_i}^i(u_{j_i})| > \frac{\epsilon_1}{2}$, $i = 2, \ldots, q^2$.

The functional

$$f = \frac{\epsilon_1}{q} \cdot \phi_1 + \sum_{i=2}^{q^2} \frac{\epsilon_i}{q} \psi_{j_i}, \quad |\epsilon_i| = 1, \quad i = 1, \ldots, q^2$$

belongs to G. We consider the vector

$$u = u_1 + \sum_{i=2}^{q^2} u_{j_i}.$$

We have that

$$|f(u)| = \left| \frac{\epsilon_1}{q} \phi_1(u_1) + \sum_{i=2}^{q^2} \frac{\epsilon_i}{q} \psi_{j_i}(u_{j_i}) \right| \geq \frac{1}{q} q^2 \frac{\epsilon_1}{2} = q \frac{\epsilon_1}{2}$$

and from (4.5) we get that $|f(u)| > M$, a contradiction.

Therefore we have proved that $\lim_n \sup\{|\phi(u_n)| : \phi \in S_{l+1}\} = 0$.

In a similar way we prove that $\lim_n \sup\{|\phi(u_n)| : \phi \in S_1\} = 0$.

It is not hard to see that

$$\lim_n \sup \left\{ |\phi(u_n)| : \phi \in \left(\bigcup_{i=1}^{l} K_i \right) \bigcup \left(\bigcup_{i=1}^{l} S_i \right) \right\} = 0, \ l \in \mathbb{N}.$$

Hence we may construct a subsequence $(u_{n_l})_{l \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ such that

$$|f(u_{n_l})| < \frac{\epsilon_0}{l 2^{l+1}}, \quad l \in \mathbb{N}, \quad f \in \left(\bigcup_{i=1}^{l} K_i \right) \bigcup \left(\bigcup_{i=1}^{l} S_i \right).$$

From (4.2) we get that $\epsilon_0 < \|u_{n_l}\|_G, l \in \mathbb{N}$, so for every $l \in \mathbb{N}$ there exist $f_l \in G$ with $\operatorname{ran}(f_l) \subset \operatorname{ran}(u_{n_l})$ and $\epsilon_0 < |f_l(u_{n_l})|$. We choose $q \in \mathbb{N}$ such that

$$q > \frac{2}{\epsilon_0} M. \tag{4.6}$$

We have that $\frac{\epsilon_0}{2} < \epsilon_0 < |f_1(u_{n_1})|$. Let

$$r_1 = \max \operatorname{ind}(f_1)$$

and

$$f_l = z_l^2 + \psi_l^2, l \geq 2$$

where $\operatorname{ind}(z_l^2) \subset \{1, \dots, r_1\}$, $l \geq 2$ and $\operatorname{ind}(\psi_l^2) \subset \{r_1 + 1, \dots\}$, $l \geq 2$.

We assume that the special functionals in $z_l^2, l \geq 2$ belong to $S_1 \cup \cdots \cup S_{i_1}$.

We have that $\epsilon_0 < |f_l(u_{n_l})|$, $l \geq 2$, hence for every $l \geq 2$ it follows that

$$\frac{\epsilon_0}{2} < |z_l^2(u_{n_l})| \text{ or } \frac{\epsilon_0}{2} < |\psi_l^2(u_{n_l})|.$$

We set

$$A^2 = \left\{ l \in \mathbb{N} : \ l \geq 2 \text{ and } |z_l^2(u_{n_l})| > \frac{\epsilon_0}{2} \right\}.$$

The set A^2 is finite. Assume the contrary. Then we may choose $l_1 \in A^2$ with $l_1 > \max\{r_1, i_i\}$ and thus $\frac{\epsilon_0}{2} < |z_{l_1}^2(u_{n_{l_1}})| < r_1 \frac{\epsilon_0}{l_1 2^{l_1+1}} < \frac{\epsilon_0}{2^{l_1+1}}$, a contradiction.

Therefore there exist $j_2 \geq 2$ such that

$$|\psi_{j_2}^2(u_{n_{j_2}})| > \frac{\epsilon_0}{2} \text{ and } \inf(\psi_{j_2}^2) \subset \{r_1 + 1, \dots\}.$$

Let

$$r_2 = \max \operatorname{ind}(\psi_{j_2}^2) \text{ and } f_l = z_l^3 + \psi_l^3, l \geq j_2 + 1$$

where $\operatorname{ind}(z_l^3) \subset \{1, \dots, r_2\}$, $l \geq j_2 + 1$ and $\operatorname{ind}(\psi_l^3) \subset \{r_2 + 1, \dots\}$, $l \geq j_2 + 1$. It is obvious that $r_2 > r_1$.

We assume that the special functionals in z_l^3, $l \geq j_2 + 1$ belong to $S_1 \cup \cdots \cup S_{i_2}$. We have that

$$|f_l(u_{n_l})| > \epsilon_0, \quad l \geq j_2 + 1$$

so for every $l \geq j_2 + 1$ it follows that

$$|z_l^3(u_{n_l})| > \frac{\epsilon_0}{2} \quad \text{or} \quad |\psi_l^3(u_{n_l})| > \frac{\epsilon_0}{2}.$$

We set

$$A^3 = \left\{ l \in \mathbb{N} : l \geq j_2 + 1 \text{ and } |z_l^3(u_{n_l})| > \frac{\epsilon_0}{2} \right\}.$$

The set A^3 is finite. If is infinite then we may choose $l_2 > \max\{r_2, i_2\}$, so

$$\frac{\epsilon_0}{2} < |z_{l_2}^3(u_{n_{l_2}})| < r_2 \frac{\epsilon_0}{l_2 2^{l_2+1}} < \frac{\epsilon_0}{2^{l_2+1}}$$

a contradiction.

Hence there exist $j_3 > j_2$ such that $|\psi_{j_3}^3(u_{n_{j_3}})| > \frac{\epsilon_0}{2}$.

We may construct functionals $\psi_{j_2}^2, \ldots, \psi_{j_{q^2}}^{q^2}$ with disjoint indices and

$$|\psi_{j_i}^i(u_{n_{j_i}})| > \frac{\epsilon_0}{2}, \quad i = 1, \ldots, q^2.$$

We consider the functional

$$f = \frac{\epsilon_1}{q} f_1 + \sum_{i=2}^{q^2} \frac{\epsilon_i}{q} \psi_{j_i}^i, \quad |\epsilon_i| = 1, \quad i = 1, \ldots, q^2$$

which belongs to G and the vector

$$u = u_1 + \sum_{i=2}^{q^2} u_{n_{j_i}}.$$

We have that

$$|f(u)| \geq \frac{\epsilon_0}{2} \frac{1}{q} q^2 = q \frac{\epsilon_0}{2}$$

and from (4.6) it follows that $|f(u)| > M$, a contradiction. Therefore $(e_n)_{n \in \mathbb{N}}$ is boundedly complete.

5. The space $\mathfrak{X}_{nqr}$ is ℓ^2 saturated. In this section we prove that the sequence $(F_j)_{j \in L_1}$ is JTG (Definition 5.1) and $\mathfrak{X}_{nqr}$ is ℓ^2 saturated.

Definition 5.1. *A sequence $(F_j)_{j \in \mathbb{N} \bigcup \{0\}}$ of subsets of $c_{00}(\mathbb{N})$ is said to be James tree generating (JTG) provided that satisfies the following conditions:*

(1) $F_0 = \{|q_n| e_n^* : |q_n| = 1, n \in \mathbb{N}\} \bigcup \{0\}$ *and each F_j is nonempty, countable, symmetric, closed in restrictions to intervals of $\mathbb{N}$ and compact in the topology of pointwise convergence.*

(2) *Setting* $\tau_j = \sup\{\|f\|_\infty : f \in F_j\}$, $j \in \mathbb{N}$, *the sequence* $(\tau_j)_{j\in\mathbb{N}}$ *is strictly decreasing and* $\sum\limits_{j=1}^{\infty} \tau_j^2 \leq 1$.

(3) *For every block sequence* $(x_k)_{k\in\mathbb{N}}$ *of* $c_{00}(\mathbb{N})$, *every* $j \in \mathbb{N}\bigcup\{0\}$ *and every* $\delta > 0$ *there exists a vector* $x \in < x_k : k \in \mathbb{N} >$ *such that*

$$\delta \sup\left\{ f(x) : f \in \bigcup_{i=0}^{\infty} F_i \right\} > \sup\{f(x) : f \in F_j\}.$$

Lemma 5.1. *The sequence* $(F_j)_{j\in L_1\bigcup\{0\}}$ *is JTG.*

Proof. **1.** Each F_j, $j \in L_1$ is countable, closed in restrictions to intervals of $\mathbb{N}$ and compact in the topology of pointwise convergence.

2. Setting $\tau_j = \sup\{\|f\|_\infty : f \in F_j\}$, $j \in L_1$ then $\sum\limits_{j\in L_1} \tau_j^2 \leq 1$.

3. For every block sequence $(x_k)_{k\in\mathbb{N}}$ of $c_{00}(\mathbb{N})$, every $j \in L_1\bigcup\{0\}$ and every $\delta > 0$ there exists a vector $x \in \langle x_k : k \in \mathbb{N}\rangle$ such that

$$\delta \cdot \sup\left\{ |f(x)| : f \in \left(\bigcup_{j\in L_1} F_j\right)\bigcup F_0 \right\} > \sup\{|f(x)| : f \in F_j\}.$$

We shall prove the last property. Assume the contrary. Then there exists a block sequence $(x_k)_{k\in\mathbb{N}}$ of $c_{00}(\mathbb{N})$ with $\|x_k\|_{K_1} = 1$, $k \in \mathbb{N}$, $j \in L_1 \cup \{0\}$ and there exists $\delta > 0$ such that for every vector $x \in \langle x_k : k \in \mathbb{N}\rangle$ with $x \neq 0$ it follows that

$$\delta \cdot \|x\|_{K_1} \leq \sup\{|f(x)| : f \in F_j\}.$$

Therefore

$$\delta \cdot \left\| \sum_{k=1}^{n} \alpha_k x_k \right\|_{K_1} \leq \left\| \sum_{k=1}^{n} \alpha_k x_k \right\|_{F_j} \quad \text{for every } n \geq 1 \text{ and } \alpha_1, \ldots, \alpha_n \text{ real numbers.}$$

It is obvious that $\frac{\delta}{2} < \|x_k\|_{F_j}$, $k \in \mathbb{N}$.

Let $j \in L_1$. We observe that

$$\|x_k\|_\infty \geq \frac{\delta m_j}{2n_j}, \quad k \in \mathbb{N}.$$

Assume that there exists $k \in \mathbb{N}$ with

$$\|x_k\|_\infty < \frac{\delta m_j}{2n_j}.$$

Let $f \in F_j$. Then $f = \frac{1}{m_j} \sum_{i \in F} \epsilon_i e_i^*$, $|F| \le n_j$, $|\epsilon_i| = 1$, $i \in F$. We have that

$$|f(x_k)| \le \frac{1}{m_j} \sum_{i \in F} \|x_k\|_\infty < \frac{1}{m_j} \cdot \frac{\delta m_j}{2n_j} \cdot |F| \le \frac{1}{m_j} \cdot \frac{\delta m_j}{2n_j} \cdot n_j = \frac{\delta}{2}$$

a contradiction.

Since $\frac{\delta m_j}{2n_j} \le \|x_k\|_\infty$, $k \in \mathbb{N}$ it follows that for every $k \in \mathbb{N}$ there exists $t_k \in \operatorname{supp}(x_k)$ such that

$$|e_{t_k}^*(x_k)| \ge \frac{\delta m_j}{2n_j}.$$

From the fact that $\lim_j \frac{n_j}{m_j} = \infty$ it follows that there exists $j_1 > j$ with $\frac{n_{j_1}}{m_{j_1}} > \frac{2n_j^2}{\delta^2 m_j^2}$.

We consider the functional $f = \frac{1}{m_{j_1}} \sum_{k=1}^{n_{j_1}} e_{t_k}^* \in F_{j_1}$ and the vector $x = \sum_{k=1}^{n_{j_1}} \varepsilon x_k$ where $|\varepsilon_k| = 1, k$.

We have that

$$\delta \cdot \|x\|_{K_1} \le \|x\|_{F_j} \le \frac{n_j}{m_j}.$$

On the other hand

$$\delta\|x\|_{K_1} \ge \delta \cdot f(x) \ge \delta \cdot \frac{n_{j_1}}{m_{j_1}} \cdot \frac{\delta m_j}{2n_j} > \delta \frac{2n_j^2}{\delta^2 m_j^2} \cdot \frac{\delta^2 m_j}{2n_j} = \frac{n_j}{m_j}$$

a contradiction. Also if $j = 0$ we come to a contradiction.

Lemma 5.2. *Let $Y = \langle y_n : n \in \mathbb{N} \rangle$ be a block subspace in $\mathfrak{X}_{nqr}$ and let $\epsilon > 0$. Then there exists a vector $y \in Y$ such that $\|y\|_G = 1$ and $|x^*(y)| < \epsilon$ for every $x^* \in K$.*

Proof. Since the sequence $(F_j)_{j \in L_1 \bigcup \{0\}}$ is JTG, it follows that the identity operator id : $\mathfrak{X}_{G_1} \longrightarrow Y_{K_1 \bigcup S_1}$ is strictly singular. For a proof we refer Lemma B.11 in [1].

The set $G_1 = \left\{ \sum_{k=1}^{d} \alpha_k x_k^* : d \in \mathbb{N}, \alpha_k \in \mathbb{Q}, \sum_{k=1}^{d} \alpha_k^2 \leq 1, x_k^* \in S_1 \bigcup (K_1 \backslash F_0) \right.$ and $\left. \text{ind}(x_k^*) \text{ pairwise disjoint} \right\} \bigcup F_0$ defines a norm $\|\cdot\|_{G_1}$ on $c_{00}(\mathbb{N})$ by the rule

$$\|x\|_{G_1} = \sup\{f(x) : \ f \in G_1\}.$$

The space $\mathfrak{X}_{G_1}$ is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{G_1})$.

Similarly the space $Y_{K_1 \bigcup S_1}$ is the completion of $(c_{00}(\mathbb{N}), \|\cdot\|_{K_1 \bigcup S_1})$, where $S_1 \bigcup K_1$ is the norming set of this space.

Hence the identity operator $\text{id} : \mathfrak{X}_{nqr} \longrightarrow Y_{K_1 \bigcup S_1}$ is strictly singular.

We will prove the lemma by induction.

Let $r \in \mathbb{N}$, $r > 1$ and we assume that the identity operators $\text{id} : \mathfrak{X}_{nqr} \longrightarrow Y_{K_i \bigcup S_i}, i \leq r$ are strictly singular. The space $Y_{K_i \bigcup S_i}$ is the completion of the space $(c_{00}(\mathbb{N}), \|\cdot\|_{K_i \bigcup S_i})$. The norming set for this norm is the set $K_i \bigcup S_i$.

We will prove that the identity operator $id : \mathfrak{X}_{nqr} \longrightarrow Y_{K_{r+1}}$ is strictly singular.

Assume the contrary. Then there exists $\epsilon_0 > 0$ and $\langle w_n : n \in \mathbb{N}\rangle$ block subspace of $\mathfrak{X}_{nqr}$ such that for every $w \in \langle w_n : n \in \mathbb{N}\rangle$ with $\|w\|_G = 1$ there exists $f \in K_{r+1}$ with $\epsilon_0 \leq |f(w)|$.

Therefore for every $w \in \langle w_n : n \in \mathbb{N}\rangle$ with $\|w\|_G = 1$ there exists $f \in K_{r+1}$ with

$$\epsilon_0 \leq |f(w)| \leq \|w\|_{K_{r+1}}. \tag{5.1}$$

It is not hard to see that the identity operator

$$\text{id} : \mathfrak{X}_{nqr} \longrightarrow Y_{\left(\bigcup\limits_{i=1}^{r} K_i\right) \bigcup \left(\bigcup\limits_{i=1}^{r} S_i\right)} \text{ is strictly singular.} \tag{5.2}$$

From (5.2) we may choose $z_1 \in \langle w_n : n \in \mathbb{N}\rangle$ with $\|z_1\|_G = 1$ such that $\|z_1\|_{K_i \bigcup S_i} < \frac{\epsilon_0}{2}$, $i = 1, \ldots, r$ and from (5.1) we get that

$$\|z_1\|_{K_i \bigcup S_i} < \frac{\epsilon_0}{2} < \|z_1\|_{K_{\lambda+1}}, \quad i = 1, \ldots, r.$$

There exists I_1 finite subset of L_{r+1} such that

$$\frac{1}{m_j} < \frac{\epsilon_0}{2} \cdot \frac{1}{\|z_1\|_1} \quad \text{for every} \quad j \in (L_{r+1} \backslash I_1),$$

where $\|\cdot\|_1$ is the l_1 norm.

Consequently

$$|f(z_1)| < \frac{\epsilon_0}{2} \text{ for every } f \in \left[K_{r+1} \Big\backslash \left(\left(\bigcup_{j\in I_1} F_j \right) \bigcup F_0 \right) \right].$$

From (5.1) and (5.2) we may choose a vector $z_2 \in \langle w_n : n \in \mathbb{N}\rangle$ with $z_2 > z_1$, $\|z_2\|_G = 1$ such that

$$\|z_2\|_{K_i \bigcup S_i} < \frac{\epsilon_0}{2^2} \cdot \frac{m_{\min I_1}}{n_{\max I_1}} \leq \frac{\epsilon_0}{2} < \|z_2\|_{K_{r+1}}, \quad i = 1, \ldots, r. \tag{5.3}$$

Let $f \in \bigcup_{j\in I_1} F_j$. Then $f = \frac{1}{m_j} \cdot \sum_{i=1}^{d} \phi_i$, $\phi_1 < \cdots < \phi_d$, $d \leq n_j$, $\phi_1, \ldots, \phi_d$ belong to $\left(\bigcup_{i=1}^{r} K_i\right) \bigcup \left(\bigcup_{i=1}^{r} S_i\right)$.

Using (5.3) we have that

$$|f(z_2)| = \frac{1}{m_j} \cdot \left| \sum_{i=1}^{d} \phi_i(z_2) \right| \leq \frac{n_j}{m_j} \cdot \frac{\epsilon_0}{2^2} \cdot \frac{m_{\min I_1}}{n_{\max I_1}} \leq \frac{\epsilon_0}{2^2}.$$

Hence

$$|f(z_2)| < \frac{\epsilon_0}{2^2} \quad \text{for every} \quad f \in \bigcup_{j\in I_1} F_j.$$

There exists I_2 finite subset of L_{r+1}, such that

$$\frac{1}{m_j} < \frac{\epsilon_0}{2^2} \cdot \frac{1}{\|z_2\|_1} \text{ for every } j \in (L_{r+1} \backslash I_2).$$

Hence

$$|f(z_2)| < \frac{\epsilon_0}{2^2} \quad \text{for every} \quad f \in \left[K_{r+1} \Big\backslash \left(\left(\bigcup_{j\in I_2} F_j \right) \bigcup F_0 \right) \right].$$

From (5.1) and (5.2) we may choose a vector $z_3 \in \langle w_n : n \in \mathbb{N}\rangle$ with $z_3 > z_2$, $\|z_3\|_G = 1$ and

$$\|z_3\|_{S_i} < \frac{\epsilon_0}{2^3} \cdot \frac{m_{\min(I_1 \bigcup I_2)}}{n_{\max(I_1 \bigcup I_2)}} \leq \frac{\epsilon_0}{2} < \|z_3\|_{K_{r+1}}, \quad i = 1, \ldots, r. \tag{5.4}$$

Using (5.4) we observe that

$$|f(z_3)| \leq \frac{\epsilon_0}{2^3} \text{ for every } f \in \bigcup_{j \in (I_1 \bigcup I_2)} F_j.$$

There exists I_3 finite subset of L_{r+1} such that

$$\frac{1}{m_j} < \frac{\epsilon_0}{2^3} \cdot \frac{1}{\|z_3\|_1} \text{ for every } j \in (L_{r+1} \backslash I_3).$$

Hence

$$|f(z_3)| \leq \frac{\epsilon_0}{2^3} \quad \text{for every} \quad f \in \left[K_{r+1} \Big\backslash \left(\left(\bigcup_{j \in I_3} F_j \right) \bigcup F_0 \right) \right].$$

Therefore we inductively construct a sequence $(z_k)_{k \in \mathbb{N}}$ such that $z_k \in \langle w_n : n \in \mathbb{N} \rangle$, $k \in \mathbb{N}$, $z_k < z_{k+1}$, $k \in \mathbb{N}$, $\|z_k\|_G = 1$, and a sequence $(I_k)_{k \in \mathbb{N}}$ of finite subsets of L_{r+1} such that

(1) $\|z_k\|_{K_i \bigcup S_i} < \frac{\epsilon_0}{2} < \|z_k\|_{K_{r+1}}$ $k \in \mathbb{N}$, $i = 1, \ldots, r$.

(2) $|f(z_k)| < \frac{\epsilon_0}{2^k}$, $f \in \left[K_{r+1} \Big\backslash \left(\left(\bigcup_{j \in I_k} F_j \right) \bigcup F_0 \right) \right]$, $k \in \mathbb{N}$.

(3) $|f(z_k)| < \frac{\epsilon_0}{2^k}$, $f \in \bigcup_{j \in I_1 \bigcup \cdots \bigcup I_{k-1}} F_j, k \in \mathbb{N}$.

We will prove that

$$\|z_1 + \cdots + z_k\|_{K_{r+1}} \leq 1 + \epsilon_0 \text{ for each } k \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ and $\phi \in K_{r+1}$.

We distinguish the following cases.

(1) Let $\phi \in \left(K_{r+1} \Big\backslash \bigcup_{j \in I_1 \bigcup \ldots \bigcup I_k} F_j \right)$. Then

$$|\phi(z_1 + \ldots + z_k)| \leq |\phi(z_1)| + \ldots + |\phi(z_k)| \leq \sum_{j=1}^{k} \frac{\epsilon_0}{2^j} \leq 1 + \epsilon_0.$$

(2) Let $\phi \in \bigcup_{j\in I_1} F_j$ or $\phi \in \bigcup_{j\in I_k\setminus(I_1\bigcup\ldots\bigcup I_{k-1})} F_j$. Then

$$|\phi(z_1+\ldots+z_k)| \leq |\phi(z_1)| + (|\phi(z_2)| + \ldots |\phi(z_k)|) \leq 1+\epsilon_0.$$

(3) Let $\phi \in \bigcup_{j\in I_{i+1}\setminus(I_1\bigcup\ldots\bigcup I_i)} F_j$, $1<i<k-1$. Then

$$\begin{aligned}|\phi(z_1+\cdots+z_k)| &\leq |\phi(z_1)| + \cdots + |\phi(z_i)| + (|\phi(z_{i+1})| + |\phi(z_{i+2}+\cdots+z_k)|)\\ &\leq \sum_{j=1}^{i}\frac{\epsilon_0}{2^j} + 1 + \sum_{j=i+2}^{k}\frac{\epsilon_0}{2^j} \leq 1+\epsilon_0\end{aligned}$$

We have that $z_1+\cdots+z_k \in \langle w_n : n\in\mathbb{N}\rangle$, $k\in\mathbb{N}$.

For the vector $v_k = \dfrac{z_1+\cdots+z_k}{\|z_1+\cdots+z_k\|_G}$, $k\in\mathbb{N}$ we have that $\dfrac{\epsilon_0}{2} < \|v_k\|_{K_{r+1}}$, hence $\dfrac{\epsilon_0}{2}\|z_1+\cdots+z_k\|_G < \|z_1+\cdots+z_k\|_{K_{r+1}}$ for every $k\in\mathbb{N}$. For the vector z_1 we have

$$\frac{\epsilon_0}{2} < |f_1(z_1)| \leq \|z_1\|_{K_{r+1}} \text{ for some } f_1\in K_{r+1}.$$

The functional f_1 belongs to $\bigcup_{j\in I_1} F_j$.

For the vector z_2 we have that

$$\frac{\epsilon_0}{2} < |f_2(z_2)| \text{ for some } f_2\in K_{r+1}.$$

We observe that $f_2 \in \left(\bigcup_{j\in I_2} F_j - \bigcup_{j\in I_1} F_j\right)$.

For each $k\in\mathbb{N}$ we choose $f_k \in \left(\bigcup_{j\in I_k} F_j \Big\backslash \bigcup_{j\in I_1\bigcup\ldots I_{k-1}} F_j\right)$ with $\operatorname{ran} f_k \subset \operatorname{ran} z_k$ and $\dfrac{\epsilon_0}{2} < |f_k(z_k)|$.

Let $q\in\mathbb{N}$. The functional

$$f = \sum_{i=1}^{q^2}\frac{\epsilon_i}{q}\cdot f_i \text{ where } |\epsilon_i| = 1$$

belongs to G. Hence

$$\|z_1+\ldots+z_{q^2}\|_G \geq f(z_1+\ldots+z_{q^2}) \geq \frac{\epsilon_0}{2}\frac{1}{q}\cdot q^2 = \frac{\epsilon_0}{2}q.$$

We have that

$$\frac{\epsilon_0^2}{4}q \leq \frac{\epsilon_0}{2}\|z_1 + \ldots + z_{q^2}\|_G < \|z_1 + \ldots + z_{q^2}\|_{K_{r+1}} \leq 1 + \epsilon_0, \quad q \in \mathbb{N}$$

a contradiction.

We have proved that the operator id : $\mathfrak{X}_{nqr} \longrightarrow Y_{K_{r+1}}$ is strictly singular. Following the same steps as in Lemma B.11 in [1] we get that if Z be a block subspace of $\mathfrak{X}_{nqr}$ then there exists $z \in Z$ with $\|z\|_G = 1$ such that $|x^*(z)| < \epsilon$ for every $x^* \in S_{r+1}$. Hence the identity operator id : $\mathfrak{X}_{nqr} \longrightarrow Y_{K_{r+1} \bigcup S_{r+1}}$ is strictly singular. We have proved that the operators id : $\mathfrak{X}_{nqr} \longrightarrow Y_{K_n \bigcup S_n}, n \in \mathbb{N}$ are strictly singular.

We shall show that the identity operator id : $\mathfrak{X}_{nqr} \longrightarrow Y_K$ is strictly singular.

Let $\epsilon > 0$ and let a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers with $\lim\limits_n \epsilon_n = 0$.

It is not hard to see that for every $n \in \mathbb{N}$ the operator id : $\mathfrak{X}_{nqr} \longrightarrow Y_{(\bigcup_{i=1}^n K_i)\bigcup(\bigcup_{i=1}^n S_i)}$ is strictly singular. Therefore there exists a block sequence $(x_n)_{n\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ with $\|x_n\|_G = 1, n \in \mathbb{N}$ and $|x^*(x_n)| < \epsilon_n$,

$$x^* \in \left[\left(K_1 \bigcup S_1\right) \bigcup \cdots \bigcup \left(K_n \bigcup S_n\right)\right], \quad n \in \mathbb{N}.$$

We claim that:

If $\delta > 0$, then there exists an infinite subset M of $\mathbb{N}$ such that for every σ branch b, it follows that the set $\{n \in M : |b^*(x_n)| \geq \delta\}$ contains at most 1 element.

Assume the contrary. Then there exists $\delta > 0$ such that for every infinite subset M of $\mathbb{N}$ there exist σ branch b such that the set $\{n \in M : |b^*(x_n)| \geq \delta\}$ contains at least 2 elements.

Applying Ramsey theorem for doubletons we may find an $L \subset \mathbb{N}$ infinite, such that for every pair $n < m \in L$ there exist σ branch $b_{n,m}$ with $|b^*_{n,m}(x_n)| \geq \delta$ and $|b^*_{n,m}(x_m)| \geq \delta$.

Since $\lim\limits_n \epsilon_n = 0$ there exist $n_0 \in \mathbb{N}$ with $\epsilon_n < \delta, n \geq n_0$.

We set $L_1 = L \bigcap [n_0, \infty)$ and let $n_1 = \min L_1$.

It is obvious that $\epsilon_n < \delta, n \geq n_1$ and for each $n < m \in L_1$ there exist σ branch $b_{n,m}$ with $|b^*_{n,m}(x_n)| \geq \delta$ and $|b^*_{n,m}(x_m)| \geq \delta$.

Let $m_1 \in L_1$ with $m_1 > n_1$.

There exist a σ branch $b_{n_1,m_1} = b_1$ with $|b^*_1(x_{n_1})| \geq \delta$ and $|b^*_1(x_{m_1})| \geq \delta$.

Let $x^*_1 = E_1 b^*_1$ where $E_1 = [\min \text{supp}(x_{n_1}), \max \text{supp}(x_{m_1})]$.

Then $|x^*_1(x_{n_1})| \geq \delta$ and $|x^*_1(x_{m_1})| \geq \delta$.

Since the functional x_1^* does not belong to $(K_1 \bigcup S_1) \bigcup \cdots \bigcup (K_{m_1} \bigcup S_{m_1})$ there exists $d_2 > m_1$ with $x_1^* \in (K_{d_2} \bigcup S_{d_2}) \backslash F_0$.

Let $m_2 \in L_1$ with $m_2 > d_2$.

There exists a σ branch $b_{n_1,m_2} = b_2$ with $|b_2^*(x_{n_1})| \geq \delta$ and $|b_2^*(x_{m_2})| \geq \delta$.

Let $x_2^* = E_2 b_2^*$ where $E_2 = [\min \operatorname{supp}(x_{n_1}), \max \operatorname{supp}(x_{m_2})]$.

Then $|x_2^*(x_{n_1})| \geq \delta$ and $|x_2^*(x_{m_2})| \geq \delta$. The functional x_2^* does not belong to $(K_1 \bigcup S_1) \bigcup \cdots \bigcup (K_{m_2} \bigcup S_{m_2})$, so the functionals x_1^*, x_2^* have disjoint indices.

We inductively construct a sequence $(x_n^*)_{n \in \mathbb{N}}$ of σ special functionals with disjoint indices and $|x_n^*(x_{n_1})| \geq \delta, n \in \mathbb{N}$.

Let $q \in \mathbb{N}$. The functional

$$f = \sum_{i=1}^{q^2} \frac{\epsilon_i}{q} x_i^*, \quad |\epsilon_i| = 1, \quad i = 1, \ldots, q^2$$

belongs to G. We have that

$$f(x_{n_1}) = \sum_{i=1}^{q^2} \frac{\epsilon_i}{q} x_i^*(x_{n_1}) = \sum_{i=1}^{q^2} \frac{1}{q} |x_i^*(x_{n_1})| \geq q\delta.$$

Therefore

$$q\delta \leq |f(x_{n_1})| \leq 1, \quad q \in \mathbb{N}$$

a contradiction.

Using again the same techniques as in Lemma B.11 in [1] and the fact that the operators id : $\mathfrak{X}_{nqr} \longrightarrow Y_{(\bigcup_{i=1}^n K_i) \bigcup (\bigcup_{i=1}^n S_i)}$, $n \in \mathbb{N}$ are strictly singular we get a vector $y \in \langle y_n, n \in \mathbb{N} \rangle$ with $\|y\|_G = 1$ and $|x^*(y)| < \epsilon$, $x^* \in K$.

The following Lemma is similar to a corresponding result in [1, Lemma B.13].

Lemma 5.3. *For every $x \in c_{00}(\mathbb{N})$ and every $\epsilon > 0$ there exists $d \in \mathbb{N}$ such that for every $g \in (G \backslash F_0)$ with $\operatorname{ind}(g) \cap \{1, \ldots, d\} = \emptyset$ we have that $|g(x)| < \epsilon$.*

Combining Lemmas 5.2 and 5.3 we may construct in every block subspace of $\mathfrak{X}_{nqr}$ a block sequence which is equivalent with the usual basis of $\ell^2(\mathbb{N})$. The proof of Theorem 5.1 follows the lines of Theorem B.14 in [1].

Theorem 5.1. *Let Y be a closed, infinite dimensional subspace of $\mathfrak{X}_{nqr}$. Then for every $\epsilon > 0$ there exists a subspace of Y, which is $1+\epsilon$ isomorphic to ℓ^2.*

6. The dual space $\mathfrak{X}^*_{nqr}$. In this section is studied the structure of $\mathfrak{X}^*_{nqr}$. Proposition 6.1 is similar to Proposition B.15 ([1]). The proof of Proposition 6.1 makes use of the fact that $\mathfrak{X}_{nqr}$ does not contain an isomorphic copy of $\ell^1(\mathbb{N})$.

Proposition 6.1. *For the dual space $\mathfrak{X}^*_{nqr}$ we have that*

$$\mathfrak{X}^*_{nqr} = \overline{\operatorname{span}}\{e^*_n : n \in \mathbb{N}\} \cup \{b^* : b\ \sigma \text{ infinite branch}\}.$$

Also the basis $(e_n)_{n\in\mathbb{N}}$ of $\mathfrak{X}_{nqr}$ is weakly null.

Proof. The space $\mathfrak{X}_{nqr}$ is ℓ^2 saturated, so does not contain an isomorphic copy of ℓ^1. From Haydon's [9] theorem we get that

$$B_{\mathfrak{X}^*_{nqr}} = \overline{\operatorname{conv}}\operatorname{ext}B_{\mathfrak{X}^*_{nqr}}. \tag{6.1}$$

Since G is the norming set of the space $\mathfrak{X}_{nqr}$ it follows that

$$B_{\mathfrak{X}^*_{nqr}} = \overline{\operatorname{conv} G}^{w^*}.$$

It is not hard to see that $B_{\mathfrak{X}^*_{nqr}} = \overline{\operatorname{conv} \overline{G}^{w^*}}^{w^*}$, hence $\operatorname{ext}B_{\mathfrak{X}^*_{nqr}} \subset \overline{G}^{w^*}$. Combining this with (6.1) we get that

$$\mathfrak{X}^*_{nqr} = \overline{\langle \overline{G}^{w^*} \rangle}.$$

As is shown in Proposition B.15 in [1] the following holds: $\overline{G}^{w^*} = F_0 \bigcup \Big\{ \sum\limits_{i=1}^{\infty} \alpha_i x^*_i :$ $\sum\limits_{i=1}^{\infty} \alpha_i^2 \leq 1, \alpha_i \in \mathbb{Q}, x^*_i$ finite or infinite σ special functionals with disjoint indices$\Big\}$, hence the first part of the proposition is proved.

For the second part of Proposition it clearly suffices to show that $\lim\limits_n b^*(e_n) = 0$ for every σ branch b.

If b is finite σ branch then the sequence $(b^*(e_n))_{n\in\mathbb{N}}$.

Let $b = (f_n)_{n\in\mathbb{N}}$ an infinite branch. From the fact that $\lim\limits_n \|f_n\|_\infty = 0$ it follows that $(b^*(e_n))_{n\in\mathbb{N}}$ is a null sequence.

7. Rapidly increasing sequences in $\mathfrak{X}_{nqr}$. We begin by the definition of a Rapidly Increasing Sequence (RIS).

Definition 7.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a block sequence in $\mathfrak{X}_{nqr}$ and C, ϵ positive numbers. The sequence $(x_n)_{n\in\mathbb{N}}$ will be called (C,ϵ) RIS (Rapidly Increasing Sequence) if the following hold*

(1) $\|x_n\|_G \leq C, n \in \mathbb{N}$.

(2) *There exists a strictly singular sequence of natural numbers* $(j_n)_{n\in\mathbb{N}}$ *such that* $\dfrac{|\operatorname{supp}(x_n)|}{m_{j_{n+1}}} < \epsilon, n \in \mathbb{N}$ *and if* $n \in \mathbb{N}, f \in (K \backslash F_0)$ *with* $w(f) = m_i$, $i < j_n$ *then* $|f(x_n)| \leq \dfrac{C}{m_i}$.

We notice that for an $f \in (K\backslash F_0)$ *of the form* $f = \frac{1}{m_j}\sum_{i=1}^{d} f_i$, $d \leq n_j$, $f_i \in (K\backslash F_0)$, $i = 1, \ldots, n_j$ *we say that* f *has weight* m_j *and we write* $w(f) = m_j$.

Definition 7.2. *Let* $j_0 \in \mathbb{N}$ *and* $(x_n)_{n\in\mathbb{N}}$ *be a* (C, ϵ) *RIS with* $0 < \epsilon < \dfrac{5}{m_{j_0}^2}$ *and* $(j_n)_n$ *its associated sequence of natural numbers. We will call* $(x_n)_{n\in\mathbb{N}}$ j_0*-separated if the following are satisfied:*

1. $j_1 > j_0$.
2. *For every functional* $f \in (K\backslash F_0)$ *with* $w(f) > m_{j_0}$ *we have that*
$$\left|\left\{n \in \mathbb{N} : |f(x_n)| \geq \frac{5}{m_{j_0}^2}\right\}\right| \leq 1.$$
3. *For every special functional* x^* *with* $\operatorname{ind}(x^*) \subset \{j_0 + 1, \ldots\}$, *we have that*
$$\left|\left\{n \in \mathbb{N} : |x^*(x_n)| \geq \frac{10}{m_{j_0}^2}\right\}\right| \leq 2.$$
4. *For every* $f \in G$ *with* $\operatorname{ind}(f) \subset \{j_0 + 1, \ldots\}$, *we have that*
$$\left|\left\{n \in \mathbb{N} : |f(x_n)| \geq \frac{2}{m_{j_0}^2}\right\}\right| \leq 257 m_{j_0}^4.$$

The next step is to prove that for a $j_0 \in \mathbb{N}$ and a bounded block sequence of averages with increasing lengths in $\mathfrak{X}_{nqr}$, there exists a subsequence which is j_0-separated.

The proof of the following lemma follows the same steps as in [5, Lemma II.23].

Lemma 7.1. *Let* $x \in \mathfrak{X}_{nqr}$ *be a* (M, k)*-average,* $M > 0, k \in \mathbb{N}$, *i.e. an average of the form* $x = \frac{1}{k}(x_1 + \cdots + x_k)$, *where*

i. $x_1, \ldots, x_k \in \langle e_n, n \in \mathbb{N} \rangle$

ii. $x_1 < \ldots < x_k$

iii. $\|x_i\|_G \leq M$, $i = 1, \ldots, k$

and $f \in (K \backslash F_0)$ *with* $w(f) = m_i$. *Then*

$$|f(x)| \leq \frac{M}{m_i}\left(1 + \frac{2n_i}{k}\right).$$

Lemma 7.2. *Let* $\epsilon > 0$ *and* $(x_n)_{n\in\mathbb{N}}$ *be a block sequence in* $\mathfrak{X}_{nqr}$ *such that each* x_n *is a* (M, l_n) *average, where* $(l_n)_{n\in\mathbb{N}}$ *is a strictly increasing sequence of natural numbers. Then there exists a subsequence of* $(x_n)_{n\in\mathbb{N}}$ *which is* $\left(\frac{3M}{2}, \epsilon\right)$ *RIS.*

The proof of Lemma 7.2 follows the lines of Proposition II.25 in [5].

Remark 7.1. Let $(z_k)_{k\in\mathbb{N}}$ be a normalized block sequence in $\mathfrak{X}_{nqr}$. We set

$$y_k = \frac{1}{n_k}\sum_{i\in F_k} z_i, \quad k \in \mathbb{N}$$

where $|F_k| = n_k, F_k < F_{k+1}, k \in \mathbb{N}$.

Since $\|y_k\|_G \leq 1, k \in \mathbb{N}$ and $\mathfrak{X}_{nqr}$ does not contain an isomorphic copy of ℓ^1 (Theorem 5.1), it follows from Rosenthal's ℓ^1 theorem [15] that there exists a subsequence of $(y_k)_{k\in\mathbb{N}}$ which is w-Cauchy.

Without loss of generality we may assume that $(y_k)_{k\in\mathbb{N}}$ is w-Cauchy.

We set $x_k = y_{2k-1} - y_{2k}$, $k \in \mathbb{N}$. The sequence $(x_k)_{k\in\mathbb{N}}$ is weakly null and $\|x_k\|_G \leq 2$, $k \in \mathbb{N}$.

If $f \in (K \backslash F_0)$ with $w(f) = m_i$ then Lemma 7.1 yields

$$|f(x_k)| \leq |f(y_{2k-1})| + |f(y_{2k})| \leq \frac{2}{m_i}\left(1 + \frac{2n_i}{F_{2k-1}}\right), \quad k \in \mathbb{N}.$$

Hence if $\epsilon > 0$, then from Lemma 7.2 it follows that there exists an $L \in [\mathbb{N}]$ such that the sequence $(x_k)_{k\in L}$ is $(3, \epsilon)$ RIS.

Therefore without loss of generality we can assume that every $(x_k)_{k\in\mathbb{N}}$ which is a $(3, \epsilon)$ RIS is weakly null.

For the rest of this paper we will assume that every $(3, \epsilon)$ RIS we consider is weakly null, unless stated otherwise.

Lemma 7.3. *Let $j_0 \in \mathbb{N}$ and $(x_k)_{k\in\mathbb{N}}$ be a $(3,\epsilon)$ RIS with $0<\epsilon<\dfrac{5}{m_{j_0}^2}$. Assume that the sequence $(j_n)_n$ associated to the RIS sequence satisfies $j_1 > j_0$. Then for every $f \in (K\backslash F_0)$ with $w(f) > m_{j_0}$ we have that*

$$\left|\left\{k\in\mathbb{N} : |f(x_k)| \geq \frac{5}{m_{j_0}^2}\right\}\right| \leq 1.$$

For a proof of Lemma 7.3 we refer Lemma 5.2 in [14].

Lemma 7.4. *Let $j_0 \in \mathbb{N}$ and $(x_k)_{k\in\mathbb{N}}$ be a $(3,\epsilon)$ RIS with $0<\epsilon<\dfrac{5}{m_{j_0}^2}$. Assume that the sequence $(j_n)_n$ associated to the RIS sequence satisfies $j_1 > j_0$. Then there exists a $L \in [\mathbb{N}]$ such that for every special functional x^*, with* $\operatorname{ind}(x*) \subset \{j_0+1,\ldots\}$, *we have that* $\left|\left\{k\in L : |x^*(x_k)| \geq \dfrac{10}{m_{j_0}^2}\right\}\right| \leq 2.$

The proof of Lemma 7.4 follows the lines of Lemma 5.3 in [14].

Proposition 7.1. *Let $j_0 \in \mathbb{N}$ and $(y_k)_{k\in\mathbb{N}}$ be a block sequence of averages with increasing lengths (as in Remark* 7.1*).*

Then there exists an $L \in [\mathbb{N}]$ such that for every $f \in (G\backslash F_0)$ with $\operatorname{ind}(f) \subset \{j_0+1,\ldots\}$ *we have that* $\left|\left\{k\in L : |f(y_k)| \geq \dfrac{2}{m_{j_0}^2}\right\}\right| \leq 257 m_{j_0}^4.$

The proof of the above Proposition is identical of Proposition 5.1 in [14].

All the above yield the following

Proposition 7.2. *Let $j_0 \in \mathbb{N}$, $0<\epsilon<\dfrac{5}{m_{j_0}^2}$ and $(y_k)_{k\in\mathbb{N}}$ be a block sequence of averages with increasing lengths. (as in Remark* 7.1*). We set $x_k = y_{2k-1} - y_{2k}, k\in\mathbb{N}$. Then there exists an $L \in [\mathbb{N}]$ such that $(x_k)_{k\in L}$ is $(3,\epsilon)$ RIS and j_0-separated.*

Analogous Proposition can be found in [14]. (Proposition 5.2)

Remark 7.2. Let $j_0 \in \mathbb{N}$, $0<\epsilon<\dfrac{5}{m_{j_0}^2}$ and $(y_k)_{k\in\mathbb{N}}$ be a block sequence of averages with increasing lengths. (as in Remark 7.1).

In the sequel we will assume without loss of generality that there exists an $L \in [\mathbb{N}]$ such that $(y_k)_{k\in L}$ is $(3,\epsilon)$ RIS and j_0-separated.

8. The basic inequality. The purpose of this section is to prove Basic Inequality, which will be used in the next chapter. Similar results exist in the papers [3], [2], [14]. The Basic Inequality is a method, which has been developed and attributes estimates of sums of block sequences with certain properties to estimates of sums of the basis of a mixed type Tsirelson space.

Specificly if $(x_k)_{k\in\mathbb{N}}$ is a (C, ϵ) R. I. S. $\left(0<\epsilon<\dfrac{5}{m_{j_0}}\right)$ sequence in $\mathfrak{X}_{nqr}$, which is j_0 separated, then calculations of the form $f\left(\sum_k \lambda_k x_k\right)$, $f\in \left(\bigcup_{n\in\mathbb{N}} K_n\right)\setminus F_0$ are transformed into calculations of the form $g_1\left(\sum_k |\lambda_k| e_k\right)$ and $g_2\left(\sum_k |\lambda_k| e_k\right)$ where $g_1\in W$, $g_2\in c_{00}(\mathbb{N})$ with $\|g_2\|_\infty\le\epsilon$.

The set W is the norming set of the space T, which is called the auxiliary space. In this space we estimate sums of the form $\dfrac{e_{k_1}+\cdots+e_{k_{n_j}}}{n_j}$ where $j\in\mathbb{N}$ and $k_1<\ldots<k_{n_j}$ are natural numbers.

Definition 8.1. *We denote by W the minimal subset of $c_{00}(\mathbb{N})$ such that:*

(1) $\{\epsilon_n e_n^*, n\in\mathbb{N}, |\epsilon_n|=1\}\subset W$.

(2) *W is closed under the operations $\left(\mathcal{A}_{2n_j},\dfrac{1}{m_j}\right)_{j\in\mathbb{N}}$, i. e for every $j\in\mathbb{N}$, $d\le 2n_j$ and for every $f_1<\cdots<f_d$ in W it follows that $\dfrac{1}{m_j}(f_1+\cdots+f_d)\in W$.*

(3) *W is closed under the operation $\left(\mathcal{A}_4,\dfrac{1}{2}\right)$, i.e. for every $d\in\mathbb{N}$ with $d\le 4$ and for every $f_1<\cdots<f_d$ in W it follows that $\dfrac{1}{2}(f_1+\cdots+f_d)\in W$.*

The set W defines a norm on $c_{00}(\mathbb{N})$ by the rule $\|x\|_W=\sup\{|f(x)|:\ f\in W\}$, $x\in c_{00}(\mathbb{N})$. The completion of $(c_{00}(\mathbb{N}),\|\cdot\|_W)$ is denoted by T.

Lemma 8.1. *Let $f\in W$, $j\in\mathbb{N}$ and $k_1<\ldots<k_{n_j}$ natural numbers. Then*

$$\left|f\left(\frac{1}{n_j}\sum_{r=1}^{n_j} e_{k_r}\right)\right|\le\begin{cases}\dfrac{2}{m_i m_j} & \text{if } w(f)=m_i,\ i<j\\[2ex] \dfrac{1}{m_i} & \text{if } w(f)=m_i,\ i\ge j\end{cases}.$$

Also $\left\|\dfrac{1}{n_j}\sum_{r=1}^{n_j} e_{k_r}\right\|_W=\dfrac{1}{m_j}$.

For a proof of the above Lemma we refer to Lemma 3.16 and Proposition 3.19 in [4].

Proposition 8.1 (basic inequality)**.** *Let $j_0 \in \mathbb{N}$, $j_0 \geq 3$ and $(x_k)_{k\in\mathbb{N}}$ a (C,ϵ) RIS sequence on $\mathfrak{X}_{nqr}$, $0<\epsilon<\dfrac{5}{m_{j_0}^2}$, $C \geq 1$ which is j_0-separated with* $\min \operatorname{supp}(x_1) > m_{j_0}$.

Let also $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of real numbers.

Then for every $f \in \left(\bigcup_{n\in\mathbb{N}} K_n\right) \backslash F_0$ there exist g_1, g_2 on $c_{00}(\mathbb{N})$ with non-negative coordinates where

(1) $g_1 \in W$ *with* $w(g_1) = w(f)$

(2) $\|g_2\|_\infty \leq \dfrac{5}{m_{j_0}^2}$

such that for every $n \in \mathbb{N}$ we have that

$$\left| f\left(\sum_{k=1}^{n} \lambda_k x_k\right)\right| \leq 2C(g_1+g_2)\left(\sum_{k=1}^{n} |\lambda_k| e_k\right).$$

The proof of the above Basic Inequality follows the arguments of Basic inequality(Proposition 6.1) in [14]. The only difference is that in the set $\left(\bigcup_{n\in\mathbb{N}} K_n\right) \backslash F_0$ do not appear ℓ^2 convex combinations as in the case of Basic Inequality in [14]. Consequently the proof of Basic Inequality in this paper is simpler than the corresponding one. Also the norming set W of the auxiliary space T is more simple. Finally we notice that the Condition 4 of Definition 7.2 is unnecessary in the proof of Basic Inequality in this paper. However this condition is necessary in the following lemma.

Lemma 8.2. *Let $j_0 \in \mathbb{N}$ with $j_0 > 1$ and $(x_k)_{k\in\mathbb{N}}$ a (C,ϵ) R.I.S sequence on $\mathfrak{X}_{nqr}$, $0<\epsilon<\dfrac{5}{m_{j_0}^2}$, $C \geq 1$ which is j_0-separated.*

Let also $k_1 < \cdots < k_{n_{j_0}}$ natural numbers.

Then there exists $M > 0$ which depends only on $C > 0$ such that

$$\left\| \frac{x_{k_1} + \cdots + x_{k_{n_{j_0}}}}{n_{j_0}} \right\|_G \leq \frac{M}{m_{j_0}}.$$

P r o o f. Let $f \in (G\backslash F_0)$. Then $f = \sum_{i=1}^{d} \alpha_i x_i^*$ where $\alpha_i \in \mathbb{Q}$, $\sum_{i=1}^{d} \alpha_i^2 \leq 1$ and $x_i^* \in S, i = 1, \ldots, d$ have pairwise disjoint indices.

Let $f = f_0+g+h$ where $w(f_0) = m_{j_0}$, $g = \sum_{i=1}^{d} \alpha_i x^*_{i,<j_0}$ and $h = \sum_{i=1}^{d} \alpha_i x^*_{i,>j_0}$.

We set $x = \dfrac{x_{k_1} + \cdots + x_{k_{n_{j_0}}}}{n_{j_0}}$.

From Lemma 7.1 we get that

$$|f_0(x)| \leq \frac{C}{m_{j_0}} \left(1 + \frac{2n_{j_0}}{n_{j_0}}\right) = \frac{3C}{m_{j_0}}.$$

Since $(x_k)_{k\in\mathbb{N}}$ is j_0-separated we have that

$$|h(x)| \leq \frac{C \cdot 257m^4_{j_0} + \frac{2}{m^2_{j_0}}(n_{j_0} - 257m^4_{j_0})}{n_{j_0}} \leq \frac{C}{m_{j_0}} + \frac{2}{m_{j_0}} = \frac{C+2}{m_{j_0}}.$$

Also

$$|g(x)| \leq \sum_{i=1}^{d} |x^*_{i,<j_0}(x)| \leq \sum_{i\in A} |f_i(x)|$$

where $f_i \in \left(\bigcup_{n\in\mathbb{N}} K_n\right) \setminus F_0$ with $w(f_i) = m_i$ and $|A| \leq j_0$.

Let $i \in A$. Using basic inequality we get that

$$|f_i(x)| \leq 2C(g^i_1 + g^i_2) \left(\frac{e_{k_1} + \cdots + e_{k_{n_{j_0}}}}{n_{j_0}}\right)$$

where, $g^i_1 \in W, w(g^i_1) = w(f_i)$ and $\|g^i_2\|_\infty \leq \dfrac{5}{m^2_{j_0}}$. Hence

$$|f_i(x)| \leq 2C(\frac{2}{m_i m_{j_0}} + \frac{5}{m^2_{j_0}}) \text{ and thus}$$

$$|g(x)| \leq 2C \left[\sum_{i\in A} \left(\frac{2}{m_i m_{j_0}} + \frac{5}{m^2_{j_0}}\right)\right] \leq 2C \left(\frac{2}{m_{j_0}} + j_0 \frac{5}{m^2_{j_0}}\right)$$

$$\leq 2C \left(\frac{2}{m_{j_0}} + \frac{5}{m_{j_0}}\right) = \frac{14C}{m_{j_0}}$$

Finally $|f(x)| \leq \dfrac{18C+2}{m_{j_0}}$, so letting $M = 18C + 2$ we get that $\|x\|_G \leq \dfrac{M}{m_{j_0}}$.

9. Every non-reflexive subspace of $\mathfrak{X}_{nqr}$ has non separable dual. We pass to the final section where we prove the next Theorem.

Theorem 9.1. *Let Y a closed, infinite and non-reflexive subspace of $\mathfrak{X}_{nqr}$. Then Y^* is non-separable.*

Proof. It is enough to prove the conclusion for the block subspaces of $\mathfrak{X}_{nqr}$.

Let $Y = \overline{\langle y_n, n \in \mathbb{N}\rangle}$ a non-reflexive block subspace of $\mathfrak{X}_{nqr}$. Since the subspace Y is non-reflexive, James classical theorem [12] yields that the sequence $(y_n)_{n\in\mathbb{N}}$ is not shrinking. Therefore there exist $(z_n)_{n\in\mathbb{N}}$ a block sequence of $(y_n)_{n\in\mathbb{N}}, \epsilon_0 > 0$ and $x^* \in \mathfrak{X}^*_{nqr}$ such that:

i. $\|z_n\|_G = 1, n \in \mathbb{N}$

ii. $\epsilon_0 < |x^*(z_n)|$, $n \in \mathbb{N}$

Using Proposition 6.1 it is not hard to prove that there exist $\epsilon_1 > 0$ and an infinite σ branch b such that for the functional b^* holds

$$\epsilon_1 < b^*(z_n), \quad n \in \mathbb{N}.$$

Let b be a σ_r branch, where $r \in \mathbb{N}$.

We will prove that Y^* is non-separable. It is enough to show that if $Z = \overline{\langle z_n, n \in \mathbb{N}\rangle}$ then Z^* is non-separable.

We consider the sequence $v_k = \frac{1}{n_k} \sum_{i\in F_k} z_i$, $k \in \mathbb{N}$ where $F_k \subset \mathbb{N}$, $|F_k| = n_k$, $F_k < F_{k+1}$, $k \in \mathbb{N}$.

We observe that

(1) $\|v_k\|_G \leq 1, k \in \mathbb{N}$.

(2) $b^*(v_k) = b^*\left(\frac{1}{n_k} \sum_{i\in F_k} z_i\right) > \epsilon_1$, $k \in \mathbb{N}$.

Since $b^*(v_k) \geq \epsilon_1$, $k \in \mathbb{N}$ it follows that for every $k \in \mathbb{N}$ there exists

$$x^*_k \in S_r \text{ with } \operatorname{ran}(x^*_k) \subset \operatorname{ran}(v_k) \text{ such that } \epsilon_1 < x^*_k(v_k).$$

We notice that x^*_k is the restriction of b^* on the interval $\operatorname{ran}(v_k) = [\min\operatorname{supp}(v_k), \max\operatorname{supp}(v_k)]$.

Using the dyadic tree we construct along of his branches uncountable σ_{r+1} special sequences such that considering these as functionals on Z, any two of them have big distance.

We notice that for a $j \in \mathbb{N}$, $0 < \epsilon < \dfrac{5}{m_j^2}$ there exists an $L \in [\mathbb{N}]$ such that $(v_k)_{k \in L}$ is $(3, \epsilon)$ RIS and j-separated (Remark 7.2).

Let D the dyadic tree. Inductively we construct $(x_\alpha, f_\alpha, j_\alpha)_{\alpha \in D}$ (the induction is in the lexicographic order of D) such that

(1) For every $\alpha \in D$ we have that $x_\alpha = \dfrac{1}{n_{j_\alpha}} \sum\limits_{i \in \Lambda_\alpha} v_i$ and $f_\alpha = \dfrac{1}{m_{j_\alpha}} \sum\limits_{i \in \Lambda_\alpha} x_i^*$ where $\Lambda_\alpha \subset \mathbb{N}$, $|\Lambda_\alpha| = n_{j_\alpha}$.

(2) $j_\emptyset \in \Omega_1$, $j_\emptyset > 1$, $m_{j_\emptyset} > 257$ and for every $\alpha \in D$ with $\alpha \neq \emptyset$ it follows that $j_\alpha = \sigma_{r+1}((f_\beta)_{\beta < \alpha})$.

(3) if $\alpha <_{lex} \beta$ then $\Lambda_\alpha < \Lambda_\beta$.

(4) if $a \in D$ and $S_a = \{b, c\}$ are the immediate successors of a then $\Lambda_a < \Lambda_b$, $\Lambda_a < \Lambda_c$ and Λ_b, Λ_c are successive.

(5) if $a \in D$ then $(v_i)_{i \in \Lambda_a}$ is $(3, \epsilon)$ RIS $\left(0 < \epsilon < \dfrac{5}{m_{j_a}^2}\right)$ and j_a separated.

We observe that $f_\alpha(x_\alpha) \geq \dfrac{\epsilon_1}{m_{j_\alpha}}$ for every $\alpha \in D$ and also from Lemma 8.2 we have that $\|x_a\|_G \leq M$ for all $\alpha \in D$.

Let $(f_\alpha)_{\alpha \in b_1}, (f_\alpha)_{\alpha \in b_2}$ 2 σ_{r+1} special sequences and b_1, b_2 different branches of the dyadic tree.

We consider the functionals $g_{b_1} = \sum\limits_{\alpha \in b_1} f_\alpha$ and $g_{b_2} = \sum\limits_{\alpha \in b_2} f_\alpha$.

Since $b_1 \neq b_2$ we may assume that there exists $\alpha \in (b_1 \backslash b_2)$. We have that

$$\|g_{b_1}|_Z - g_{b_2}|_Z\| \geq \frac{(g_{b_1} - g_{b_2})(x_\alpha)}{\|x_\alpha\|_G} = \frac{g_{b_1}(x_\alpha)}{\|x_\alpha\|_G} = \frac{f_\alpha(x_\alpha)}{\|x_\alpha\|_G} \geq \frac{\epsilon_0}{m_{j_\alpha}} \frac{m_{j_\alpha}}{M} = \frac{\epsilon_0}{M}.$$

Corollary 9.1. *The space $\mathfrak{X}_{nqr}$ does not contain any quasi-reflexive and non-reflexive subspace.*

Proof. Assume the contrary. Then there exists a quasi-reflexive and non-reflexive subspace Y of $\mathfrak{X}_{nqr}$, i. e. $0 < \dim Y^{**}/Y < \infty$. From this it follows that Y^{**}/Y is separable and since Y is also separable we get that Y^{**} is separable. Therefore Y^* is separable which contradicts to Theorem 9.1.

REFERENCES

[1] S. A. Argyros, A. D. Arvanitakis, A. G. Tolias. Saturated exensions, the attractors method and hereditarily James tree spaces. In: Methods in Banach spaces (Eds J. M. Castillo, et al.), Proceedings of the V conference on Banach spaces, Cáceres, Spain, September 13–18, 2004. Cambridge, Cambridge University Press, London Mathematical Society Lecture Note Series, vol. **337** (2006), 1–90.

[2] S. A. Argyros, J. Lopez-Abad, S. Todorcevic. A class of Banach spaces with few non-strictly singular operators., *J. Funct. Anal.* **222**, *2* (2005), 306–384.

[3] S. A. Argyros, A. Manoussakis. An indecomposable and unconditionally saturated Banach space. Dedicated to Professor Aleksander Pełczyński on the occasion of his 70th birthday. *Studia Math.* **159**, *1* (2003), 1–32.

[4] S. A. Argyros, A. G. Tolias. Methods in the theory of hereditarily indecomposable Banach spaces. *Mem. Amer. Math. Soc.* **170**, *806*, (2004), vi + 114 pp.

[5] S. A. Argyros, S. Todorcevic. Ramsey methods in analysis. Birkhauser, CRM, Barcelona, 2004.

[6] Steven F. Bellenot. More quasi-reflexive subspaces. *Proc. Amer. Math. Soc.* **101** (1987), 693–696.

[7] P. Civin, B. Yood. Quasi reflexive spaces. *Proc. Amer. Math. Soc.* **8** (1957), 906–911.

[8] C. Finet. Subspaces of Asplund Banach spaces with the point of continuity property. *Israel J. Math.* **60** (1987), 191–198.

[9] R. G. Haydon. Some more characterizations of Banach spaces not containing l^1. *Math. Proc. Cambridge Phil. Soc.* **80** (1976), 269–276.

[10] R. C. James. A separable somewhat reflexive Banach space with non separable dual. *Bull. Amer. Math. Soc.* **80** (1974), 738–743.

[11] R. C. James. A non reflexive Banach space isometric with its second conjugate. *Proc. Natl. Acad. Sci. USA* **37** (1951), 174–177.

[12] R. C. James. Bases and reflexivity of Banach spaces. *Ann. of Math.* **52** (1950), 518–527.

[13] W. B. Johnson, H. P. Rosenthal. On w^* basic sequences and their applications to the study of Banach spaces. *Studia. Math.* **43** (1972), 77–92.

[14] G. Petsoulas, T. Raikoftsalis. A Gowers tree like space and the space of its bounded linear operators. *Studia Math.* (to appear).

[15] H. P. Rosenthal. A characterization of Banach spaces containing ℓ^1. *Proc. Natl. Acad. Sci. USA* **71** (1974), 2411–2413.

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