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Integral Representations of Functional Series with Members Containing Jacobi Polynomials

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In this article we establish a double definite integral representation, and two other indefinite integral expressions for a functional series and its derivative with members containing Jacobi polynomials.

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1. Introduction and motivation

A family

$$y(x) = p_n(x) \equiv p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0, \qquad p_n \neq 0,$$

of polynomials of degree exactly $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ is a family of classical continuous orthogonal polynomials if it is the solution of a differential equation of the type

$$p_2(x)y''(x) + p_1(x)y'(x) + \pi_n y(x) = 0\,, \tag{1}$$

where $p_2(x) = ax^2 + bx + c$ is a polynomial of at most second order and $p_1(x) = dx + e$ is a linear polynomial [8, 13]. Since the polynomial $p_n(x)$ has exact degree n, by equating the highest coefficients of x^n in (1) one gets

$$\pi_n = -\left(an(n-1) + dn\right)\,.$$

The most widely used orthogonal polynomials are the *classical orthogonal polynomials*, consisting of the Hermite polynomials, the Laguerre polynomials and the Jacobi polynomials, together with their special cases which are ultraspherical polynomials, the Čebyšev polynomials and the Legendre polynomials.

In this article, our main aim is to derive integral representations for the functional series with members containing Jacobi polynomials. This will be realized in a similar manner as the authors have done it in the articles on Neumann series [3, 4] and on the Kapteyn series, [2].

The Jacobi polynomials, which are also called hypergeometric polynomials, can be represented with the following formula [18]

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left[\begin{array}{c} -n,\, 1+\alpha+\beta+n \\ 1+\alpha \end{array} \Big| \frac{1-z}{2}\right]. \tag{2}$$

When $\alpha = \beta = 0$, the polynomial (2) becomes the Legendre polynomial. The Gegenbauer polynomials, and also the Čebyšev polynomials, are special cases of the Jacobi polynomials. From (2) it follows that $P_n^{(\alpha,\beta)}(z)$ is a polynomial of degree precisely n and that

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}\,.$$

The Jacobi polynomials are orthogonal with respect to the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ on the interval $[-1, 1]$. Assurance of the integrability of $w(x)$ is achieved by requiring $\alpha > -1$ and $\beta > -1$, see [22].

The orthogonal polynomials with the weight function $(b-x)^\alpha(x-a)^\beta$, on the finite interval $[a, b]$ can be expressed in the form [22]

$$\text{constant} \cdot P_n^{(\alpha,\beta)}\Big(2\frac{x-a}{b-a} - 1\Big)\,.$$

It is worth mentioning that Luke and Wimp [15] proved that if we have continuous function $f(x)$, which has a piecewise continuous derivative for $0 \le x \le \lambda$, then $f(x)$ may be expanded into a uniformly convergent series of shifted Jacobi polynomials in the form

$$f(x) = \sum_{n=0}^\infty a_n(\lambda) P_n^{(\alpha,\beta)}(2x/\lambda - 1)\,,$$

where $\epsilon \le x/\lambda \le 1-\epsilon$, $\epsilon > 0$, $\alpha > -1$, $\beta > -1$. Various techniques are available for the determination of the coefficients $a_n(\lambda)$.

Let us define a functional series in the following form

$$\mathfrak{P}_{\alpha,\beta}(z) := \sum_{n=1}^\infty a_n P_n^{(\alpha,\beta)}(z), \qquad z \in \mathbb{C}, \tag{3}$$

where a_n are constants and $P_n^{(\alpha,\beta)}$ stands for the Jacobi polynomial. We point out that the Bulgarian mathematician P. Rusev studied in [20] the convergence of the series $\mathfrak{P}_{\alpha,\beta}(z)$ (precisely, he considered $a_0 + \mathfrak{P}_{\alpha,\beta}(z)$). For that purpose, he used the asymptotic formula by Darboux (see [6], [22, Eq. (8.21.9)]):

$$P_n^{(\alpha,\beta)}(z) = P^{(\alpha,\beta)}(z)\, n^{-1/2}\, \omega^n(z)\, (1 + p_n^{(\alpha,\beta)}(z))\,,$$

where $\omega(z)$ is the inverse of Žukovsky transformation $z = \frac{1}{2}(\omega + \omega^{-1})$ for which $\omega(\infty) = \infty$, $P^{(\alpha,\beta)}(z) \neq 0$ and $\big(p_n^{(\alpha,\beta)}(z)\big)_{n\in\mathbb{N}}$ are analytic functions holomorphic in the region $\mathbb{C} \setminus [-1,1]$ and such that $\lim\limits_{n\to\infty} p_n^{(\alpha,\beta)}(z) = 0$ uniformly on every compact subset of this region. Further, for $1 < r < +\infty$, he denoted by $E(r) :=$ Int$\gamma(r)$, where $\gamma(r) := \{z \in \mathbb{C} \colon |\omega(z)| = r\}$; thus *ad definitionem*, $E(\infty) = \mathbb{C}$. He obtained the following result (written in our present notation).

Theorem [20, Proposition 1.1.] *Let* $\eta = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. *Then:*

(i) *if* $\eta \geq 1$, *the series* $\mathfrak{P}_{\alpha,\beta}(z)$ *is divergent in the whole region* $\mathbb{C} \setminus [-1,1]$;

(ii) *if* $0 \leq \eta < 1$, *the* $\mathfrak{P}_{\alpha,\beta}(z)$ *is absolutely uniformly convergent on every compact subset of the region* $E(\eta^{-1})$ *and diverges at every point of the region* $\mathbb{C} \setminus \overline{E(\eta^{-1})}$.

2. Integral representation

In this section we will derive the double integral representation for the Rusev series (3). For that purpose, we will replace $z \in \mathbb{C}$ with $x \in \mathbb{R}$ and assume that the behavior of $(a_n)_{n\in\mathbb{N}}$ ensures the convergence of our main series.

We would also need some symbols and formulae which we present as follows. By convention, $[a]$ and $\{a\} = a - [a]$ denote the integer and fractional part of some real number a, respectively.

The Laplace integral representation for the Dirichlet series, which is given below, following mainly [10], [12, **C. §V**]:

$$\mathcal{D}_\lambda(x) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n x} = x \int_0^\infty \mathrm{e}^{-xt} \left(\sum_{n=1}^{[\lambda^{-1}(t)]} a_n \right) \mathrm{d}t\,,$$

where the convention is followed that the real sequence $\big(\lambda_n\big)_{n\in\mathbb{N}}$ monotonically increases and tends to infinity; equivalently

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \uparrow \infty\,.$$

Also taking a function $x \mapsto a_x = a(x)$, where $a \in \mathrm{C}^1[k,m]$, $k, m \in \mathbb{Z}$, $k < m$, then by using the operator

$$\mathfrak{d}_x := 1 + \{x\} \frac{\mathrm{d}}{\mathrm{d}x}\,,$$

we get the following condensed form of the Euler–Maclaurin summation formula [17, p. 2365]:

$$\sum_{j=k+1}^{m} a_j = \int_k^m \big(a(x) + \{x\}a'(x)\big)\mathrm{d}x = \int_k^m \mathfrak{d}_x a(x)\mathrm{d}x\,.$$

Now, we are ready to formulate the following theorem.

Theorem 1. *Let $a \in \mathrm{C}^1(\mathbb{R}_+)$ and $a|_{\mathbb{N}} = (a_n)_{n\in\mathbb{N}}$. Then for all $\alpha > -1/2$, $\alpha+\beta > -1$ and for all x of the domain*

$$\mathcal{I}_a := \big(\max\{0, 2\eta - 1\}, 1\big] \tag{4}$$

we have the integral representation

$$\begin{aligned}\mathfrak{P}_{\alpha,\beta}(x) = &-\int_1^\infty \int_0^{[s]} \frac{\partial}{\partial s}\left(\frac{\Gamma(2s+1)\,P_s^{(\alpha,\beta)}(x)}{\Gamma(\alpha+s+\frac12)\Gamma(\beta+s+\frac12)}\right)\\ &\times \mathfrak{d}_w\left(\frac{a(w)\,\Gamma(\alpha+w+\frac12)\Gamma(\beta+w+\frac12)}{\Gamma(2w+1)}\right)\mathrm{d}s\,\mathrm{d}w\,.\end{aligned}$$

P r o o f. First, we begin by establishing the convergence conditions for the series $\mathfrak{P}_{\alpha,\beta}(x)$. For that purpose, let us consider the integral representation given by Feldheim [9]:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(\alpha+\beta+n+1)}\int_0^\infty t^{\alpha+\beta+n}\mathrm{e}^{-t}\,L_n^{(\alpha)}\left(\frac12(1-x)t\right)\mathrm{d}t\,, \tag{5}$$

valid for all $n \in \mathbb{N}_0, \alpha+\beta > -1$, where $L_n^{(\alpha)}$ is the Laguerre polynomial. We estimate (5) *via* the bounding inequality for Laguerre functions $L_\nu^{(\mu)}(x)$, given by Love [14, p. 396, Theorem 2]:

$$|L_\nu^{(\mu)}(x)| \le \frac{\Gamma(\Re(\nu+\mu+1))}{|\Gamma(\nu+1)|\Gamma(\Re(\mu)+1)}\,\frac{\Gamma(\Re(\mu)+\frac12)}{|\Gamma(\mu+\frac12)|}\,\mathrm{e}^x\,, \tag{6}$$

where $\nu \in \mathbb{C}$, $x > 0$, $\Re(\mu) > -\frac12$ and $\Re(\mu+\nu) > -1$, which has been generalized by Pogány and Srivastava [16]. Specifying $\mu = \alpha \in \mathbb{R}, \nu = n \in \mathbb{N}_0$ the bound (6) reduces to

$$|L_n^{(\alpha)}(x)| \le \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\,\mathrm{e}^x, \qquad x > 0\,. \tag{7}$$

Now, applying bound (7) to the integrand of (5), we have that

$$|\mathfrak{P}_{\alpha,\beta}(x)| \le \frac{1}{\Gamma(\alpha+1)}\left(\frac{2}{1+x}\right)^{\alpha+\beta+1}\sum_{n=1}^\infty \frac{|a_n|\Gamma(\alpha+n+1)}{n!}\left(\frac{2}{1+x}\right)^n.$$

The resulting power series converges uniformly for all x satisfying constraint (4).

A more convenient integral representation for the Jacobi polynomials has been given by Braaksma and Meulenbeld [5], [7, p. 191]

$$P_n^{(\alpha,\beta)}(1-2z^2) = \frac{(-1)^n 4^n (\alpha+\frac12)_n(\beta+\frac12)_n}{\pi(2n)!} \int_{-1}^{1}\int_{-1}^{1} \Big(zu \pm \mathrm{i}\sqrt{1-z^2}\,v\Big)^{2n}$$
$$\times (1-u^2)^{\alpha-\frac12}(1-v^2)^{\beta-\frac12}\,\mathrm{d}u\mathrm{d}v, \qquad 0\le z\le 1,$$

where $2\min(\alpha,\beta) > -1$. This expression in an obvious way one reduces to

$$P_n^{(\alpha,\beta)}(x) = \frac{2^n(\alpha+\frac12)_n(\beta+\frac12)_n}{\pi(2n)!}\int_{-1}^{1}\int_{-1}^{1}\big(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v\big)^{2n}$$
$$\times (1-u^2)^{\alpha-\frac12}(1-v^2)^{\beta-\frac12}\,\mathrm{d}u\mathrm{d}v, \qquad |x|\le 1. \tag{8}$$

Thus, combining (3) and (8) we get

$$\mathfrak{P}_{\alpha,\beta}(x) = \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1}(1-u^2)^{\alpha-\frac12}(1-v^2)^{\beta-\frac12}\,\mathcal{D}_a(u,v)\,\mathrm{d}u\mathrm{d}v, \tag{9}$$

where $\mathcal{D}_a(u,v)$ is the Dirichlet series

$$\mathcal{D}_a(u,v) = \sum_{n=1}^{\infty}\frac{a_n\,(\alpha+\frac12)_n(\beta+\frac12)_n}{(2n)!}\,\mathrm{e}^{-n\ln\left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\right)^{-2}}.$$

The Dirichlet series possesses Laplace integral representation when its parameter has positive real part, therefore we are looking for the two-dimensional region $\mathcal{S}_{uv}(x)$ in the uv–plane where

$$\Re\Big\{\ln 2\big(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v\big)^2\Big\} = \ln 2\big((1+x)v^2+(1-x)u^2\big) < 0\,.$$

So, we get the ellipse

$$\mathcal{S}_{uv}(x) = \big\{(u,v)\in\mathbb{R}^2\colon (1+x)v^2+(1-x)u^2 < 1/2\big\}\,,$$

such that is nonempty for all $x\in\mathcal{I}_a$, so $\mathcal{D}_a(u,v)$ converges in $\mathcal{I}_a$.

Now, the related Laplace–integral and the Euler–Maclaurin summation formula (see for instance [3], [2]) give us:

$$\mathcal{D}_a(u,v) = -\frac{\ln\left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\right)^2}{\Gamma(\alpha+\frac12)\Gamma(\beta+\frac12)}$$
$$\times\int_0^\infty\int_0^{[s]}\big(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\big)^{2s}$$
$$\times\,\mathfrak{d}_w\left(\frac{a(w)\,\Gamma(\alpha+w+\frac12)\Gamma(\beta+w+\frac12)}{\Gamma(2w+1)}\right)\mathrm{d}s\,\mathrm{d}w. \tag{10}$$

Substituting (10) into (9) we get

$$\begin{aligned}\mathfrak{P}_{\alpha,\beta}(x) = &-\frac{1}{\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})}\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{\infty}\int_{0}^{[s]}(1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\\ &\times \ln\big(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\big)^2\cdot\big(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\big)^{2s}\\ &\times \mathfrak{d}_w\left(\frac{a(w)\,\Gamma(\alpha+w+\frac{1}{2})\Gamma(\beta+w+\frac{1}{2})}{\Gamma(2w+1)}\right)\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}w\,. \end{aligned}\tag{11}$$

Denoting

$$\begin{aligned}\mathcal{I}_x(s) := &\int_{-1}^{1}\int_{-1}^{1}\ln\big(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\big)^2\\ &\times\big(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\big)^{2s}(1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\,\mathrm{d}u\,\mathrm{d}v,\end{aligned}$$

we get

$$\begin{aligned}\int\mathcal{I}_x(s)\mathrm{d}s &= \int_{-1}^{1}\int_{-1}^{1}\big(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\big)^{2s}(1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\mathrm{d}u\mathrm{d}v\\ &= \pi\,\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(2s+1)\,P_s^{(\alpha,\beta)}(x)}{\Gamma(\alpha+s+\frac{1}{2})\Gamma(\beta+s+\frac{1}{2})}\,.\end{aligned}$$

Therefore, we can easily conclude that

$$\mathcal{I}_x(s) = \pi\Gamma(\alpha+\tfrac{1}{2})\Gamma(\beta+\tfrac{1}{2})\,\frac{\partial}{\partial s}\left(\frac{\Gamma(2s+1)\,P_s^{(\alpha,\beta)}(x)}{\Gamma(\alpha+s+\frac{1}{2})\Gamma(\beta+s+\frac{1}{2})}\right). \tag{12}$$

Finally, by using (11) and (12), we immediately get the proof of the theorem, with the assertion that the integration domain $\mathbb{R}_+$ becomes $[1,\infty)$ because $[s]$ is equal to zero for all $s\in[0,1)$. ■

Remark 2. In the previous theorem, we used Love's bound [14] for the Laguerre function $L_\nu^{(\mu)}(x)$. Similar results one can get using some other bounds for Laguerre polynomials, i.e. for the Laguerre functions. Let us mention some of them.

Pogány and Srivastava [16, p. 354, Theorem 2] derived an extension of Love's bounding inequality. The magnitude of their bounds is $\mathcal{O}(x^{-\mu/2-c}\mathrm{e}^x))$, $c\ge 0$ (see [16]) which results in a convergence region similar to $\mathcal{I}_a$.

There are two, well–known (see, e.g. [1]), classical global uniform estimates for the Jacobi polynomials, given by Szegő [22], subsequently improved by Rooney [19]. However both these bounds, having magnitudes $\mathcal{O}(\mathrm{e}^{x/2})$, are inferior to Love's and to the one by Pogány and Srivastava in [16, p. 354].

Other inequalities for the Laguerre functions, that is for the Jacobi function and the Jacobi polynomials can also be found in [14, 16, 21]. Asymptotic estimates for the Jacobi functions can be found e.g. in [1, 11].

3. Indefinite integral representations for $\mathfrak{P}_{\alpha,\beta}(x)$

In this section, we will deduce another, indefinite type integral representations for the functional series (3), by using the fact that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the linear homogeneous ODE of the second order [18, 22]:

$$(1-x^2)y'' + (\beta-\alpha-(2+\alpha+\beta)x)\,y' + n(1+\alpha+\beta+n)y = 0\,. \tag{13}$$

Now, multiplying (13) with a_n and then summing up that expression in $n \in \mathbb{N}$ we immediately get the following equality

$$\begin{aligned}(1-x^2)\mathfrak{P}''_{\alpha,\beta}(x) &+ (\beta-\alpha-(2+\alpha+\beta)x)\,\mathfrak{P}'_{\alpha,\beta}(x)\\ &= -\sum_{n=1}^{\infty} a_n\, n(1+\alpha+\beta+n)P_n^{(\alpha,\beta)}(x) =: \mathfrak{R}_{\alpha,\beta}(x)\,,\end{aligned}$$

where the right–hand side expression $\mathfrak{R}_{\alpha,\beta}(x)$ is the *functional series associated with the series* $\mathfrak{P}_{\alpha,\beta}(x)$. In the following theorem, the first main result of this section is given.

Theorem 2. *For all* $\alpha > -\frac{1}{2}$, $\alpha+\beta > -1$ *the particular solution of the linear* ODE*:*

$$(1-x^2)y' + (\beta-\alpha-(2+\alpha+\beta)x)\,y = \mathfrak{R}_{\alpha,\beta}(x)\,, \tag{14}$$

represents the first derivative $\frac{\partial}{\partial x}\mathfrak{P}_{\alpha,\beta}(x)$ *of the functional series* (3)*. Here for* $a \in \mathrm{C}^1(\mathbb{R}_+)$, $a|_{\mathbb{N}} = (a_n)_{n\in\mathbb{N}}$ *and letting* $\sum_{n=1}^{\infty} n^2 a_n$ *absolutely converges, for all* $x \in \mathcal{I}_a$ *we have the integral representation*

$$\begin{aligned}\mathfrak{R}_{\alpha,\beta}(x) = \int_1^{\infty}\int_0^{[s]} &\frac{\partial}{\partial s}\left(\frac{\Gamma(2s+1)\,P_s^{(\alpha,\beta)}(x)}{\Gamma(\alpha+s+\frac{1}{2})\Gamma(\beta+s+\frac{1}{2})}\right)\\ &\times \mathfrak{d}_w\left(\frac{a(w)\,w\,(1+\alpha+\beta+w)\,\Gamma(\alpha+w+\frac{1}{2})\Gamma(\beta+w+\frac{1}{2})}{\Gamma(2w+1)}\right)\mathrm{d}s\,\mathrm{d}w\,.\end{aligned}$$

P r o o f. Equation (14) was established in the beginning of this section. Further, the uniform convergence of the series $\mathfrak{R}_{\alpha,\beta}(x)$ can be easily recognized, using the convergence conditions of the series $\mathfrak{P}_{\alpha,\beta}(x)$, to be such that $\sum_{n=1}^{\infty} n^2|a_n| < \infty$. Then, using an integral representation derived in Theorem 1, with $a_n \mapsto -n\,(1+\alpha+\beta+n)\,a_n$, we readily get the statement. ■

Below, we shall introduce another indefinite integral representation for the series $\mathfrak{P}_{\alpha,\beta}(x)$.

Theorem 3. *Let the situation be the same as in* Theorem 2. *Then we have*

$$\mathfrak{P}_{\alpha,\beta}(x) = \int \frac{1}{(1-x)^{\alpha+1}(1+x)^{\beta+1}} \left(\int \mathfrak{R}_{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta}\, \mathrm{d}x \right) \mathrm{d}x\,,$$

where $\mathfrak{R}_{\alpha,\beta}(x)$ *is the series associated with the series* $\mathfrak{P}_{\alpha,\beta}(x)$.

Proof. It is easy to see that the Jacobi polynomial $P_0^{(\alpha,\beta)}(x) = 1$ is a solution of the homogeneous differential equation

$$(1-x^2)y'' + (\beta - \alpha - (2+\alpha+\beta)x)\, y' = 0\,. \tag{15}$$

So, a guess of the particular solution is $\mathfrak{P}_{\alpha,\beta}(x) = P_0^{(\alpha,\beta)}(x)w(x) = w(x)$. Substituting this form into nonhomogeneous differential equation (14), we get

$$(1-x^2)w''(x) + (\beta - \alpha - (2+\alpha+\beta)x)\, w'(x) = \mathfrak{R}_{\alpha,\beta}(x)\,.$$

It is easy to check that the previous equation can be rewritten into

$$\Big[(1-x)^{\alpha+1}(1+x)^{\beta+1}w'(x)\Big]' = \mathfrak{R}_{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta}\,;$$

so we have that

$$w'(x) = \frac{1}{(1-x)^{\alpha+1}(1+x)^{\beta+1}} \left(\int \mathfrak{R}_{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta}\, \mathrm{d}x + C_1 \right).$$

Finally, the desired particular solution is

$$\begin{aligned} w(x) = \int & \frac{\int \mathfrak{R}_{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta}\, \mathrm{d}x}{(1-x)^{\alpha+1}(1+x)^{\beta+1}}\, \mathrm{d}x \\ & + C_1\, 2^{-\alpha-\beta-1}\mathrm{B}\Big(\frac{1+x}{2}; -\beta, -\alpha\Big) + C_2\,, \end{aligned}$$

where $\mathrm{B}(t;p,q) = \int t^{p-1}(1-t)^{q-1}\mathrm{d}t$ denotes the so–called Čebyšev integral (incomplete Beta function).

As $P_0^{(\alpha,\beta)}$ is a solution of homogeneous differential equation (15), it does not contribute to the particular solution, so the constants C_1, C_2 can be taken to be zero and we immediately get the assertion of the theorem. ■

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