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DENSITY OF POLYNOMIALS IN THE L^2 SPACE ON THE REAL AND THE IMAGINARY AXES AND IN A SOBOLEV SPACE

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Abstract. In this paper we consider an L^2 type space of scalar functions $L^2_{\mathbf{M},A}(\mathbb{R}\cup i\mathbb{R})$ which can be, in particular, the usual L^2 space of scalar functions on $\mathbb{R}\cup i\mathbb{R}$. We find conditions for density of polynomials in this space using a connection with the L^2 space of square-integrable matrix-valued functions on $\mathbb{R}$ with respect to a non-negative Hermitian matrix measure. The completness of $L^2_{\mathbf{M},A}(\mathbb{R}\cup i\mathbb{R})$ is also established.

1. Introduction. Let $\mathbb{C}_{n\times n}$ be the algebra of all $n\times n$ matrices with complex elements and $\mathbb{C}^{\geq}_{n\times n}$ be the cone of all non-negative Hermitian matrices from $\mathbb{C}_{n\times n}$, $n\in\mathbb{N}$. Denote by $\mathfrak{B}(\mathbb{R})$ the Borel subsets of $\mathbb{R}$. By a $\mathbb{C}^{\geq}_{n\times n}$-valued measure on $\mathfrak{B}(\mathbb{R})$ we will mean a σ-additive function from $\mathfrak{B}(\mathbb{R})$ into $\mathbb{C}^{\geq}_{n\times n}$.

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For each $\mathbb{C}^{\geq}_{n\times n}$-valued measure M on $\mathfrak{B}(\mathbb{R})$ one can define a non-decreasing left-continuous (in the matrix norm) $\mathbb{C}_{n\times n}$-valued function (distribution function), see [1]:

$$M(x) := \begin{cases} M([0,x)), & x>0 \\ 0, & x=0 \\ M([x,0)), & x<0 \end{cases}. \tag{1}$$

Conversely, each non-decreasing left-continuous $\mathbb{C}_{n\times n}$-valued function $M(x)$, $x\in\mathbb{R}$, defines a $\mathbb{C}^{\geq}_{n\times n}$-valued measure on $\mathfrak{B}(\mathbb{R})$ [1].

Let M be a $\mathbb{C}^{\geq}_{n\times n}$-valued measure on $\mathfrak{B}(\mathbb{R})$ with finite matrix moments

$$S_k = \int_{\mathbb{R}} x^k dM, \quad k\in\mathbb{Z}_+. \tag{2}$$

The set of all $\mathbb{C}^{\geq}_{n\times n}$-valued measures on $\mathfrak{B}(\mathbb{R})$ with moments $\{S_k\}_{k=0}^{\infty}$ we denote by V. The problem of the description of V if one knows moments $\{S_k\}_{k=0}^{\infty}$ is the matrix Hamburger moment problem [14, p. 52]. This problem is called determinate if V consists of a unique measure and indeterminate in the opposite case.

Let $L^2_M = L^2_M(\mathbb{R})$ be the space of square-integrable with respect to M $\mathbb{C}_{n\times n}$-valued functions (see [12]). Denote by $\mathbb{P}_{n\times n}$ the set of all $(n\times n)$ matrices whose entries are complex polynomials. Elements of $\mathbb{P}_{n\times n}$ we call matrix polynomials. The set of all complex polynomials we denote by $\mathbb{P}$ $(=\mathbb{P}_{1\times 1})$. Is the set $\mathbb{P}_{n\times n}$ dense in L^2_M? The theorem of Riesz states that in the scalar case $(n=1)$ polynomials will be dense if and only if M is a N-extremal solution of moment problem (2) [2, p. 59]. A similar situation is in the case of an arbitrary n. We suppose that for every $P(x) = x^k I_n + A_{k-1}x^{k-1} + A_{k-2}x^{k-2} + \cdots + A_0 \in \mathbb{P}_{n\times n}$, where $A_i \in \mathbb{C}_{n\times n}$, $i=0,1,\dots,k-1$; $k\in\mathbb{Z}_+$; $I_n = (\delta_{i,j})_{i,j=1}^n$, the following condition holds true:

$$\int_{\mathbb{R}} P(x) dM P^*(x) \qquad \text{is an invertible matrix.} \tag{3}$$

Applying the process of pseudo-orthogonalization to the sequence $I_n,\ xI_n,\ x^2I_n,\dots,$ (see [3, pp. 577–578]) we obtain a sequence of orthonormal matrix polynomials $\{P_k(x)\}_{k=0}^{\infty}$ (where $P_k(x)$ has degree n):

$$\int_{\mathbb{R}} P_k(x) dM P_l^*(x) = I_n\delta_{k,l}, \quad k,l\in\mathbb{Z}_+. \tag{4}$$

For these polynomials one can construct a generalized Jacobi matrix J which is a $(n \times n)$ block tridiagonal matrix. The corresponding linear operator $\mathbf{J}$ in $l^2 = \{(x_l)_{l \in \mathbb{Z}_+} : \ x_l \in \mathbb{C}, \ \sum_{k=0}^{\infty} |x_k|^2 < \infty\}$ is symmetric. Let (m_-, m_+) be its deficiency indices. The matrix Hamburger moment problem is called completely indeterminate if $m_+ = m_- = n$. In this case $\mathbb{P}_{n\times n}$ will be dense in L^2_M iff (see [4, Theorem 1.1, p. 249]):

$$\int_{\mathbb{R}} \frac{dM(x)}{x-\lambda} = \tag{5}$$
$$= -\{C^*(\lambda)[I+U] - iA^*(\lambda)[I-U]\}\{D^*(\lambda)[I+U] - iB^*(\lambda)[I-U]\}^{-1},$$
$$\lambda \in \mathbb{C}\backslash\mathbb{R},$$

where U is a constant unitary matrix and $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are holomorphic matrix functions which can be computed explicitly using moments $\{S_k\}_{k=0}^{\infty}$.

Measures which satisfy condition (5) are called N-extremal.

Notice also that in the scalar case ($n = 1$) Riesz proved that polynomials are dense in L^2_M iff the measure $(1+|x|)^{-2}dM(x)$ is determinate (i.e. defines a determinate moment problem (2)) [5]. Consequently, several criteria for the determinacy of a measure can be used to study the question of density of polynomials [6].

The main purpose of our present investigation is to study the question of density of $\mathbb{P}$ in a general L^2 space which in special cases can be the L^2 space on $\mathbb{R} \cup i\mathbb{R}$ or a discrete Sobolev type space related to the discrete Sobolev orthogonal polynomials [7]. These general L^2 spaces are related to orthogonal polynomials on rays [8].

If $f = f(\lambda)$ is a $\mathbb{C}$-valued function on $\mathbb{R} \cup i\mathbb{R}$, we set

$$\vec{f_s}(\lambda) = (f(\lambda), f(-\lambda)), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}.$$

If there exist derivatives of $f(\lambda)$ at $\lambda = 0$ in directions of $\mathbb{R}$, $i\mathbb{R}$ and they coincide, we denote by $f'(0)$ their common value.

If there exists $f'(0)$, we set $\vec{f_d} = (f(0), f'(0))$.

Let M and M_i be $\mathbb{C}^{\geq}_{n\times n}$-valued measures on $\mathfrak{B}(\mathbb{R})$ and $M(x)$, $M_i(x)$ be their distribution functions, respectively. Denote by $\mathfrak{B}(i\mathbb{R})$ the set $\{iA, \ A \in \mathfrak{B}(\mathbb{R})\}$. For a set $\hat{A} \in \mathfrak{B}(i\mathbb{R})$ we set $M_I(\hat{A}) = M_i(\frac{1}{i}\hat{A})$. Then M_I will be a σ-additive function from $\mathfrak{B}(i\mathbb{R})$ into $\mathbb{C}^{\geq}_{n\times n}$. Such functions we call $\mathbb{C}^{\geq}_{n\times n}$-valued measures on $\mathfrak{B}(i\mathbb{R})$. The function

$$M_I(x) := \begin{cases} M([0,x)), & \frac{1}{i}x > 0 \\ 0, & x = 0 \\ M([x,0)), & \frac{1}{i}x < 0 \end{cases} \tag{6}$$

will be called a distribution function of M_I.

The Radon-Nikodym derivatives of the measures M, M_I we will denote by M' and M_I', respectively [12]. The corresponding trace measures we denote by τ_M and τ_{M_I}, respectively. We set $\mathbf{M} = (M, M_I)$. Let $A \in \mathbb{C}^{\geq}_{n\times n}$.

Definition 1.1. *Consider a set $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$ of $\mathbb{C}$-valued functions on $\mathbb{R} \cup i\mathbb{R}$, $f \in L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R}) \Leftrightarrow$*

(i) f is τ_M-measurable on $\mathbb{R}$, f is τ_{M_I}-measurable on $i\mathbb{R}$;

(ii) $\vec{f}_s(\lambda) M' \vec{f}_s^*(\lambda) \in L^1_{\tau_M}(\mathbb{R})$, $\vec{f}_s(\lambda) M_I'(\lambda) \vec{f}_s^*(\lambda) \in L^1_{\tau_{M_I}}(i\mathbb{R})$;

(iii) if $A \neq 0$ then $f(0)$ is finite and there exists $f'(0)$.

Obviously, $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$ is a linear vector space. We define the inner product in $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$ in the following way:

$$\langle f, g \rangle_{2;\mathbf{M},A} = \int_{\mathbb{R}} \vec{f}_s(\lambda) M'(\lambda) \vec{g}_s^*(\lambda) d\tau_M + \int_{i\mathbb{R}} \vec{f}_s(\lambda) M_I'(\lambda) \vec{g}_s^*(\lambda) d\tau_{M_I} + \\ + \vec{f}_d A \vec{g}_d^*, \quad f, g \in L^2_{\mathbf{M},A}. \tag{7}$$

For $f \in L^2_{\mathbf{M},A}$ the norm is defined as

$$\|f\|^2_{2;\mathbf{M},A} = \sqrt{\langle f, f \rangle_{2;\mathbf{M},A}}. \tag{8}$$

As usual, we shall consider equivalence classes of such functions with respect to $\langle \cdot, \cdot \rangle_{2;\mathbf{M},A}$. Then $L^2_{\mathbf{M},A}$ becomes a unitary space with inner product $\langle \cdot, \cdot \rangle_{2;\mathbf{M},A}$.

To study density of $\mathbb{P}$ in $L^2_{\mathbf{M},A}$ we will construct a non-decreasing $\mathbb{C}_{2\times 2}$-valued function M_1 on $\mathbb{R}$ depending on $\mathbf{M}$, A and establish an isometric isomorphism $\mathbf{U}$ between $L^2_{\mathbf{M},A}$ and the Hilbert space $L^2_v(M_1)$ of $\mathbb{C}^2$-valued functions square integrable with respect to M_1, see [12]. Then we use a connection between $L^2_v(M_1)$ and $L^2_{M_1}$ and apply the mentioned above results about density to $L^2_{M_1}$. As a by-product, we obtain that the space $L^2_{\mathbf{M},A}$ is complete.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real, complex, positive integer, integer, non-negative integer numbers, respectively, and $i\mathbb{R}$ stands for the imaginary axis in the complex plane. Besides the definitions given above we should note the following notations. The set of complex vectors (c_1, c_2), $c_1, c_2 \in \mathbb{C}$ we denote by $\mathbb{C}^2$. We identify $\mathbb{C}_{1\times 1}$ with $\mathbb{C}$ and $\mathbb{C}^{\geq}_{1\times 1}$ with the set of non-negative real numbers. If $A \in \mathbb{C}_{n\times n}$ then A^* stands for its adjoint, $n \in \mathbb{N}$. If $A \in \mathbb{C}_{n\times n}$ is non-degenerate, A^{-1} means its inverse. The set of all vectors (p_1, p_2) where $p_1, p_2 \in \mathbb{P}$ we denote by $\mathbb{P}_v$. These vectors will be called vector polynomials.

2. Density of polynomials. Let C_0 be a subset of $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$ which consists of continuous functions which have the first derivative at zero (in the directions of $\mathbb{R}$, $i\mathbb{R}$ and they coincide). For $f, g \in C_0$ we can write

$$\langle f, g\rangle_{2;\mathbf{M},A} = \int_{\mathbb{R}} \vec{f}_s M' \vec{g}_s^{\,*} d\tau_M + \int_{i\mathbb{R}} \vec{f}_s M_I' \vec{g}_s^{\,*} d\tau_{M_I} + \vec{f}_d A \vec{g}_d^{\,*} =$$

$$= \int_{\mathbb{R}\setminus(-\varepsilon_1,\varepsilon_2)} \vec{f}_s M' \vec{g}_s^{\,*} d\tau_M + \int_{(i\mathbb{R})\setminus(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s M_I' \vec{g}_s^{\,*} d\tau_{M_I} + \int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_s M' \vec{g}_s^{\,*} d\tau_M +$$

$$+ \int_{(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s M_I' \vec{g}_s^{\,*} d\tau_{M_I} + \vec{f}_d A \vec{g}_d^{\,*}, \tag{9}$$

where $\varepsilon_i > 0$, $i = \overline{1,4}$.

Define

$$\hat{\tau}_M(\lambda) = \begin{cases} \tau_M(\lambda), & \lambda \leq 0 \\ \tau_M(\lambda) - \Delta\tau_M(0), & \lambda > 0 \end{cases},$$

where $\Delta\tau_M(0) := \tau_M(+0) - \tau_M(0)$ is a jump of $\tau_M(\lambda)$ at zero;

$$\hat{\tau}_{M_I}(\lambda) = \begin{cases} \tau_{M_I}(\lambda), & \lambda \in (-i\infty, 0] \\ \tau_{M_I}(\lambda) - \Delta_i\tau_{M_I}(0), & \lambda \in (0, i\infty) \end{cases},$$

where $\Delta_i\tau_{M_I}(0) := \tau_{M_I}(+i0) - \tau_{M_I}(0) = \lim_{\varepsilon\to+0} \tau_{M_I}(\varepsilon i) - \tau_{M_I}(0)$, is a jump of $\tau_{M_I}(\lambda)$ at zero.

Then

$$\int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_s M' \vec{g}_s^{\,*} d\tau_M = \int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_s M' \vec{g}_s^{\,*} d\hat{\tau}_M = \vec{f}_s(0) M'(0) \vec{g}_s^{\,*}(0) \Delta\tau_M(0) +$$

$$= \int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_s M' \vec{g}_s^* d\hat{\tau}_M + f(0)\overline{g(0)}\tilde{a},$$

where $\tilde{a} := (1,1)M'(0)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\Delta\tau_M(0)$.

Since functions f, g and entries of $M'(\lambda)$ are τ_M-a.e. bounded functions in any neighbourhood of 0, we have

$$\int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_s M' \vec{g}_s^* d\hat{\tau}_M \to 0, \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0.$$

Consequently,

$$\int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_s M' \vec{g}_s^* d\tau_M \to f(0)\overline{g(0)}\tilde{a}, \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0. \tag{10}$$

In a similar manner we can write

$$\int_{(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s M_I' \vec{g}_s^* d\tau_{M_I} = \int_{(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s M_I' \vec{g}_s^* d\hat{\tau}_{M_I} + \vec{f}_s(0) M_I'(0) \vec{g}_s^*(0) \Delta_i \tau_{M_I}(0) =$$

$$= \int_{(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s M_I' \vec{g}_s^* d\hat{\tau}_M + f(0)\overline{g(0)}\tilde{a}_I,$$

where $\tilde{a}_I := (1,1)M_I'(0)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\Delta_i\tau_{M_i}(0)$.

Since functions f, g and entries of $M_I'(\lambda)$ are τ_M-a.e. bounded functions in any neighbourhood of 0, we have

$$\int_{(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s M_I' \vec{g}_s^* d\hat{\tau}_{M_I} \to 0, \quad \text{as } \varepsilon_3, \varepsilon_4 \to 0.$$

Thus, we have

$$\int_{(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s M_I' \vec{g}_s^* d\tau_{M_I} \to f(0)\overline{g(0)}\tilde{a}_I, \quad \text{as } \varepsilon_3, \varepsilon_4 \to 0. \tag{11}$$

Set $J_\lambda := \begin{pmatrix} 1 & \frac{1}{\lambda} \\ 1 & -\frac{1}{\lambda} \end{pmatrix}$. Notice that $J_\lambda^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix}$.
Then we can write

$$\int_{\mathbb{R}\setminus(-\varepsilon_1,\varepsilon_2)} \vec{f}_s M' \vec{g}_s^* d\tau_M = \int_{\mathbb{R}\setminus(-\varepsilon_1,\varepsilon_2)} \vec{f}_s J_\lambda J_\lambda^{-1} M' (J_\lambda^*)^{-1} J_\lambda^* \vec{g}_s^* d\tau_M =$$

$$= \int_{\mathbb{R}\backslash(-\varepsilon_1,\varepsilon_2)} \vec{f}_s J_\lambda \tilde{M}(\lambda) J_\lambda^* \vec{g}_s^* d\tau_M \to$$

$$\to \int_{\mathbb{R}}' \vec{f}_s J_\lambda \tilde{M}(\lambda) J_\lambda^* \vec{g}_s^* d\tau_M, \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0, \tag{12}$$

where $\tilde{M}(\lambda) := J_\lambda^{-1} M'(\lambda) (J_\lambda^*)^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix} M'(\lambda) \begin{pmatrix} 1 & \overline{\lambda} \\ 1 & -\overline{\lambda} \end{pmatrix}$ and the prime in $\int'$ means that the integral is understood in the improper sense at zero.

In an analogous manner we write

$$\int_{(i\mathbb{R})\backslash(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s M_I' \vec{g}_s^* d\tau_{M_I} = \int_{(i\mathbb{R})\backslash(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s J_\lambda J_\lambda^{-1} M_I' (J_\lambda^*)^{-1} J_\lambda^* \vec{g}_s^* d\tau_{M_I} =$$

$$= \int_{(i\mathbb{R})\backslash(-i\varepsilon_3,i\varepsilon_4)} \vec{f}_s J_\lambda \tilde{M}_I(\lambda) J_\lambda^* \vec{g}_s^* d\tau_{M_I} \to$$

$$\to \int_{i\mathbb{R}}' \vec{f}_s J_\lambda \tilde{M}_I(\lambda) J_\lambda^* \vec{g}_s^* d\tau_{M_I}, \quad \text{as } \varepsilon_3, \varepsilon_4 \to 0, \tag{13}$$

where $\tilde{M}_I(\lambda) := J_\lambda^{-1} M_I'(\lambda) (J_\lambda^*)^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix} M_I'(\lambda) \begin{pmatrix} 1 & \overline{\lambda} \\ 1 & -\overline{\lambda} \end{pmatrix}$ and the prime in $\int'$ means that the integral is understood in the improper sense at zero.

Passing to the limit in (9) and using (10), (11), (12), (13) we get

$$\langle f, g \rangle_{2;\mathbf{M},A} = \int_{\mathbb{R}}' \vec{f}_s J_\lambda \tilde{M} J_\lambda^* \vec{g}_s^* d\tau_M + \int_{i\mathbb{R}}' \vec{f}_s J_\lambda \tilde{M}_I J_\lambda^* \vec{g}_s^* d\tau_{M_I} +$$

$$+ \vec{f}_d \left(A + \begin{pmatrix} \tilde{a} + \tilde{a}_I & 0 \\ 0 & 0 \end{pmatrix} \right) \vec{g}_d^*. \tag{14}$$

For $f \in C_0$ we set

$$f^+(\lambda) := \frac{f(\lambda) + f(-\lambda)}{2}, \ \lambda \in \mathbb{R} \cup i\mathbb{R};$$

$$f^-(\lambda) := \frac{f(\lambda) - f(-\lambda)}{2\lambda}, \ \lambda \in (\mathbb{R} \cup i\mathbb{R}) \backslash \{0\}.$$

Functions f^+, f^- are continuous. Moreover,

$$\lim_{\lambda\to 0} f^-(\lambda) = \lim_{\lambda\to 0} \frac{f(\lambda) - f(-\lambda)}{2\lambda} = \lim_{\lambda\to 0} \frac{f(\lambda) - f(0)}{2\lambda} +$$

$$+ \lim_{\lambda\to 0} \frac{f(-\lambda) - f(0)}{-2\lambda} = f'(0),$$

and also $\lim_{\lambda\to i0} f^-(\lambda) = f'(0)$. Then we set

$$f^-(0) := f'(0),$$

and get a continuous function f^- on $\mathbb{R} \cup i\mathbb{R}$.

We set

$$\vec{f}_p(\lambda) := (f^+(\lambda), f^-(\lambda)),\ \ \lambda \in \mathbb{R} \cup i\mathbb{R}.$$

Using these definitions we can rewrite (14) in the following form:

$$\langle f, g\rangle_{2;\mathbf{M},A} = 4\int_{\mathbb{R}}' \vec{f}_p \tilde{M} \vec{g}_p^{\,*} d\tau_M + 4\int_{i\mathbb{R}}' \vec{f}_p \tilde{M}_I \vec{g}_p^{\,*} d\tau_{M_I} + \\ + \vec{f}_d \left(A + \begin{pmatrix} \tilde{a} + \tilde{a}_I & 0 \\ 0 & 0 \end{pmatrix} \right) \vec{g}_d. \tag{15}$$

Consider the following integrals:

$$I_1 := 4\int_{\mathbb{R}} \vec{f}_p \tilde{M} \vec{g}_p^{\,*} d\tau_M, \quad I_{1,I} := 4\int_{i\mathbb{R}} \vec{f}_p \tilde{M}_I \vec{g}_p^{\,*} d\tau_{M_I},$$

which exist in view of the definitions of $\tilde{M}$ and $\tilde{M}_I$.

Let us study the integral I_1:

$$I_1 = 4\int_{\mathbb{R}\setminus(-\varepsilon_1,\varepsilon_2)} \vec{f}_p \tilde{M} \vec{g}_p^{\,*} d\tau_M + 4\int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_p \tilde{M} \vec{g}_p^{\,*} d\tau_M, \tag{16}$$

where $\varepsilon_1, \varepsilon_2 > 0$.

We can write

$$4\int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_p \tilde{M} \vec{g}_p^{\,*} d\tau_M = 4\int_{(-\varepsilon_1,\varepsilon_2)} \vec{f}_p \tilde{M} \vec{g}_p^{\,*} d\hat{\tau}_M + 4\vec{f}_p(0)\tilde{M}(0)\vec{g}_p^{\,*}(0)\Delta\tau_M(0).$$

Since functions f^+, f^-, g^+, g^- and entries of $\tilde{M}$ are τ_M-a.e. bounded in any neighbourhood of 0, the first addend on the right of the latter equality tends to zero as $\varepsilon_1, \varepsilon_2 \to 0$.

The first addend on the right-hand side of (16) tends to $4\int'_{\mathbb{R}} \vec{f}_p \tilde{M} \vec{g}_p^* d\tau_M$, as $\varepsilon_1, \varepsilon_2 \to 0$, by the definition of the improper integral.

Consequently, passing to the limit in (16) we get

$$I_1 = 4\int'_{\mathbb{R}} \vec{f}_p \tilde{M} \vec{g}_p^* d\tau_M + 4\vec{f}_p(0)\tilde{M}(0)\vec{g}_p^*(0)\Delta\tau_M(0).$$

In a similar way we get that

$$I_{1,I} = 4\int'_{i\mathbb{R}} \vec{f}_p \tilde{M}_I \vec{g}_p^* d\tau_{M_I} + 4\vec{f}_p(0)\tilde{M}_I(0)\vec{g}_p^*(0)\Delta_i\tau_{M_I}(0).$$

Therefore

$$4\int'_{\mathbb{R}} \vec{f}_p \tilde{M} \vec{g}_p^* d\tau_M + 4\int'_{i\mathbb{R}} \vec{f}_p \tilde{M}_I \vec{g}_p^* d\tau_{M_I} = I_1 + I_{1,I} -$$

$$-4\vec{f}_p(0)\tilde{M}(0)\vec{g}_p^*(0)\Delta\tau_M(0) - 4\vec{f}_p(0)\tilde{M}_I(0)\vec{g}_p^*(0)\Delta_i\tau_{M_I}(0) =$$

$$= I_1 + I_{1,I} - 4\vec{f}_d\tilde{M}(0)\vec{g}_d^*\Delta\tau_M(0) - 4\vec{f}_d\tilde{M}_I(0)\vec{g}_d^*\Delta_i\tau_{M_I}(0).$$

Substituting the last expression in (15) we get

$$\langle f, g\rangle_{2;\mathbf{M},A} = 4\int_{\mathbb{R}} \vec{f}_p \tilde{M} \vec{g}_p^* d\tau_M + 4\int_{i\mathbb{R}} \vec{f}_p \tilde{M}_I \vec{g}_p^* d\tau_{M_I} + \vec{f}_d A \vec{g}_d^*, \tag{17}$$

since

$$\begin{pmatrix} \tilde{a} + \tilde{a}_I & 0 \\ 0 & 0 \end{pmatrix} - 4\tilde{M}(0)\Delta\tau_M(0) - 4\tilde{M}_I(0)\Delta_i\tau_{M_I}(0) = 0.$$

Set

$$\widehat{M}(\lambda) = \begin{cases} 4\int_0^\lambda \tilde{M}(\lambda) d\tau_M(\lambda), & \lambda \in \mathbb{R} \\ 4\int_0^\lambda \tilde{M}_I(\lambda) d\tau_{M_I}(\lambda), & \lambda \in i\mathbb{R} \end{cases}.$$

Since functions f^+, f^-, g^+, g^- are continuous, we can rewrite (17) in the following form:

$$(18) \qquad \langle f, g\rangle_{2;\mathbf{M},A} = \int_{\mathbb{R}} \vec{f}_p d\widehat{M}(\lambda)\vec{g}_p^* + \int_{i\mathbb{R}} \vec{f}_p d\widehat{M}(\lambda)\vec{g}_p^* + \vec{f}_d A\vec{g}_d^*,$$

where integrals are understood as a sum of integrals of each scalar addend.

Denote the sum of integrals on the right by I_2. Then we can write

$$(19) \qquad I_2 = \int_{(-\infty,0]} \vec{f}_p d\widehat{M}(\lambda)\vec{g}_p^* + \int_{[0,\infty)} + \cdots + \int_{(-i\infty,0]} + \cdots + \int_{[0,i\infty)} + \cdots .$$

In the first two terms on the right we make the changes of variable: $x = \lambda^2,\ \lambda = -\sqrt{x}$ and $x = \lambda^2,\ \lambda = \sqrt{x}$, respectively. Here $\sqrt{x} = |x|^{\frac{1}{2}}$.

In the last two terms on the right of (19) we make the changes of variable: $x = \lambda^2,\ \lambda = -\sqrt{x}$ and $x = \lambda^2,\ \lambda = \sqrt{x}$, respectively. Here $\sqrt{x} = |x|^{\frac{1}{2}} i$. Note that functions f^+, f^-, g^+, g^- are even and do not depend on the choice of a branch of the square root. Define the following matrix-function:

$$(20) \qquad \widehat{M}_1(x) = \begin{cases} \widehat{M}(-\sqrt{x}) - \widehat{M}(\sqrt{x}), & x \leq 0 \\ \widehat{M}(\sqrt{x}) - \widehat{M}(-\sqrt{x}) + A, & x > 0 \end{cases} .$$

This function is a non-decreasing $\mathbb{C}_{2\times 2}$-valued function on $\mathbb{R}$. Then from (18) and (19) we get

$$\langle f, g\rangle_{2;\mathbf{M},A} = \int_{\mathbb{R}} \vec{f}_p(\sqrt{x}) d\widehat{M}_1(x)\vec{g}_p^*(\sqrt{x}).$$

We set $M_1(x) := \widehat{M}_1(x-0),\ x \in \mathbb{R}$, to get a left-continuous function and

$$(21) \qquad \langle f, g\rangle_{2;\mathbf{M},A} = \int_{\mathbb{R}} \vec{f}_p(\sqrt{x}) dM_1(x)\vec{g}_p^*(\sqrt{x}).$$

The function $M_1(x)$ defines a $\mathbb{C}_{2\times 2}^{\geq}$-valued measure M_1 on $\mathfrak{B}(\mathbb{R})$. The space of square-integrable vector functions $\vec{v} = (v_1(x), v_2(x))$ on $\mathbb{R}$ with respect to the measure M_1 we shall denote $L_v^2(M_1)$. The space of square-integrable matrix-functions $v = \begin{pmatrix} v_1(x) & v_2(x) \\ v_3(x) & v_4(x) \end{pmatrix}$ on $\mathbb{R}$ with respect to the measure M_1 we denote $L^2_{M_1}$. For the general definition of such spaces we refer to [12]. Notice that in fact

$$(22) \qquad L^2_{M_1} = L_v^2(M_1) \oplus L_v^2(M_1),$$

according to the map

$$L^2_{M_1} \ni \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \to (v_1, v_2) \oplus (v_3, v_4) \in L^2_v(M_1) \oplus L^2_v(M_1).$$

From this fact it follows a corollary:

Corollary 2.1. *The set $\mathbb{P}_v$ is dense in $L^2_v(M_1)$ iff the set $\mathbb{P}_{2\times 2}$ is dense in $L^2(M_1)$.*

By $\langle \cdot, \cdot \rangle_{2,v;M_1}$ and $\| \cdot \|_{2,v;M_1}$ we denote the inner product and the norm in $L^2_v(M_1)$, respectively. We can define an operator $\mathbf{V} : C_0 \to L^2_v(M_1)$:

$$(\mathbf{V}f)(x) = \vec{f}_p(\sqrt{x}), \ x \in \mathbb{R}, \ f \in C_0. \tag{23}$$

From (21) it follows that

$$\langle \mathbf{V}f, \mathbf{V}g \rangle_{2,v;M_1} = \langle f, g \rangle_{2;\mathbf{M},A}, \ f, g \in C_0. \tag{24}$$

Let R_0 be a subset of $L^2_v(M_1)$ which consists of functions $\vec{v}(x) = (v_1(x), v_2(x))$, such that v_1, v_2 are continuous and $w(\lambda) := v_1(\lambda^2)$, $\lambda \in \mathbb{R} \cup i\mathbb{R}$ has the first derivative at zero (in the directions of $\mathbb{R}$ and $i\mathbb{R}$ and they coincide). We shall show that $\mathbf{V}C_0 = R_0$.

Notice that for $f \in C_0$ the functions $f^+(\sqrt{x}), f^-(\sqrt{x})$ are continuous and the function $f^+(\lambda) = \frac{1}{2}(f(\lambda) + f(-\lambda))$ has the derivative $\frac{d}{d\lambda} f^+ \Big|_{\lambda=0} (= 0)$. Hence, $\mathbf{V}C_0 \subseteq R_0$. On the other hand, take $\vec{v}(x) = (v_1(x), v_2(x)) \in R_0$ and set $f_v(\lambda) := v_1(\lambda^2) + \lambda v_2(\lambda^2)$, $\lambda \in \mathbb{R} \cup i\mathbb{R}$. Then $f_v^+(\lambda) = v_1(\lambda^2)$, $f_v^-(\lambda) = v_2(\lambda^2)$ and

$$(f_v^+(\sqrt{x}), f_v^-(\sqrt{x})) = (v_1(x), v_2(x)), \ x \in \mathbb{R}. \tag{25}$$

Moreover,

$$\exists f_v'(0) = \lim_{\Delta\lambda \to 0} \frac{f_v(\Delta\lambda) - f_v(0)}{\Delta\lambda} =$$

$$= \lim_{\Delta\lambda \to 0} \frac{v_1((\Delta\lambda)^2) + \Delta\lambda v_2((\Delta\lambda)^2) - v_1(0)}{\Delta\lambda} = v_2(0) + (v_1(\lambda^2))'|_{\lambda=0},$$

$$(\Delta\lambda \in \mathbb{R} \text{ or } \Delta\lambda \in i\mathbb{R}).$$

Using (25) we can write

$$\|\vec{v}\|_{2,v;M_1} = \int_{\mathbb{R}} (\vec{f_v})_p(\sqrt{x})dM_1(x)(\vec{f_v})_p^*(\sqrt{x}).$$

Moving backwards from (21) to (9) we obtain

$$\|\vec{v}\|_{2,v;M_1} = \|f_v\|_{2;\mathbf{M},A} < \infty.$$

Hence, $f_v \in C_0$. Moreover,

$$(\mathbf{V}f_v)(x) = (\vec{f_v})_p(\sqrt{x}) = \vec{v}(x). \tag{26}$$

Consequently, $\mathbf{V}C_0 = R_0$.

Define an operator $\mathbf{G} : R_0 \to C_0$:

$$(\mathbf{G}\vec{v})(\lambda) = v_1(\lambda^2) + \lambda v_2(\lambda^2),\ \lambda \in \mathbb{R} \cup i\mathbb{R},\ \vec{v} = (v_1, v_2) \in R_0. \tag{27}$$

From (27) it follows that $\mathbf{VG}\vec{v} = \vec{v}$, $\vec{v} \in R_0$. On the other hand, $\mathbf{GV}f = \mathbf{G}\vec{f_p}(\sqrt{x}) = f^+(\lambda) + \lambda f^-(\lambda) = f$, $f \in C_0$. Hence, $\mathbf{G} = \mathbf{V}^{-1}$.

Lemma 2.2. *The subset C_0 is dense in $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$.*

P r o o f. *1) Case $A = 0$.* Here we shall follow the ideas of Kats in [13]. Take $f \in L^2_{\mathbf{M},A}$. Set

$$f_N(\lambda) = \begin{cases} f(\lambda), & \text{if } |f(\lambda)| < N,\ |f(-\lambda)| < N \\ 0, & \text{otherwise} \end{cases},\ \lambda \in \mathbb{R} \cup i\mathbb{R},\ N \in \mathbb{N}. \tag{28}$$

Let $E_N := \{\lambda \in \mathbb{R} \cup i\mathbb{R} :\ f_N(\lambda) \neq f(\lambda)\}$. Since $f(\lambda)$ has almost everywhere with respect to τ_M on $\mathbb{R}$ and with respect to τ_{M_I} on $i\mathbb{R}$ finite values, we get that

$$\tau_M(E_N \cap \mathbb{R}) \to 0 \text{ and } \tau_{M_I}(E_N \cap (i\mathbb{R})) \to 0 \tag{29}$$

as $N \to \infty$. We shall denote

$$\Psi(\lambda; g) := \vec{g}_s(\lambda)M'(\lambda)\vec{g}_s^*(\lambda),\ \lambda \in \mathbb{R},$$

$$\Psi_I(\lambda; g) := \vec{g}_s(\lambda)M_I'(\lambda)\vec{g}_s^*(\lambda),\ \lambda \in i\mathbb{R}, \quad g \in L^2_{\mathbf{M},A}.$$

Then

$$\|f_N - f\|^2_{2;\mathbf{M},A} = \int_{E_N \cap \mathbb{R}} \Psi(\lambda; f)d\tau_M + \int_{E_N \cap (i\mathbb{R})} \Psi_I(\lambda; f)d\tau_{M_I} \to 0,$$

as $N \to \infty$, since f is integrable and (29) holds.

Hence, bounded functions are dense in $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$.

Consider a bounded function $f_b(\lambda) \in L^2_{\mathbf{M},A}$: $|f_b(\lambda)| \leq N,\ N \in \mathbb{N}$. Take an arbitrary $\varepsilon :\ 0 < \varepsilon < 1$. Choose $K > 0$ such that τ_M is continuous at points $-K, K$ and

$$\left| \int_{\mathbb{R} \setminus [-K,K]} \Psi(\lambda; f) d\tau_M \right| < \varepsilon.$$

Let $\tau_M(\lambda),\ \lambda \in \mathbb{R}$ be the distribution function of the measure τ_M. Consider the function

$$\hat{\tau}_M(\lambda) := \begin{cases} \tau_M(\lambda), & \lambda \leq 0 \\ \tau_M(\lambda) + 1, & \lambda > 0 \end{cases}.$$

The function $\hat{\tau}_M(\lambda)$ defines a $\mathbb{C}^{\geq}$-valued measure $\hat{\tau}_M$ on $\mathfrak{B}(\mathbb{R})$. It is important for us that

$$\hat{\tau}_M(A) = \tau_M(A), \quad A \in \mathfrak{B}(\mathbb{R}) :\ 0 \notin A; \tag{30}$$

$$\hat{\tau}_M(A) = \tau_M(A) + 1 \geq 1, \quad A \in \mathfrak{B}(\mathbb{R}) :\ 0 \in A. \tag{31}$$

Applying Luzin's theorem (see [9, pp. 227–229]) we can find a continuous function $g_N(\lambda)$ on $[-K, K]$ such that for $Q_N := \{\lambda \in [-K, K] :\ g_N(\lambda) \neq f_b(\lambda)\}$ we have $\hat{\tau}_M(Q_N) < \varepsilon$. Also $g_N(\lambda)$ can be chosen such that $|g_N(\lambda)| < N$.

From (31) it follows that $0 \notin Q_N$, therefore $g_N(0) = f_b(0)$. From (31) it follows also that $\tau_M(Q_N) < \varepsilon$.

There exists $\delta > 0$ such that $\tau_M([-K, -K - \delta]) < \varepsilon$ and $\tau_M([K, K + \delta]) < \varepsilon$, since τ_M is continuous at points $-K, K$.

We continue the function $g_N(\lambda)$ on intervals $[-K, -K - \delta]$ and $[K, K + \delta]$ by intervals of a straight line $a\lambda + b$ connecting the known value of $g_N(\lambda)$ with 0. Namely,

$$g_N(\lambda) := \begin{cases} \dfrac{g_N(-K)}{\delta}(\lambda + K + \delta), & \lambda \in [-K, -K - \delta] \\ -\dfrac{g_N(K)}{\delta}(\lambda - K - \delta), & \lambda \in [K, K + \delta] \end{cases}.$$

Thus, we have

$$|g_N(\lambda)| \leq |g_N(-K)| < N,\ \lambda \in [-K, -K - \delta],$$

$$|g_N(\lambda)| \le |g_N(K)| < N,\ \lambda \in [K, K-\delta].$$

On the rest of $\mathbb{R}$ we set $g_N(\lambda) = 0$.

Consequently,

$$\left| \int_{\mathbb{R}} \Psi(\lambda; f - g_N) d\tau_M \right| < \varepsilon + \left| \int_{[-K-\delta,-K]\cup[K,K+\delta]} \Psi(\lambda; f - g_N) d\tau_M \right| +$$

$$+ \left| \int_{[-K,K]} \Psi(\lambda; f - g_N) d\tau_M \right| < \varepsilon + 16N^2 \tau_M([-K-\delta, -K]) +$$

$$+16N^2\tau_M([K, K+\delta]) + 16N^2\tau_M(Q_N) < (48N^2+1)\varepsilon.$$

In a similar way we can construct a continuous function $g_{N,I}(\lambda)$ on $i\mathbb{R}$ such that $g_{N,I}(0) = f_b(0)$ and

$$\left| \int_{i\mathbb{R}} \Psi_I(\lambda; f - g_{N,I}) d\tau_{M_I} \right| < (48N^2+1)\varepsilon.$$

We set

$$d_N(\lambda) := \begin{cases} g_N(\lambda), & \lambda \in \mathbb{R} \\ g_{N,I}(\lambda), & \lambda \in i\mathbb{R} \end{cases}.$$

It is not hard to see that $d_N(\lambda)$ will approximate $f_b(\lambda)$. Consequently, continuous functions are dense in $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$.

Take a continuous function $f_c \in L^2_{\mathbf{M},A}$. For $a = -1; 1; -i; i$ set

$$\hat{f}(\lambda) = (\lambda - a)g(\lambda) + f_c(a), \tag{32}$$

where

$$g(\lambda) = \frac{1}{a}(f_c(a) - f_c(0)) + \lambda\left(\frac{1}{a^2}(f_c(a) - f_c(0)) - \frac{d}{a}\right),\ \lambda \in [0, a]$$

and d is an arbitrary complex number. It is not hard to see that $\hat{f}(\lambda)$ is a continuos function on $[-1, 1] \cup [-i, i]$ such that

$$\hat{f}(-1) = f_c(-1),\ \hat{f}(1) = f_c(1),\ \hat{f}(-i) = f_c(-i),\ \hat{f}(i) = f_c(i); \tag{33}$$

$$\hat{f}(0) = f_c(0), \quad \exists \hat{f}'(0) = d. \tag{34}$$

Since $f_c(\lambda) \to f_c(0)$ and $\hat{f}(\lambda) \to f_c(0)$ as $\lambda \to 0$, we get that

$$\forall \varepsilon > 0\ \exists \delta_1 > 0\ (\delta_1 < 1):\ \lambda \in [-1,1] \cup [-i,i],\ |\lambda| \le \delta \Rightarrow$$

$$|\hat{f}(\lambda) - f_c(\lambda)| < \varepsilon.$$

Set

$$f_1(\lambda) = \begin{cases} \hat{f}(\lambda), & |\lambda| < \delta_1 \\ f_c(\lambda) + \dfrac{|\lambda|-1}{\delta_1 - 1}(\hat{f}(\delta_1) - f_c(\delta_1)), & \delta_1 \le |\lambda| \le 1 \\ f_c(\lambda), & |\lambda| > 1 \end{cases}, \ \lambda \in \mathbb{R} \cup i\mathbb{R}. \tag{35}$$

Then $f_1(\lambda)$ is a continuous function on $\mathbb{R} \cup i\mathbb{R}$ which has the first derivative at zero and such that:

$$|f_1(\lambda) - f_c(\lambda)| < \varepsilon,\ \lambda \in [-1,1] \cup [-i,i]. \tag{36}$$

Then

$$\|f_1 - f_c\|^2_{2;\mathbf{M},A} = \int_{[-1,1]} \Psi(\lambda; f_1 - f_c) d\tau_M + \int_{[-i,i]} \Psi_I(\lambda; f_1 - f_c) d\tau_{M_I} \le$$

$$\le 4\varepsilon^2 \tau_M([-1,1]) + 4\varepsilon^2 \tau_{M_I}([-i,i]). \tag{37}$$

Consequently, C_0 is dense in $L^2_{\mathbf{M},A}$ in this case.

2) Case $A \neq 0$. Take a function $f \in L^2_{\mathbf{M},A}$. Consider the space $L^2_{\mathbf{M},0}$. Evidently, $f \in L^2_{\mathbf{M},0}$ as well. Let us apply constructions from the previous case. Since $f(0)$ is finite, we can for an arbitrary $\varepsilon > 0$ construct a bounded function $f_N(\lambda)$ ($N > |f(0)|$) such that $f_N(0) = f(0)$ and $\|f - f_N\|_{2;\mathbf{M},0} < \varepsilon$. Then we construct a continuous function $f_c(\lambda)$ such that $f_c(0) = f_N(0) = f(0)$ and $\|f_N - f_c\|_{2;\mathbf{M},0} < \varepsilon$. For the continuous function f_c we construct a function $\hat{f}(\lambda)$ as in (32) with $d = f'(0)$. Then we repeat considerations after (32). Define a function $f_1(\lambda)$ as in (35). The function f_1 will posess property (36) and (37) holds with $A = 0$. Notice that $f_1 \in L^2_{\mathbf{M},A}$ by the definition of $L^2_{\mathbf{M},A}$. Moreover, $f_1 \in C_0$. Consequently,

$$\|f - f_1\|_{2;\mathbf{M},A} = \|f - f_1\|_{2;\mathbf{M},0} \le \|f - f_N\|_{2;\mathbf{M},0} + \|f_N - f_c\|_{2;\mathbf{M},0} +$$

$$+\|f_c - f_1\|_{2;\mathbf{M},0} \leq 2\varepsilon + 4\varepsilon^2 \tau_M([-1,1]) + 4\varepsilon^2 \tau_{M_I}([-i,i]).$$

Hence, C_0 is dense in $L^2_{\mathbf{M},A}$ in this case as well. $\square$

Lemma 2.2. *The subset R_0 is dense in $L^2_v(M_1)$.*

P r o o f. Since continuous matrix-functions are dense in $L^2(M_1)$ (see [13]), continuous vector functions are dense in $L^2_v(M_1)$ as follows from (22). Let $\vec{f} = (f_1, f_2) \in L^2_v(M_1)$ be continuous.

For $a = -1, 1$ we set

$$\hat{f}_k(x) = (x-a)g_k(x) + f_k(a),$$

where

$$g_k(x) = \frac{1}{a}(f_k(a) - f_k(0)) + x\left(\frac{1}{a^2}(f_k(a) - f_k(0)) - \frac{1}{a}\right), \quad x \in [0,a],$$

$$k = 1,2.$$

It is not hard to see that $\hat{f}_1(x)$ and $\hat{f}_2(x)$ are continuous functions on $[-1,1]$ such that

$$\hat{f}_1(-1) = f_1(-1),\ \hat{f}_1(1) = f_1(1),\ \hat{f}_2(-1) = f_2(-1),\ \hat{f}_2(1) = f_2(1); \tag{38}$$

$$\hat{f}_1(0) = f_1(0),\ \hat{f}_2(0) = f_2(0), \quad \exists \hat{f}_1'(0) = 1. \tag{39}$$

Since $f_k(x) \to f_k(0)$ and $\hat{f}_k(x) \to f_k(0)$ as $x \to 0$, we obtain that

$$\forall \varepsilon > 0\ \exists \delta_1 > 0\ (\delta_1 < 1):\ x \in [-1,1],\ |x| \leq \delta_1 \Rightarrow$$

$$|\hat{f}_k(x) - f_k(x)| < \varepsilon,\ k = 1,2.$$

Set

$$\tilde{f}_k(x) = \begin{cases} \hat{f}_k(x), & |x| < \delta_1 \\ f_k(x) + \dfrac{|x|-1}{\delta_1 - 1}(\hat{f}_k(\delta_1) - f_k(\delta_1)), & \delta_1 \leq |x| \leq 1 \\ f_k(x), & |x| > 1 \end{cases}, \ x \in \mathbb{R};\ k = 1,2.$$

Then $\vec{\tilde{f}}(x) := (\tilde{f}_1(x), \tilde{f}_2(x))$ is a continuous function on $\mathbb{R}$ such that $\tilde{f}_1(\lambda^2),\ \lambda \in \mathbb{R} \cup i\mathbb{R}$ has the first derivative at zero and

$$|\tilde{f}_k(x) - f_k(x)| < \varepsilon,\ x \in [-1,1],\ k = 1,2. \tag{40}$$

Let $M_1'(x)$ be the matrix of the Radon-Nikodym derivatives of the measure M_1 with respect to the trace measure τ_{M_1}. Then

$$\|\vec{f} - \vec{\tilde{f}}\|^2_{2,v;M_1} = \int_{[-1,1]} (f_1(x) - \tilde{f}_1(x), f_2(x) - \tilde{f}_2(x)) M_1'(x) *$$

$$*(f_1(x) - \tilde{f}_1(x), f_2(x) - \tilde{f}_2(x))^* d\tau_{M_1} \le 4\varepsilon^2 \tau_{M_1}([-1,1]).$$

Consequently, R_0 is dense in $L^2_v(M_1)$. $\square$

We can extend the operator $\mathbf{V}$ by continuity to an isometric isomorphism $\mathbf{U}$ between $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$ and $L^2_v(M_1)$. Hence, we have obtained the following result:

Lemma 2.3. *There exists an isometric isomorphism* $\mathbf{U}$ *between* $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$ *and* $L^2_v(M_1)$.

Since $L^2_v(M_1)$ is a complete space (see [12, Theorem, p. 295]), from Lemma 2.3 we get the following theorem.

Theorem 2.1. $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$ *is a complete Hilbert space under the inner product* $\langle \cdot, \cdot \rangle_{2;\mathbf{M},A}$.

Special choices of $\mathbf{M}$ and A in the definition of $L^2_{\mathbf{M},A}$ lead to the following important L^2 spaces:

(a) $M = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$, $M_I = 0,\ A = 0$. This leads to the usual $L^2_m(\mathbb{R})$.

(b) $M = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$, $M_I = 0,\ A \neq 0$. That leads to a discrete Sobolev space related to the discrete Sobolev orthogonal polynomials [7].

(c) $M = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}$, $M_I = \begin{pmatrix} m_I & 0 \\ 0 & 0 \end{pmatrix}$: $m_I(\{0\}) = 0,\ A = 0$. Define a measure $\mathbf{m}$ on sets $[a,b] \cup [ic, id],\ a,b,c,d > 0$:

$$\mathbf{m}([a,b] \cup [ic, id]) := m([a,b]) + m_I([ci, di]),$$

and in a similar manner for sets with $(a,b]$, $[a,b)$, (a,b), $(ci,di]$, $[ci,di)$, (ci,di). Then we define a measure μ on rectangles in $\mathbb{C}$ (with sides parallel to the axes):

$$\mu(R) = \mathbf{m}((\mathbb{R} \cup i\mathbb{R}) \cap R), \quad \text{where } R \text{ is a rectangle.}$$

Finaly, we apply Lebesque's extention to μ to obtain a positive measure $\tilde{\mu}$ defined on the Borel subsets $\mathfrak{B}(\mathbb{C})$ in the complex plane. It is not hard to see that $L_{2;\mathbf{M},A} = L^2_{\tilde{\mu}}$ in this case. The measure $\tilde{\mu}$ has support on $\mathbb{R} \cup i\mathbb{R}$.

Conversely, each positive measure $\tilde{\mu}$ on $\mathfrak{B}(\mathbb{C})$ with support on $\mathbb{R} \cup i\mathbb{R}$ defines measures $\tilde{M}$ and $\tilde{M}_I$ on intervals:

$$\tilde{M}([a,b]) := \lim_{\varepsilon \to +0} R(a - \varepsilon i, a + \varepsilon i, b - \varepsilon i, b + \varepsilon i), \tag{41}$$

$$\tilde{M}_I([ci,di]) := \lim_{\varepsilon \to +0} R(ci - \varepsilon, ci + \varepsilon, di - \varepsilon, di + \varepsilon), \tag{42}$$

where $R(z_1, z_2, z_3, z_4)$ is a closed rectangle with vertices at points $z_i,\ i = 1,2,3,4$; $a,b,c,d > 0$.

For intervals $[a,b)$, $(a,b]$, (a,b), $[ci,di)$, $(ci,di]$, (ci,di) we use the obvious modifications of definitions (41),(42). Applying Lebesgue's extension we obtain $\mathbb{C}^{\geq}$-valued measures $\hat{M}$ and $\hat{M}_I$ on $\mathfrak{B}(\mathbb{R})$ and $\mathfrak{B}(i\mathbb{R})$, respectively. Finally, we set $M := \hat{M}$ and for $A \in \mathfrak{B}(i\mathbb{R})$

$$M_I(A) := \begin{cases} \hat{M}_I(A), & \text{if } 0 \notin A \\ \hat{M}_I(A) - \hat{M}(\{0\}), & \text{if } 0 \in A \end{cases}.$$

It is not hard to see that $L^2_{\tilde{\mu}} = L_{2;\mathbf{M},A}$.

Let us now turn to the question of density of polynomials in $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$. It is natural to assume that complex polynomials belong to $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$. This condition is equivalent to existance of the following integrals:

$$\int_{\mathbb{R}} (\vec{\lambda^n})_s M'(\lambda) (\vec{\lambda^n})_s^* d\tau_M + \int_{i\mathbb{R}} (\vec{\lambda^n})_s M_I'(\lambda) (\vec{\lambda^n})_s^* d\tau_{M_I} < \infty,\ n \in \mathbb{Z}_+. \tag{43}$$

It is not hard to see that $\mathbf{V}\mathbb{P} = \mathbb{P}_v$. Let polynomials be dense in $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$. Therefore they are dense in C_0. Thus, $\forall \vec{v} = (v_1, v_2) \in R_0$ we can find a polynomial $p(\lambda)$ such that

$$\varepsilon > \|\mathbf{V}^{-1} v - p\|_{2;\mathbf{M},A} = \|v - \mathbf{V}p\|_{2,v;M_1}.$$

Hence, vector polynomials are dense in R_0 and from Lemma 2.2 it follows that they are dense in $L^2_v(M_1)$.

Let vector polynomials be dense in $L^2_v(M_1)$. Therefore they are dense in R_0. Thus, $\forall f(\lambda) \in C_0$ we can find a vector $\vec{p} = (p_1, p_2)$, where p_1, p_2 are polynomials such that

$$\varepsilon > \|\mathbf{V}f - \vec{p}\|_{2,v;M_1} = \|f - \mathbf{V}^{-1}\vec{p}\|_{2;\mathbf{M},A}.$$

Hence, polynomials are dense in C_0 and from Lemma 2.2 it follows that they are dense in $L^2_{\mathbf{M},A}(\mathbb{R} \cup i\mathbb{R})$.

We get that polynomials are dense in $L^2_{\mathbf{M},A}$ iff vector polynomials are dense in $L^2_v(M_1)$.

Taking into account Corollary 2.1 we obtain the following theorem:

Theorem 2.2. *Consider an $L^2_{\mathbf{M},A}$ space such that for the measure M condition* (43) *is satisfied. Then polynomials are dense in $L^2_{\mathbf{M},A}$ iff matrix polynomials are dense in $L^2_{M_1}$ where M_1 is constructed as above.*

We shall suppose that the measure M is not degenerate in the following sense:

$$\langle p, p\rangle_{2;\mathbf{M},A} > 0, \tag{44}$$

for any complex polynomial $p(\lambda) \neq 0$.

Applying the Gramm-Schmidt orthogonalization method to the sequence $1,\ \lambda,\ \lambda^2,\ \ldots$, with respect to $\langle\cdot,\cdot\rangle_{2;\mathbf{M},A}$ we get a system of orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$. They satisfy the orthonormality property

$$\int_{\mathbb{R}} (\vec{p_n})_s(\lambda) M'(\lambda) (\vec{p_m})_s^*(\lambda) d\tau_M + \int_{i\mathbb{R}} (\vec{p_n})_s(\lambda) M_I'(\lambda) (\vec{p_m})_s^*(\lambda) d\tau_{M_I} +$$

$$+(\vec{p_n})_d A (\vec{p_m})_d^* = \delta_{n,m}, \quad n, m \in \mathbb{Z}_+. \tag{45}$$

Special choices of $\mathbf{M}$ and A in the definition of $L^2_{\mathbf{M},A}$, see cases **(a)**,**(b)**,**(c)** after Theorem 2.1, lead to the following important systems of orthogonal polynomials:

(a): Orthogonal polynomials on the real line, see [10];

(b): Discrete Sobolev orthogonal polynomials, see [7];

(c): Orthogonal polynomials on radial rays in the complex plane, see [11].

For orthonormal polynomials from (45) we define

$$\vec{p}_n(x) := (\mathbf{V}p_n)(x) = (p_n^+(\sqrt{x}), p_n^-(\sqrt{x})),\ x \in \mathbb{R},\ n \in \mathbb{Z}_+.$$

Then $\{\vec{p}_n(x)\}_{n\in\mathbb{Z}_+}$ will be an orthonormal system in $L^2_v(M_1)$. Define matrix polynomials

$$P_k(x) := \begin{pmatrix} p_{2k}^+(\sqrt{x}) & p_{2k}^-(\sqrt{x}) \\ p_{2k+1}^+(\sqrt{x}) & p_{2k+1}^-(\sqrt{x}) \end{pmatrix} = \begin{pmatrix} \vec{p}_{2k}(x) \\ \vec{p}_{2k+1}(x) \end{pmatrix},\ k \in \mathbb{Z}_+. \tag{46}$$

Notice that $P_k(x)$ has degree exactly k. Moreover, we have

$$\int_{\mathbb{R}} P_k(x) dM_1(x) P_l^*(x) = I\delta_{k,l},\ k,l \in \mathbb{Z}_+, \tag{47}$$

where $I = (\delta_{i,j})_{i,j=1}^2$.

Hence, $\{P_k(x)\}_{k=0}^\infty$ are matrix orthonormal polynomials. Denote by J the corresponding Jacobi matrix and let $\mathbf{J}$ be the corresponding symmetric operator in l^2. Suppose that $\mathbf{J}$ has indices of deficiency $(2,2)$. In other words that means that the matrix Hamburger moment problem corresponding to M_1 is completely indeterminate.

In this case we get the following theorem:

Theorem 2.3. *Consider an $L^2_{\mathbf{M},A}$ space such that for the measure M conditions* (43), (44) *are satisfied. Construct the measure M_1 as above. Suppose that the corresponding symmetric operator $\mathbf{J}$ has indices of deficiency* $(2,2)$. *Then polynomials are dense in $L^2_{\mathbf{M},A}$ iff condition* (5) *holds true for $M = M_1$ with a constant unitary matrix U.*

Note that there are several criteria for the completely indeterminate case of the matrix moment problem. For example, one can use a generalization of Hamburger's criterion [14, p. 56], see also [15]–[17] and references therein.

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