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Some Completely Monotonic Properties for the (p, q)-Gamma Function

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It is defined a $\Gamma_{p,q}$-function, as a generalization of the Γ-function. Also, we define a $\psi_{p,q}$-analogue of the psi-function as the log derivative of $\Gamma_{p,q}$. For the $\Gamma_{p,q}$-function, we give some properties related to convexity, log-convexity and completely monotonic functions. Also, some properties of $\psi_{p,q}$ analog of the ψ-function are established. As an application, when $p \to \infty, q \to 1$, we obtain all results from [12] and [21].

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1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [3]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)}, \tag{1}$$

where $\gamma = 0.57721 \cdots$ denotes Euler's constant.

Euler gave another equivalent definition for the $\Gamma(x)$ (see [2],[19])

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdot \ldots \cdot (x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdot \ldots \cdot (1+\frac{x}{p})}, \quad x > 0, \tag{2}$$

where p is positive integer, and

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x). \tag{3}$$

The p-analogue of the psi function, as the logarithmic derivative of the Γ_p function (see [12]), is

$$\psi_p(x) = \frac{d}{dx}\ln \Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{4}$$

The following representations are valid:

$$\Gamma_p(x) = \int_0^p \Big(1-\frac{t}{p}\Big)^p t^{x-1}dt, \tag{5}$$

$$\psi_p(x) = \ln p - \int_0^\infty \frac{e^{-xt}(1-e^{-(p+1)t})}{1-e^{-t}}dt, \tag{6}$$

and

$$\psi_p^{(m)}(x) = (-1)^{m+1}\cdot \int_0^\infty \frac{t^m\cdot e^{-xt}}{1-e^{-t}}(1-e^{-pt})dt. \tag{7}$$

Jackson (see [8, 9, 10, 20]) defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0<q<1, \tag{8}$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}},\ q>1, \tag{9}$$

where $(a;q)_\infty = \prod_{j\geq 0}(1-aq^j)$.

The q-gamma function has the following integral representation

$$\Gamma_q(t) = \int_0^\infty x^{t-1}E_q^{-qx}d_qx,$$

where $E_q^x = \sum_{j=0}^\infty q^{\frac{j(j-1)}{2}}\frac{x^j}{[j]!} = (1+(1-q)x)_q^\infty$, which is the q-analogue of the classical exponential function. The q-analogue of the psi function is defined for $0<q<1$ as the logarithmic derivative of the q-gamma function, that is,

$\psi_q(x) = \frac{d}{dx} \log \Gamma_q(x)$. Many properties of the q-gamma function were derived by Askey [4]. It is well known that $\Gamma_q(x) \to \Gamma(x)$ and $\psi_q(x) \to \psi(x)$ as $q \to 1^-$. From (8), for $0 < q < 1$ and $x > 0$ we get

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n\geq 0} \frac{q^{n+x}}{1-q^{n+x}} = -\log(1-q) + \log q \sum_{n\geq 1} \frac{q^{nx}}{1-q^n} \quad (10)$$

and from (9) for $q > 1$ and $x > 0$ we obtain

$$\begin{aligned} \psi_q(x) &= -\log(q-1) + \log q \left(x - \tfrac{1}{2} - \sum_{n\geq 0} \tfrac{q^{-n-x}}{1-q^{-n-x}} \right) \\ &= -\log(q-1) + \log q \left(x - \tfrac{1}{2} - \sum_{n\geq 1} \tfrac{q^{-nx}}{1-q^{-n}} \right). \end{aligned} \quad (11)$$

A Stieltjes integral representation for $\psi_q(x)$ with $0 < q < 1$ is given in [16]. It is well-known that $\psi^{'}$ is strictly completely monotonic on $(0, \infty)$, that is,

$$(-1)^n (\psi'(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0,$$

see [3, p. 260]. From (10) and (11) we conclude that ψ_q' has the same property for any $q > 0$,

$$(-1)^n (\psi_q'(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0.$$

If $q \in (0,1)$, using the second representation of $\psi_q(x)$ given in (10) can be shown that

$$\psi_q^{(k)}(x) = \log^{k+1} q \sum_{n\geq 1} \frac{n^k \cdot q^{nx}}{1-q^n} \quad (12)$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 1$, for all $k \geq 1$. If $q > 1$, from the second representation of $\psi_q(x)$ given in (11) we obtain

$$\psi_q'(x) = \log q \Big(1 + \sum_{n\geq 1} \frac{nq^{-nx}}{1-q^{-nx}} \Big) \quad (13)$$

and for $k \geq 2$,

$$\psi_q^{(k)}(x) = (-1)^{k-1} \log^{k+1} q \sum_{n\geq 1} \frac{n^k q^{-nx}}{1-q^{-nx}} \quad (14)$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 0$, for all $q > 1$.

Definition 1. For $x > 0, p \in N$ and for $q \in (0,1)$

$$\Gamma_{p,q}(x) = \frac{[p]_q^x [p]_q!}{[x]_q [x+1]_q \cdots [x+p]_q}, \tag{15}$$

where $[p]_q = \frac{1-q^p}{1-q}$.

It is easy to see that $\Gamma_{p,q}(x)$ fits into the following commutative diagrams:

$$\begin{array}{ccc}
\Gamma_{p,q}(x) & \xrightarrow{p\to\infty} & \Gamma_q(x) \\
\big\downarrow{\scriptstyle q\to 1} & & \big\downarrow{\scriptstyle q\to 1} \\
\Gamma_p(x) & \xrightarrow{p\to\infty} & \Gamma(x)
\end{array}$$

We define a (p,q)-analogue of the psi function as the logarithmic derivative of the p,q-gamma function, that is,

$$\psi_{p,q}(x) = \frac{d}{dx} \log \Gamma_{p,q}(x). \tag{16}$$

Definition 2. The function f is called log-convex if for all $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and for all $x, y > 0$ the following inequality holds

$$\log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y),$$

or equivalently

$$f(\alpha x + \beta y) \leq (f(x))^\alpha \cdot (f(y))^\beta.$$

Now, we will give some definitions about completely monotonic function:

A function f is said to be *completely monotonic* on an open interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, (x \in I, n = 0, 1, 2, \ldots). \tag{17}$$

If the inequality (17) is strict, then f is said to be *strictly completely monotonic* on I.

A positive function f is said to be *logarithmically completely monotonic* (see [18]) on an open interval I, if f satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, (x \in I, n = 1, 2, \ldots). \tag{18}$$

If the inequality (18) is strict, then f is said to be *strictly logarithmically completely monotonic.*

Let C and L denote the set of completely monotonic functions and the set of logarithmically completely monotonic functions, respectively. The relationship between completely monotonic functions and logarithmically completely monotonic functions can be presented (see [2]) by $L \subset C$.

The following theorem gives an integral characterization of the completely monotone functions.

Theorem 3. *(Hausdorff-Bernstein-Widder Theorem) A function $\varphi : [0, \infty) \to R$ is completely monotone on $[0, \infty)$ if and only if it is the Laplace transform of a finite non-negative Borel meassure μ on $[0, \infty)$, i.e., φ is of the form*

$$\varphi(r) = \int_0^\infty e^{-rt} d\mu(t) \tag{19}$$

- A non-negative finite linear combination of completely monotone functions is completely monotone.
- The product of two completely monotone functions is completely monotone.

2. Main results

Lemma 4. *For $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ we have:*

$$[1 + x]_q^\alpha [1 + y]_q^\beta \leq [1 + \alpha x + \beta y]_q \tag{20}$$

Proof. From Young's inequality

$$[x]_q^\alpha [y]_q^\beta \leq \alpha [x]_q + \beta [y]_q, \tag{21}$$

we have:

$$\begin{aligned}[1 + x]_q^\alpha [1 + y]_q^\beta &\leq \alpha [1 + x]_q + \beta [1 + y]_q \\ &= \alpha \Big(\frac{1 - q^{1+x}}{1 - q} \Big) + \beta \Big(\frac{1 - q^{1+y}}{1 - q} \Big) \\ &= \frac{1}{1 - q} \Big[1 - (\alpha q^{1+x} + \beta q^{1+y}) \Big].\end{aligned}$$

We have to prove:

$$\alpha q^{1+x} + \beta q^{1+y} \geq q^{1+\alpha x + \beta y}. \tag{22}$$

From Young's inequality we have:

$$q^{1+\alpha x+\beta y} = q((q^x)^\alpha (q^x)^\beta) \leq q(\alpha q^x + \beta q^y) = \alpha q^{1+x} + \beta q^{1+y}. \quad \blacksquare$$

Theorem 5. *The function*

$$\Gamma_{p,q}(x) = \frac{[p]_q^x [p]_q!}{[x]_q [x+1]_q \cdots [x+p]_q}$$

is log-convex.

Proof. We have to prove that for all $\alpha, \beta > 0, \alpha + \beta = 1, x, y > 0$

$$\log \Gamma_{p,q}(\alpha x + \beta y) \leq \alpha \log \Gamma_{p,q}(x) + \beta \log \Gamma_{p,q}(y) \tag{23}$$

which is equivalent to

$$\Gamma_{p,q}(\alpha x + \beta y) \leq (\Gamma_{p,q}(x))^{\alpha} \cdot (\Gamma_{p,q}(y))^{\beta}. \tag{24}$$

By Lemma 4 we obtain:

$$\left[1 + \frac{x}{k}\right]_q^{\alpha} \cdot \left[1 + \frac{y}{k}\right]_q^{\beta} \leq \alpha \left[1 + \frac{x}{k}\right]_q + \beta \left[1 + \frac{y}{k}\right]_q = \left[1 + \frac{\alpha x + \beta y}{k}\right]_q \tag{25}$$

for all $k \geq 1, k \in \mathbf{N}$.

Multiplying (25) for $k = 1, 2, \ldots, p$ one obtains

$$\left[1+\frac{x}{1}\right]_q^{\alpha} \cdots \left[1+\frac{x}{p}\right]_q^{\alpha} \cdot \left[1+\frac{y}{1}\right]_q^{\beta} \cdots \left[1+\frac{y}{p}\right]_q^{\beta} \leq \left[1+\frac{\alpha x + \beta y}{1}\right]_q \cdots \left[1+\frac{\alpha x + \beta y}{p}\right]_q.$$

Now, taking the reciprocal values and multiplying by $[p]_q^{\alpha x + \beta y}$ one obtains (24) and thus the proof is completed. ■

Lemma 6. *a) The function $\psi_{p,q}$ defined by* (16) *has the following series representation and integral representation:*

$$\psi_{p,q}(x) = -\ln[p]_q - \log q \sum_{k=0}^{p} \frac{q^{x+k}}{1 - q^{x+k}}, \tag{26}$$

$$\psi_{p,q}(x) = -\ln[p]_q - \int_0^{\infty} \frac{e^{-xt}}{1 - e^{-t}} (1 - e^{-(p+1)t}) d\gamma_q(t), \tag{27}$$

where $\gamma_q(t)$ is a discrete measure with positive masses $-\log q$ at the positive points $-k \log q, k = 1, 2, \ldots,$ i.e.

$$\gamma_q(t) = -\log q \sum_{k=1}^{\infty} \delta(t + k \log q), \quad 0 < q < 1. \tag{28}$$

b) The function $\psi_{p,q}$ is increasing on $(0, \infty)$.

c) The function $\psi'_{p,q}$ *is strictly completely monotonic on* $(0,\infty)$.

P r o o f. a) After logarithmical differentiation of (15) we take (26). Using

$$\int_0^\infty e^{-xt}d\gamma_q(t) = \frac{-q^x \log q}{1-q^x}, \quad 0<q<1$$

(see [7]), we take (27).

b) Let $0<x<y$. Using (26) we obtain

$$\begin{aligned}\psi_{p,q}(x) - \psi_{p,q}(y) &= \log q \sum_{k=0}^{p}\Big(\frac{1-q^{y+k}}{q^{y+k}} - \frac{1-q^{y+k}}{q^{y+k}}\Big)\\ &= \log q \sum_{k=0}^{p}\Big(\frac{q^x - q^{x+y+k} - q^y + q^{x+y+k}}{q^{x+y+k}}\Big)\\ &= \log q \sum_{k=0}^{p}\Big(\frac{q^x - q^y}{q^{x+y+k}}\Big) < 0.\end{aligned}$$

c) Deriving n times the relation (26) one finds that:

$$\psi_{p,q}^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})d\gamma_q(t). \tag{29}$$

hence $(-1)^n(\psi'_{p,q}(x))^{(n)} > 0$, for $x>0, n\geq 0$, ■

Remark 7. *The function* $\psi_{p,q}(x)$ *fits into the following commutative diagrams*

$$\begin{array}{ccc} \psi_{p,q}(x) & \xrightarrow{p\to\infty} & \psi_q(x) \\ \big\downarrow{\scriptstyle q\to 1} & & \big\downarrow{\scriptstyle q\to 1} \\ \psi_p(x) & \xrightarrow{p\to\infty} & \psi(x) \end{array}$$

3. Logarithmically completely monotonic function

Theorem 8. *The function* $G_{p,q}(x;a_1,b_1,\dots,a_n,b_n)$ *given by*

$$G_{p,q}(x) = G_{p,q}(x;a_1,b_1,\dots,a_n,b_n) = \prod_{i=1}^{n}\frac{\Gamma_{p,q}(x+a_i)}{\Gamma_{p,q}(x+b_i)}, \quad q\in(0,1) \tag{30}$$

is a completely monotonic function on $(0,\infty)$, *for any* a_i *and* b_i, $i=1,2,\dots,n$, *real numbers such that* $0<a_1\leq\cdots\leq a_n$, $0<b_1\leq b_2\leq\cdots\leq b_n$ *and* $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ *for* $k=1,2,\dots,n$.

Proof. Let $h(x) = \sum_{i=1}^{n}(\log\Gamma_{p,q}(x+b_i) - \log\Gamma_{p,q}(x+a_i))$. Then for $k \geq 0$ we have

$$\begin{aligned}
(-1)^k(h'(x))^{(k)} &= (-1)^k\sum_{i=1}^{n}(\psi_{p,q}^{(k)}(x+b_i) - \psi_{p,q}^{(k)}(x+a_i))\\
&= (-1)^k\sum_{i=1}^{n}(-1)^{k+1}\int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})(e^{-bi} - e^{-ai})d\gamma_q(t)\\
&= (-1)^{2k+1}\int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}}(1-e^{-(p+1)t})\sum_{i=1}^{n}(e^{-bi} - e^{-ai})d\gamma_q(t).
\end{aligned}$$

Alzer [1] showed that if f is a decreasing and convex function on R, then it holds

$$\sum_{i=1}^{n} f(b_i) \leq \sum_{i=1}^{n} f(a_i). \tag{31}$$

Thus, since the function $z \mapsto e^{-z}, z > 0$ is decreasing and convex on R, we have that $\sum_{i=1}^{n}(e^{-ai} - e^{-bi}) \geq 0$, so $(-1)^k(G'_{p,q}(x))^{(k)} \geq 0$ for $k \geq 0$. Hence h' is completely monotonic on $(0,\infty)$. Using the fact that if h' is completely monotonic function on $(0,\infty)$, then $\exp(-h)$ is also completely monotonic function on $(0,\infty)$ (see [5]), we get the desired result. ∎

Theorem 9. *The function*

$$f(x) = \frac{1}{[\Gamma_{p,q}(x+1)]^{\frac{1}{x}}} \tag{32}$$

is logarithmically completely monotonic in $(0,\infty)$.

Proof. Using the Leibnitz rule

$$[u(x)v(x)]^{(n)} = \sum_{k=0}^{n}\binom{n}{k}u^{(k)}(x)v^{(n-k)}(x),$$

we obtain

$$\begin{aligned}
[\ln f(x)]^{(n)} &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{x}\right)^{(k)}\Big(-\ln\Gamma_{p,q}(x+1)\Big)^{(n-k)}\\
&= -\frac{1}{x^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k k! x^{n-k}\psi_{p,q}^{(n-k-1)}(x+1)\\
&= -\frac{1}{x^{n+1}}g(x),
\end{aligned}$$

$$
\begin{aligned}
g^{'}(x) &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k k!(n-k) x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1)+ \\
&+ \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \psi_{p,q}^{(n-k)}(x+1) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k!(n-k) x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1)+ \\
&+ x^n \psi_{p,q}^{(n)}(x+1) + \sum_{k=0}^{n} \binom{n}{k} (-1)^k k! x^{n-k} \psi_{p,q}^{(n-k)}(x+1) \\
&= \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k k!(n-k) x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1)+ \\
&+ x^n \psi_{p,q}^{(n)}(x+1) + \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^{k+1} (k+1)! x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1) \\
&= \sum_{k=0}^{n-1} \left[\binom{n}{k}(n-k) - \binom{n}{k+1}(k+1) \right] (-1)^k k! x^{n-k-1} \psi_{p,q}^{(n-k-1)}(x+1) \\
&+ x^n \psi_{p}^{(n)}(x+1) = x^n \psi_{p,q}^{(n)}(x+1).
\end{aligned}
$$

If n is odd, then for $x > 0$,

$$
g^{'}(x) > 0 \Rightarrow g(x) > g(0) = 0 \Rightarrow (\ln f(x))^{(n)} < 0 \Rightarrow
$$

$$
\Rightarrow (-1)^n (\ln f(x))^{(n)} > 0.
$$

If n is even, then for $x > 0$,

$$
g^{'}(x) < 0 \Rightarrow g(x) < g(0) = 0 \Rightarrow (\ln f(x))^{(n)} > 0 \Rightarrow
$$

$$
\Rightarrow (-1)^n (\ln f(x))^{(n)} > 0.
$$

Hence,

$$
(-1)^n (\ln f(x))^{(n)} > 0
$$

for all real $x \in (0, \infty)$ and all integers $n \geq 1$. The proof is completed. ■

Remark 10. *Let p tend to ∞, then we obtain Theorem 1 from [6]. Let q tend to 1, then we obtain Theorem 2.1 from [14].*

Let s and t be two real numbers with $s \neq t, \alpha = \min\{s, t\}$ and $\beta \geq -\alpha$, for $x \in (-\alpha, \alpha)$, define

$$h_{\beta,p,q}(x) = \begin{cases} \left[\frac{\Gamma_{p,q}(\beta+t)}{\Gamma_{p,q}(\beta+s)} \cdot \frac{\Gamma_{p,q}(x+s)}{\Gamma_{p,q}(x+t)}\right]^{\frac{1}{x-\beta}} & x \neq \beta, \\ \exp[\psi_{p,q}(\beta+s) - \psi_{p,q}(\beta+t)] & x = \beta. \end{cases}$$

The following theorem is a generalization of a result of [15].

Theorem 11. *The function $h_{\beta,p,q}(x)$ is logarithmically completely monotonic on $(-\alpha, +\infty)$ if $s > t$.*

Proof. For $x \neq \beta$, taking logarithm of the function $h_{\beta,p}(x)$ gives

$$\begin{aligned} \ln h_{\beta,p,q}(x) &= \frac{1}{x-\beta}\Big[\ln\frac{\Gamma_{p,q}(\beta+t)}{\Gamma_{p,q}(\beta+s)} - \ln\frac{\Gamma_{p,q}(x+s)}{\Gamma_{p,q}(x+t)}\Big] \\ &= \frac{\ln\Gamma_{p,q}(x+s) - \ln\Gamma_{p,q}(\beta+s)}{x-\beta} - \frac{\ln\Gamma_{p,q}(x+t) - \ln\Gamma_{p,q}(\beta+t)}{x-\beta} \\ &= \frac{1}{x-\beta}\int_\beta^x \psi_{p,q}(u+s)du - \frac{1}{x-\beta}\int_\beta^x \psi_{p,q}(u+t)du \\ &= \frac{1}{x-\beta}\int_\beta^x [\psi_{p,q}(u+s) - \psi_{p,q}(u+t)]du \\ &= \frac{1}{x-\beta}\int_\beta^x\int_t^s \psi'_{p,q}(u+v)dvdu \\ &= \frac{1}{x-\beta}\int_\beta^x \varphi_{p,q,s,t}(u)du = \int_0^1 \varphi_{p,q,s,t}((x-\beta)u+\beta)du, \end{aligned}$$

and by differentiating $\ln h_{\beta,p,q}(x)$ with respect to x,

$$[\ln h_{\beta,p,q}(x)]^{(k)} = \int_0^1 u^k \varphi^{(k)}_{p,q,s,t}((x-\beta)u+\beta)du. \tag{33}$$

If $x = \beta$, formula (33) is valid. Since functions $\psi'_{p,q}$ and $\varphi_{p,q,s,t}$ are completely monotonic in $(0,\infty)$ and $(-t,\infty)$ respectively, then $(-1)^i[\varphi_{p,q,s,t}(x)]^{(i)} \geq 0$ holds for $n \in (-t,\infty)$ for any nonnegative integer i. Thus

$$(-1)^k[\ln h_{\beta,p,q}(x)]^{(k)} = \int_0^1 u^k(-1)^k\varphi^{(k)}_{p,q,s,t}((x-\beta)u+\beta)du \geq 0$$

in $(-t,\infty)$ for $k \in N$. The proof is completed. ∎

4. Application of the $\Gamma_{p,q}(x)$ function

In the following, we give some results for the $\Gamma_{p,q}$ analogous to these from [21]. Since the proofs are almost similar, we omit them.

Lemma 13. *Let a, b, c, d, e be real numbers such that $a+bx > 0$, $d+ex > 0$ and $a+bx \le d+ex$. Then*

$$\psi_{p,q}(a+bx) - \psi_{p,q}(d+ex) \le 0. \tag{34}$$

Lemma 14. *Let a, b, c, d, e, f be real numbers such that $a+bx > 0, d+ex > 0, a+bx \le d+ex$ and $ef \ge bc > 0$. If*
(i) $\psi_{p,q}(a+bx) > 0$, or
(ii) $\psi_{p,q}(d+ex) > 0$,
then

$$bc\psi_{p,q}(a+bx) - ef\psi_{p,q}(d+ex) \le 0. \tag{35}$$

Lemma 15. *Let a, b, c, d, e, f be real numbers such that $a+bx > 0, d+ex > 0, a+bx \le d+ex$ and $bc \ge ef > 0$. If*
(i) $\psi_{p,q}(d+ex) < 0$, or
(ii) $\psi_{p,q}(a+bx) < 0$,
then

$$bc\psi_{p,q}(a+bx) - ef\psi_{p,q}(d+ex) \le 0. \tag{36}$$

Theorem 16. *Let f_1 be a function defined by*

$$f_1(x) = \frac{\Gamma_{p,q}(a+bx)^c}{\Gamma_{p,q}(d+ex)^f}, \quad x \ge 0, \tag{37}$$

where a, b, c, d, e, f are real numbers such that: $a+bx > 0, d+ex > 0, a+bx \le d+ex, ef \ge bc > 0$. If $\psi_{p,q}(a+bx) > 0$ or $\psi_{p,q}(d+ex) > 0$, then the function f_1 is decreasing for $x \ge 0$ and for $x \in [0,1]$ the following double inequality holds:

$$\frac{\Gamma_{p,q}(a+b)^c}{\Gamma_{p,q}(d+e)^f} \le \frac{\Gamma_{p,q}(a+bx)^c}{\Gamma_{p,q}(d+ex)^f} \le \frac{\Gamma_{p,q}(a)^c}{\Gamma_{p,q}(d)^f}. \tag{38}$$

In a similar way, using Lemma 15, it is easy to prove the following theorem.

Theorem 17. *Let f_1 be a function defined by*

$$f_1(x) = \frac{\Gamma_{p,q}(a+bx)^c}{\Gamma_{p,q}(d+ex)^f}, \quad x \ge 0, \tag{39}$$

where a, b, c, d, e, f *are real numbers such that:* $a + bx > 0, d + ex > 0, a + bx \leq d + ex, bc \geq ef > 0$. *If* $\psi_{p,q}(d + ex) < 0$ *or* $\psi_{p,q}(a + bx) < 0$*, then the function* f_1 *is decreasing for* $x \geq 0$ *and for* $x \in [0, 1]$ *the inequality* (38) *holds.*

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