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Conflict-Controlled Processes Involving Fractional Differential Equations with Impulses

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Here we investigate a problem of approaching terminal (target) set by a system of impulse differential equations of fractional order in the sense of Caputo. The system is under control of two players pursuing opposite goals. The first player tries to bring the trajectory of the system to the terminal set in the shortest time, whereas the second player tries to maximally put off the instant when the trajectory hits the set, or even avoid this meeting at all. We derive analytical solution to the initial value problem for a fractional-order system involving impulse effects. As the main tool for investigation serves the Method of Resolving Functions based on the technique of inverse Minkowski functionals. By constructing and investigating special set-valued mappings and their selections, we obtain sufficient conditions for the game termination in a finite time. In so doing, we substantially apply the technique of $\mathcal{L} \times \mathcal{B}$-measurable set-valued mappings and their selections to ensure, as a result, superpositional measurability of the first player's controls.

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1. Introduction

There exists a number of definitions of the fractional order derivative. The classical one is the definition by Riemann-Liouville [1]. The Riemann–Liouville fractional derivatives have a singularity at zero. That is why differential equations involving these derivatives require initial conditions of special form lacking clear physical interpretation. These shortcomings do not occur with the regularized fractional derivative in the sense of Caputo. The fundamental theory of fractional differential equations is developed in the monographs [2, 3]. The papers [4, 5, 6] are devoted to solving fractional differential and differintegral equations including those with partial derivatives.

The impulse (integer-order) differential equations have become important in recent years as mathematical models of processes where some parameters can change instantly in a jump-like manner. The monographs [7, 8] are devoted to the impulse differential equations and related issues, i.e. stability, control etc.

Fractional differential equations with impulses were first addressed in the paper [9], where sufficient existence and uniqueness conditions for solutions of a class of impulse differential equations involving the Caputo fractional derivative were derived.

Here we investigate a problem of approaching terminal (target) set by a system of impulse differential equations of fractional order in the sense of Caputo. As the main tool for investigation serves the Method of Resolving Functions based on the technique of inverse Minkowski functionals [10]. This method provides full substantiation of the Method of Parallel Pursuit well-known from practice. The Method of Resolving Functions is closely related to the Pontryagin First Direct Method (see [11]).

2. Linear systems of fractional differential equations in the sense of Caputo

Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. Let $t_0 \geq 0$ and $f : [t_0, \infty) \to \mathbb{R}^n$ be an absolutely continuous function. The Caputo fractional derivative [2] of order α, $0 < \alpha < 1$, is defined as

$$D_{t_0}^{(\alpha)} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau.$$

Consider a dynamic system of fractional order in the sense of Caputo described by the equation:

$$D_{t_0}^{(\alpha)} z = Az + g, \quad 0 < \alpha < 1, \tag{1}$$

under initial conditions

$$z(t_0) = z_0. \tag{2}$$

Here $z \in \mathbb{R}^n$, A is a $n \times n$-matrix, and $g : [t_0, \infty) \to \mathbb{R}^n$ is a measurable function bounded almost everywhere.

Lemma 1. *The trajectory of the system* (1), (2) *has the form:*

$$z(t) = E_{\alpha}(A(t-t_0)^{\alpha}) z_0 + \int_{t_0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha}) g(\tau) d\tau, \tag{3}$$

where

$$E_\rho(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k\rho+1)},\ E_{\rho,\mu}(B) = \sum_{k=0}^{\infty} \frac{B^k}{\Gamma(k\rho+\mu)}$$

are the classical and generalized Mittag–Leffler matrix functions, respectively.

Let us note that for $t_0 = 0$ the result formulated in Lemma 1 was first obtained in [12]. In [13], it is extended to the case of arbitrary α using Laplace transformation method.

3. Linear systems of impulse fractional differential equations

Let $\{\tau_k\}_0^\infty$ be a sequence such that $\tau_0 = t_0$ and $\lim\limits_{k\to\infty} \tau_k = \infty$.

Now let us consider an impulsive system governed by

$$D_{\tau_k}^{(\alpha)} z = Az + g,\ t \in (\tau_k, \tau_{k+1}],\ k = 0, 1, 2, \dots \tag{4}$$

Assume that the initial conditions (2) are fulfilled and at the time moments τ_k, $k = 1, 2, 3, \dots$ the following equalities hold true:

$$\Delta z\big|_{t=\tau_k} = B_k z + a_k, \tag{5}$$

where B_k are square matrices of order n, $a_k \in \mathbb{R}^n$, and $\Delta z\big|_{t=\tau_k} = z(\tau_k^+) - z(\tau_k)$.

Suppose $t \in (\tau_i, \tau_{i+1}]$, let us introduce matrix function $Z_j(t)$:

$$Z_j(t) = \begin{cases} E_\alpha(A(t-\tau_i)^\alpha), & j = i, \\ E_\alpha(A(t-\tau_i)^\alpha) \prod\limits_{k=i-1}^{j} (B_{k+1}+I) E_\alpha(A(\tau_{k+1}-\tau_k)^\alpha), & j < i. \end{cases} \tag{6}$$

We will also assume that $B_0 = 0$.

It can be readily seen that $Z_0(t)$ is a solution to the following homogeneous matrix initial value problem:

$$\begin{aligned} D_{\tau_k}^{(\alpha)} Z &= AZ,\ t \in (\tau_k, \tau_{k+1}], \\ \Delta Z\big|_{t=\tau_k} &= B_k Z, \\ Z(t_0) &= I. \end{aligned}$$

Theorem 1. *The trajectory of the impulsive system governed by (4) on the time intervals* $(\tau_k, \tau_{k+1}]$*, and satisfying (5) at* $t = \tau_k$ *is given by the formula:*

$$z(t) = Z_0(t)z_0 + \sum_{j=1}^{k} Z_j(t)a_j$$

$$+\sum_{j=1}^{k} \int_{\tau_{j-1}}^{\tau_j} (\tau_j - \tau)^{\alpha-1} Z_j(t)(B_j + I)E_{\alpha,\alpha}(A(\tau_j - \tau)^{\alpha})g(\tau)d\tau$$

$$+\int_{\tau_k}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha})g(\tau)d\tau, \ t \in (\tau_k, \tau_{k+1}].$$

P r o o f. It can be readily seen that $Z_j(t)$ satisfies the following recurrence relation:

$$\begin{aligned} Z_j(t) =& E_\alpha(A(t-\tau_i)^\alpha)(B_i + I)Z_j(\tau_i) && \text{if } i > j, \\ Z_j(t) =& E_\alpha(A(t-\tau_i)^\alpha) && \text{if } i = j, \end{aligned} \tag{7}$$

where $t \in (\tau_i, \tau_{i+1}]$. Using this recurrence relation and the equation (3) one can easily prove the theorem with the aid of the method of mathematical induction.

■

4. Method of resolving functions

Consider the following dynamic game. Let a dynamic system be described by the equations

$$\begin{aligned} D^{(\alpha)}_{\tau_k} z =& Az + u - v, \ t \in (\tau_k, \tau_{k+1}], \ k = 0, 1, 2, \dots, \\ \Delta z\big|_{t=\tau_k} =& B_k z + a_k, \end{aligned} \tag{8}$$

where the controls of the players $u(\tau)$, $u : \mathbb{R}_+ \to U$, and $v(\tau)$, $v : \mathbb{R}_+ \to V$, are measurable functions of time taking their values from the nonempty compact sets U and V, respectively.

In addition to the system (8), consider a terminal cylindrical set M^* of the form

$$M^* = M_0 + M, \tag{9}$$

where M_0 is a linear subspace in $\mathbb{R}^n$ and M is a nonempty compact set from the orthogonal complement L of M_0 in $\mathbb{R}^n$.

The goals of the first (u) and second (v) players are opposite. The first player tries to bring the trajectory of the system (8) to the terminal set in the shortest time, whereas the second player tries to maximally delay the instant when the trajectory reaches the set M^*, or even avoid this meeting at all.

Let us take the side of the first player and assume that the opponent chooses an arbitrary V-valued measurable function as a control. We also assume that the first player decides on its control at time t depending on the information about the initial position z_0 and $v(t)$:

$$u(t) = u(z_0, v(t)), \quad u(t) \in U. \tag{10}$$

Let us note that if some admissible controls $u(\tau)$, $v(\tau)$, are chosen by the players, then the solution to the latter initial value problem is given by

$$z(t) = Z_0(t)z_0 + \sum_{j=1}^{k} Z_j(t)a_j + \int_{t_0}^{t} \Phi(t,\tau)[u(\tau) - v(\tau)]d\tau, \ t \in (\tau_k, \tau_{k+1}], \tag{11}$$

where

$$\Phi(t,\tau) = \begin{cases} (\tau_j - \tau)^{\alpha-1} Z_j(t) E_{\alpha,\alpha}(A(\tau_j - \tau)^{\alpha})(B_j + I), \text{ if } \tau \in [\tau_{j-1}, \tau_j), \ j \le k, \\ (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-\tau)^{\alpha}), \text{ if } \tau \in [\tau_k, t). \end{cases}$$

Denote by π the orthogonal projection from $\mathbb{R}^n$ to L. Let $t \in (\tau_k, \tau_{k+1}]$, $\tau \in [\tau_{j-1}, \tau_j)$, consider the multivalued mappings

$$W(t,\tau,v) = \pi\Phi(t,\tau)(U - v),$$
$$W(t,\tau) = \bigcap_{v\in V} W(t,\tau,v) = \pi\Phi(t,\tau)U \overset{*}{-} \pi\Phi(t,\tau)V.$$

Pontryagin's Condition. *The multivalued mapping $W(t,\tau)$ takes on nonempty values for all $t_0 \le \tau < t$.*

Taking into account the properties of the Mittag-Leffler and $Z_j(t)$ matrix functions, we can conclude that for any fixed $t > t_0$ the mapping $W(t,\tau,v)$ is measurable in τ on the interval $[t_0, t]$ and closed in v, $v \in V$. Then [14] the mapping $W(t,\tau)$ is a closed-valued mapping measurable in $\tau \in [t_0, t]$. The Pontryagin condition and the measurable selection theorem [14] imply that for any $t \ge 0$ there exists at least one selection $\gamma(t,\tau)$ measurable in τ such that $\gamma(t,\tau) \in W(t,\tau)$, $t_0 \le \tau < t$. By the assumptions, $\gamma(t,\tau)$ is integrable in τ, $\tau \in [t_0, t]$, for any $t > t_0$. Let us fix some selection $\gamma(t,\tau)$ and introduce a function

$$\xi(t) = \pi Z_0(t)z_0 + \sum_{j=1}^{k} \pi Z_j(t)a_j + \int_{t_0}^{t} \gamma(t,\tau)d\tau. \tag{12}$$

Consider the multivalued mapping

$$\mathfrak{R}(t,\tau,v) = \left\{\rho \ge 0 : [W(t,\tau,v) - \gamma(t,\tau)] \bigcap \rho\,[M - \xi(t)] \ne \emptyset\right\},$$

and its support function in the direction +1: $\rho(t,\tau,v) = \sup\{\rho : \rho \in \mathfrak{R}(t,\tau,v)\}$, $t_0 \le \tau \le t$, $v \in V$. This function is called a resolving function [10]. Due to Pontryagin's condition, the multivalued mapping $\mathfrak{R}(t,\tau,v)$ has a nonempty closed image in its domain. Note also that if $\xi(t) \in M$, then $\mathfrak{R}(t,\tau,v) = [0,\infty)$ and, hence, $\rho(t,\tau,v) = \infty$ for all $t_0 \le \tau < t$ and $v \in V$.

Taking into account the properties of the parameters of the conflict-controlled process (8), (9) and applying the characterization and inverse image theorems [14], we can show that the multivalued mapping $\mathfrak{R}(t,\tau,v)$ is jointly $\mathcal{L} \times \mathcal{B}$-measurable [15] with respect to the variables τ, v, $\tau \in [t_0,t]$, $v \in V$; the resolving function $\rho(t,\tau,v)$ is jointly $\mathcal{L} \times \mathcal{B}$-measurable in the variables τ, v by the theorem on the support function ([14]) for $\xi(t) \notin M$.

Denote

$$\mathfrak{T} = \left\{ t \ge t_0 : \int_{t_0}^{t} \inf_{v \in V} \rho(t,\tau,v)d\tau \ge 1 \right\}. \tag{13}$$

If for some $t > 0$ $\xi(t) \notin M$, we assume the function $\inf_{v \in V} \rho(t,\tau,v)$ to be measurable with respect to τ, $\tau \in [t_0,t]$. If it is not the case, then let us define the set $\mathfrak{T}$ as follows

$$\mathfrak{T} = \left\{ t \ge t_0 : \inf_{v(\cdot) \in \Omega_V} \int_{t_0}^{t} \rho(t,\tau,v(\tau))d\tau \ge 1 \right\},$$

where Ω_V is the set of all measurable functions taking values in V.

Since the function $\rho(t,\tau,v)$ is $\mathcal{L} \times \mathcal{B}$-measurable with respect to τ, v, it is superpositionally measurable [14]. If $\xi(t) \in M$, then $\rho(t,\tau,v) = +\infty$ for $\tau \in [t_0,t]$ and in this case it is natural to set the value of the integral in (13) to be equal $+\infty$. Then the inequality in (13) is fulfilled by default. In the case when the inequality in braces in (13) fails for all $t > t_0$, we set $\mathfrak{T} = \emptyset$. Let $T \in \mathfrak{T} \ne \emptyset$.

Condition 1. *The set $\mathfrak{R}(T,\tau,v)$ is convex-valued for all $\tau \in [t_0,T]$, $v \in V$.*

Theorem 1. *Let for the game problem (8), (9) Pontryagin's Condition hold true and the set M be convex. If there exists a finite number T, $T \in \mathfrak{T} \ne \emptyset$, such that Condition 1 is fulfilled, then the trajectory of the process (8) can be brought to the set (9) from the initial position z_0 at the time instant T using the control of the form (10).*

P r o o f. Let $v(\tau)$, $v : [t_0,T] \to V$, be an arbitrary measurable function. We first consider the case when $\xi(T) \notin M$. Denote $K = \max\{k \in \mathbb{N} : \tau_k < T\}$, $\rho(T) = \int_0^T \inf_{v \in V} \rho(T,\tau,v)d\tau$, and set

$$\rho^*(T,\tau) = \frac{1}{\rho(T)} \inf_{v \in V} \rho(T,\tau,v).$$

Since $\rho(T) \geq 1$ due to (13) and Condition 1 is fulfilled, the function $\rho^*(T, \tau)$, $0 \leq \rho^*(T, \tau) \leq \rho(T, \tau, v)$, $\tau \in [t_0, T]$, $v \in V$, is a measurable selection for each of the set-valued mappings $\mathfrak{R}(T, \tau, v)$, $v \in V$, i.e. $\rho^*(T, \tau) \in \mathfrak{R}(T, \tau, v)$, $\tau \in [t_0, T]$, $v \in V$.

Consider the multivalued mapping

$$U(\tau, v) = \{u \in U : \ \pi\Phi(T, \tau)(u - v) - \gamma(T, \tau) \in \rho^*(T, \tau)[M - \xi(T)]\}. \qquad (14)$$

Since the function $\rho^*(T, \tau)$ is measurable due to the assumptions made, $M \in K(\mathbb{R}^n)$, and the vector $\xi(T)$ is bounded, it follows that the mapping $\rho^*(T, \tau)[M - \xi(T)]$ is measurable with respect to τ. Moreover, the left-hand side of the inclusion in (14) is $\mathcal{L} \times \mathcal{B}$-measurable with respect to (τ, v) and continuous in u. This implies that the mapping $U(\tau, v)$ is $\mathcal{L} \times \mathcal{B}$-measurable. Thus, according the theorem on measurable selection it contains an $\mathcal{L} \times \mathcal{B}$-measurable selection $u(\tau, v)$, which, in its turn, is a superpositionally measurable function. Let us set the first player's control to be $u(\tau) = u(\tau, v(\tau))$, $\tau \in [t_0, T]$.

In the case when $\xi(T) \in M(T)$ we construct the first player's control as follows. Let us set $\rho^*(T, \tau) \equiv 0$ in (14) and denote by $U_0(\tau, v)$ the set-valued mapping thus obtained from $U(\tau, v)$. Let us choose the first player's control in the form $u_0(\tau) = u_0(\tau, v(\tau))$, $\tau \in [0, T]$, where $u_0(\tau, v)$ is a measurable selection of the mapping $U_0(\tau, v)$.

Let us show that in each case treated above the trajectory of the process (8) hits the terminal set at the time instant T.

According to (11), we have

$$\pi z(T) = \pi Z_0(T) z_0 + \sum_{j=1}^{K} \pi Z_j(T) a_j + \int_{t_0}^{T} \pi\Phi(T, \tau)[u(\tau) - v(\tau)] d\tau. \qquad (15)$$

Consider the case $\xi(T) \notin M$. Let us add and subtract from the right-hand side of (15) the vectors

$$\int_{t_0}^{T} \gamma(T, \tau) d\tau. \qquad (16)$$

Taking into account the control rule of the first player, we obtain from (15) the following inclusion

$$\pi z(T) \in \xi(T) \left[1 - \int_{t_0}^{T} \rho^*(T, \tau) d\tau\right] + \int_{t_0}^{T} \rho^*(T, \tau) M d\tau.$$

As M is a convex compact set, and $\rho^*(T, \tau)$ is a non-negative function and $\int_{t_0}^{T} \rho^*(T, \tau) d\tau = 1$, it follows that $\int_{t_0}^{T} \rho^*(T, \tau) M d\tau = M$; hence $\pi z(T) \in M$.

Now assume $\xi(T) \in M$. Adding and subtracting the vectors (16) from the right-hand side of (15) and taking into account the first player's control rule we obtain $\pi z(T) = \xi(T) \in M$. ■

Example 1.

Consider a dynamic system whose evolution is described by the equations

$$\begin{aligned} D^{(\alpha)}_{\tau_k} z &= u - v, \ t \in (\tau_k, \tau_{k+1}], \\ \Delta z\big|_{t=\tau_k} &= a_k, \end{aligned} \tag{17}$$

where $u \in aS$, $v \in S$, $S = \{x \in \mathbb{R}^n : \ \|x\| \leq 1\}$, $a > 1$, $t_0 = 0$, and $\tau_k = kh$, $h > 0$.

Suppose that the terminal set is $M^* = M = \{0\}$.

Then $\pi = I$, $Z_j(t) \equiv I$, and solution to (17) under initial condition (2) is given by

$$z(t) = z_0 + \sum_{j=1}^{k} a_j + \int_{t_0}^{t} \Phi(t,\tau)[u(\tau) - v(\tau)]d\tau, \quad t \in (\tau_k, \tau_{k+1}],$$

where

$$\Phi(t,\tau) = \phi(t,\tau)I = \begin{cases} \frac{1}{(\tau_j - \tau)^{1-\alpha}\Gamma(\alpha)} I, & \text{if } \tau \in [\tau_{j-1}, \tau_j), \ j \leq k, \\ \frac{1}{(t-\tau)^{1-\alpha}\Gamma(\alpha)} I, & \text{if } \tau \in [\tau_k, t). \end{cases}$$

Hence, the multivalued mappings $W(t,\tau,v)$ and $W(t,\tau)$ are of the form

$$W(t,\tau,v) = \phi(t,\tau)(aS - v),$$

$$W(t,\tau) = \phi(t,\tau)aS \overset{*}{-} \phi(t,\tau)S = \phi(t,\tau)(a-1)S \neq \emptyset.$$

Thus, Pontryagin's Condition holds true. Let us set $\gamma(t,\tau) \equiv 0$. Then

$$\xi(t) = z_0 + \sum_{j=1}^{k} a_j,$$

and the resolving function is given by

$$\rho(t,\tau,v) = \sup\left\{\rho \geq 0 : \phi(t,\tau)v - \rho\left(z_0 + \sum_{j=1}^{k} a_j\right) \in \phi(t,\tau)aS\right\}.$$

Denote $\xi = \xi(t) = z_0 + \sum_{j=1}^{k} a_j$, then the resolving function can be found analytically as the greatest positive root of the quadratic equation with respect to ρ:

$$\|\phi(t,\tau)v - \rho\xi\| = \phi(t,\tau)a.$$

This yields that the resolving function is given by

$$\rho(t,\tau,v) = \frac{\phi(t,\tau)v\cdot\xi + \sqrt{(\phi(t,\tau)v\cdot\xi)^2 + \|\xi\|^2(\phi(t,\tau)^2a^2 - \phi(t,\tau)^2\|v\|^2)}}{\|\xi\|^2}.$$

Then $\min_{\|v\|\le 1}\rho(t,\tau,v) = \frac{a-1}{\|\xi\|}\phi(t,\tau)$ and the minimum is attained at $v = -\frac{\xi}{\|\xi\|}$. Since

$$\int_{t_0}^{t}\min_{\|v\|\le 1}\rho(t,\tau,v)d\tau = \frac{a-1}{\|\xi\|}\left(\frac{kh^\alpha}{\Gamma(\alpha+1)} + \frac{(t-kh)^\alpha}{\Gamma(\alpha+1)}\right),\ t\in(\tau_k,\tau_{k+1}],$$

it follows that the time T of the game termination can be estimated as

$$\left\lfloor\frac{\|\xi\|\Gamma(\alpha+1)}{(a-1)h^\alpha}\right\rfloor h \le T < \left\lfloor\frac{\|\xi\|\Gamma(\alpha+1)}{(a-1)h^\alpha}\right\rfloor h + h,$$

where $\lfloor\cdot\rfloor$ stands for the floor function.

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