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GEOMETRY OF WARPED PRODUCT SEMI-INVARIANT SUBMANIFOLDS OF A LOCALLY RIEMANNIAN PRODUCT MANIFOLD*

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Abstract. In this article, we have studied warped product semi-invariant submanifolds in a locally Riemannian product manifold and introduced the notions of a warped product semi-invariant submanifold. We have also proved several fundamental properties of a warped product semi-invariant in a locally Riemannian product manifold.

1. Introduction. It is well-known that the notion of warped products plays some important role in differential geometry as well as in physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [3, 4, 5, 6].

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The geometry warped product CR-submanifolds in complex manifolds was introduced in [3, 4]. It was proved in [3] that there exist no a proper CR-warped product in the form $N = N_{\perp} \times_f N_T$ in any Kaehler manifold M, where $N_{\perp}$ is a totally real submanifold and N_T is a holomorphic submanifold of M. On the other hand, in this article we have proved that there exist no a proper warped product semi-invariant submanifold in the form $N = N_T \times_f N_{\perp}$ in any locally Riemannian product manifold M, where N_T is an invariant submanifold and $N_{\perp}$ is an anti-invariant submanifold of M.

Let M be an m-dimensional manifold with a tensor of type (1,1) such that $F \neq I$, $F^2 = I$, then M is said to be an almost product manifold with almost product structure F. If an almost product manifold M has a Riemannian metric g such that $g(FX, Y) = g(X, FY)$, for any $X, Y \in \Gamma(TM)$, then M is called an almost Riemannian product manifold, where $\Gamma(TM)$ means the set of all differentiable vector fields on M. We denote the Levi-Civita connection on M by $\bar{\nabla}$ with respect to g. If $(\bar{\nabla}_X F)Y = 0$, for any $X, Y \in \Gamma(TM)$, then M is called a locally Riemannian product manifold[2].

Let M be a Riemannian manifold with almost Riemannian product structure F and let N be a Riemannian manifold isometrically immersed in M. For each $x \in N$, we denote by D_x the maximal invariant subspace of the tangent space T_xN of N. If the diemnsion of D_x is the same for all x in N, then D_x gives an invariant distribution D on N.

A submanifold N in a locally Riemannian product manifold is called semi-invariant submanifold if there exists on N a differentiable invariant distribution D whose orthogonal complementary $D^{\perp}$ is an anti-invariant distribution, i.e., $F(D^{\perp}) \subset TN^{\perp}$. A semi-invariant submanifold is called an anti-invariant(resp. an invariant) submanifold if $\dim(D_x) = 0$(resp. $\dim(D_x^{\perp}) = 0$). It is called a proper semi-invariant submanifold if it is a neither invariant nor an anti-invariant.

A semi-invariant submanifold N of a locally Riemannian product manifold M is called a semi-Riemannian product of an invariant submanifold N_T and an anti-invariant submanifold $N_{\perp}$ of M are totally geodesic submanifolds in N. The notion of semi-invariant in a locally Riemannian product manifolds was introduced in [2, 9].

Let N_1 and N_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and f is a differentiable and positive definite function on N_1. Consider the product manifold $N_1 \times N_2$ with its projection $\pi : N_1 \times N_2 \longrightarrow N_1$ and $\eta : N_1 \times N_2 \longrightarrow N_2$. The warped product manifold $N = N_1 \times_f N_2$ is the manifold $N_1 \times N_2$ equipped with the Riemannian metric structure such that

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + f^2(\pi(x)) g_2(\eta_* X, \eta_* Y)$$

for any $X, Y \in \Gamma(TN)$, where $*$ the symbol stand for the differential. Thus we have $g = g_1 + f^2 g_2$, where f is called the warping function of the warped product. The warped product manifold $N = N_1 \times_f N_2$ is characterized by N_1 and N_2 are totally geodesic and totally umbilical submanifolds of N, respectively[8].

In this paper, we defined and studied a new class of semi-invariant submanifolds, called warped product semi-invariant submanifold, in a Locally Riemannian product manifold. Firstly, we prove that if $N = N_T \times_f N_\perp$ is a warped product semi-invariant submanifold of locally Riemannian product manifold M such that N_T is an invariant and $N_\perp$ is an anti-invariant submanifold of M, then N is a Riemannian product. By contrast, we show that there exist many warped product semi-invariant submanifolds in the form $N = N_\perp \times_f N_T$ in a locally Riemannian product manifold which are not Riemannian product by reversing the two factor manifolds N_T and $N_\perp$ and it called warped product semi-invariant submanifold. So we have investigated the class of warped product semi-invariant submanifold and we establish the fundamental theory of such submanifolds.

2. Preliminaries. If N is an isometrically immersed submanifold in a Riemannian manifold M, then the formulas of Gauss and Weingarten for N in M are given, respectively, by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{2}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^\perp)$, where $\bar{\nabla}$ and ∇ denote the Riemannian connections on M and N, respectively, h is the second fundamental form, $\nabla^\perp$ is the normal connection on normal bundle and A is the shape operator of N in M. The second fundamental form and the shape operator are related by

$$g(A_V X, Y) = g((h(X, Y), V), \tag{3}$$

where, g denotes the Riemannian metric on M as well as N. For any a submanifold N of a Riemannian manifold M, the equation of Gauss is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\bar{\nabla}_X h)(Y, Z) \\ &- (\bar{\nabla}_Y h)(X, Z), \end{aligned} \tag{4}$$

for any $X, Y, Z \in \Gamma(TN)$, where $\bar{R}$ and R denote the Riemannian curvature tensors of M and N, respectively. The covariant derivative of h is defined by

$$(5) \qquad (\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y).$$

The equation of Codazzi is also given by

$$(6) \qquad (\bar{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),$$

where $(\bar{R}(X,Y)Z)^{\perp}$ denotes the normal component of $\bar{R}(X,Y)Z$.

If $(\bar{R}(X,Y)Z)^{\perp} = 0$, then N is said to be curvature-invariant submanifold of M.

We recall the following general lemma from [8] for later use.

Lemma 2.1. *Let $N = N_1 \times_f N_2$ be a warped product manifold with warping function f, then we have*

1.) $\nabla_X Y \in \Gamma(TN_1)$ *for each* $X, Y \in \Gamma(TN_1)$

2.) $\nabla_X Z = \nabla_Z X = X(\ln f)Z$, *for each* $X \in \Gamma(TN_1)$, $Z \in \Gamma(TN_2)$

3.) $\nabla_Z W = \nabla^{N_2}_Z W - g(Z,W)\dfrac{\text{grad} f}{f}$, *for each* $Z, W \in \Gamma(TN_2)$,

where ∇ and ∇^{N_2} denote the Levi-Civita connections on N and N_2, respectively.

In this section, we study semi-invariant submanifolds in a locally Riemannian product manifold M which are warped products of the form $N_1 \times_f N_2$. Here firstly, we suppose that N_1 is an invariant and N_2 is anti-invariant, after then, N_1 is an anti-invariant submanifold and N_2 is an invariant submanifold of M with respect to F. Now, we denote the orthogonal complementary of $F(T(N))$ in $TN^{\perp}$ by V, then we have direct sum

$$(7) \qquad TN^{\perp} = F(TN) \oplus V.$$

We can easily see that V is an invariant distribution with respect to F.

Now, let N be any submanifold of a locally Riemannian product manifold M. Then for any $X \in \Gamma(TN)$, FX can be written the following way:

$$(8) \qquad FX = tX + nX,$$

where tX and nX denote the tangential and normal components of FX, respectively. Similarly, for any $V \in \Gamma(TN^{\perp})$, FV can be written the following way:

$$(9) \qquad FV = BV + CV,$$

where BV and CV denote the tangential and normal components of FX, respectively. By direct calculations, from the (8) and (9), we can derive

$$t^2 + Bn = I, \quad nt + Cn = 0, \tag{10}$$

and

$$tB + BC = 0, \quad nB + C^2 = I. \tag{11}$$

3. Warped product semi-invariant submanifolds in locally Riemannian product manifolds. Useful characterizations of warped product semi-invariant submanifolds in locally Riemannian product manifolds will be given the following theorems.

Theorem 3.1. *If $N = N_T \times_f N_\perp$ is a warped product semi-invariant submanifold of a locally Riemannian product manifold M such that N_T is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold of M, then N is a locally Riemannian product.*

Proof. We suppose that $N = N_T \times_f N_\perp$ be a warped product semi-invariant submanifold in a locally Riemannian product manifold M such that N_T is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold M. Then from the Lemma 2.1, we know that

$$\nabla_X Z = \nabla_Z X = X(\ln f)Z, \tag{12}$$

for any $X \in \Gamma(TN_T)$ and $Z \in TN_\perp$. By using symmetric of h, A, taking into account of (1) and (2), we get

$$\begin{aligned} g(\nabla_X Z, W) &= g(\nabla_Z X, W) = g(\bar{\nabla}_Z X, W) = g(\bar{\nabla}_Z FX, FW) \\ X(\ln f)g(Z, W) &= g(h(Z, FX), FW) = g(\bar{\nabla}_{FX} Z, FW) = g(\bar{\nabla}_{FX} FZ, W) \\ &= -g(A_{FZ} FX, W) = -g(A_{FZ} W, FX) = -g(h(FX, W), FZ) \\ &= -X(\ln f)g(W, Z), \end{aligned}$$

for any $W \in \Gamma(TN_\perp)$, that is, $X(\ln f)g(Z, W) = 0$. It follows that $X(\ln f) = 0$, for any $X \in \Gamma(TN_T)$, that is, f is a constant function on N_T. Thus N is a locally Riemannian product. $\square$

Now, we will give two examples warped products in a locally Riemannian product manifolds to illustrate our results such that $N_\perp$ is an anti-invariant and N_T is an invariant.

Example 3.1. Let N be a submanifold in $\mathbb{R}^4$ with coordinates (x_1, x_2, y_1, y_2) given by

$$x_1 = u\cos\theta, \quad x_2 = u\sin\theta, \quad y_1 = u\cos\beta, \quad \textit{and} \quad y_2 = u\sin\beta,$$

where $u > 0$, θ and β denote arbitrary parameters.

It is easily to check that the tangent bundle of N is spanned by the vectors

$$\begin{aligned} Z_1 &= \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos\beta\frac{\partial}{\partial y_1} + \sin\beta\frac{\partial}{\partial y_2}, \\ Z_2 &= -u\sin\theta\frac{\partial}{\partial x_1} + u\cos\theta\frac{\partial}{\partial x_2}, \\ Z_3 &= -u\sin\beta\frac{\partial}{\partial y_1} + u\cos\beta\frac{\partial}{\partial y_2}. \end{aligned}$$

Next, we will define the almost Riemannian product structure of $\mathbb{R}^4$ by

$$F\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i} \quad \text{and} \quad F\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial y_i} \quad i = 1, 2.$$

Then the space $F(TN)$ becomes

$$\begin{aligned} FZ_1 &= -\cos\theta\frac{\partial}{\partial x_1} - \sin\theta\frac{\partial}{\partial x_2} + \cos\beta\frac{\partial}{\partial y_1} + \sin\beta\frac{\partial}{\partial y_2}, \\ FZ_2 &= u\sin\theta\frac{\partial}{\partial x_1} - u\cos\theta\frac{\partial}{\partial x_2}, \\ FZ_3 &= -u\sin\beta\frac{\partial}{\partial y_1} + u\cos\beta\frac{\partial}{\partial y_2}. \end{aligned}$$

Since FZ_1 is orthogonal to TN, FZ_2 and FZ_3 are tangent to TN, $TN_\perp$ and TN_T can be choosen subspaces $\mathrm{sp}\{Z_1\}$ and $\mathrm{sp}\{Z_2, Z_3\}$, respectively. Furthermore the Riemannian metric tensor of $M = N_\perp \times_f N_T$ is given by

$$g = 2du^2 + u^2(d\theta^2 + d\beta^2) = g_{N_\perp} \times_{u^2} g_{N_T}.$$

Thus $N = N_\perp \times_{u^2} N_T$ is a warped product semi-invariant submanifold with 3-dimensional of Riemannian product manifold $\mathbb{R}^4$ with warping function $f = u$.

Example 3.2. Consider in the Riemannian product manifold $\mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R}^2$ with coordinates $(x_1, x_2, x_3, x_4, x_5)$ the submanifold N given by the equations

$$x_4^2 = x_2^2 + x_3^2, \quad x_1 - x_5 = 0.$$

Then we have

$$TN = \operatorname{span}\left\{ Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, \quad Z_2 = \cos\alpha \frac{\partial}{\partial x_2} + \sin\alpha \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, \right.$$
$$\left. Z_3 = -v \sin\alpha \frac{\partial}{\partial x_2} + v\cos\alpha \frac{\partial}{\partial x_3} \right\},$$

v,α denote the arbitrary parameters. It follow that

$$FZ_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_5}, \quad FZ_2 = \cos\alpha \frac{\partial}{\partial x_2} + \sin\alpha \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \ \text{ and } \ FZ_3 = Z_3.$$

Since the vector fields FZ_1 and FZ_2 are orthogonal to TN and FZ_3 is tangent to TN, $TN_\perp$ and TN_T can be taken as $\operatorname{sp}\{Z_1, Z_2\}$ and $\operatorname{sp}\{Z_3\}$, respectively. Moreover, the metric of N is given

$$g = 2(du^2 + dv^2) + v^2 d\alpha^2 = 2g_{N_\perp} + v^2 g_{N_T}.$$

Thus $N = N_\perp \times_{u^2} N_T$ is a warped product semi-invariant submanifold of $\mathbb{R}^5$ with warping function $f = v$.

Now, let $N = N_\perp \times_f N_T$ be a warped product semi-invariant submanifolds in a locally Riemannian product manifold M which are warped products of the form $N_\perp \times_f N_T$, where $N_\perp$ is an anti-invariant submanifold and N_T is an invariant submanifold of M with respect to F. If we denote the Levi-Civita connections on M and N by $\bar{\nabla}$ and ∇, respectively, then from (1), (2), (8) and (9), we have

$$\begin{aligned} \bar{\nabla}_X FY &= F\bar{\nabla}_X Y \\ \nabla_X tY + h(X, tY) - A_{nY}X + \nabla^\perp_X nY &= t(\nabla_X Y) + n(\nabla_X Y) \\ &+ Bh(X,Y) + Ch(X,Y), \end{aligned} \tag{13}$$

for any $X, Y \in \Gamma(TN)$. Taking account of the tangential and normal components of (13), we have

$$(\nabla_X t)Y = A_{nY}X + Bh(X,Y) \tag{14}$$

and

$$(\nabla_X n)Y = Ch(X,Y) - h(X,tY), \tag{15}$$

where the derivations of t and n are, respectively, defined by

$$\begin{aligned}(\nabla_X t)Y &= \nabla_X tY - t(\nabla_X Y),\\ (\nabla_X n)Y &= \nabla^{\perp}_X nY - n(\nabla_X Y).\end{aligned}$$

Next, we will give the following theorems.

Theorem 3.3. *Let $N = N_\perp \times_f N_T$ be a warped product semi-invariant submanifold of a locally Riemannian product manifold M such that $N_\perp$ is an anti-invariant submanifold and N_T is an invariant submanifold of M. Then the integral manifolds $N_\perp$ and N_T are always integrable.*

Proof. Taking $X \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\perp)$ in the equation (13) and consider Lemma 2.1, then we have

$$-A_{nU}X + \nabla^{\perp}_X nU = F(U(\ln f)X) + Bh(X,U) + Ch(X,U).$$

It follows that

$$-A_{nU}X = U(\ln f)tX + Bh(X,U) \quad \textit{and} \quad \nabla^{\perp}_X nU = Ch(X,U). \tag{16}$$

Furthermore, taking $U \in \Gamma(TN_\perp)$ and $X \in \Gamma(TN_T)$ in (13) and since F is also linear, we have

$$Bh(U,X) = 0 \tag{17}$$

and

$$h(tX,U) = Ch(U,X). \tag{18}$$

Then the equation (16) becomes

$$A_{nU}X = -U(\ln f)tX. \tag{19}$$

On the other hand, the formulas of Gauss and Weingarten and consider Lemma 2.1, we can derive

$$A_{nU}V = -Bh(V,U), \tag{20}$$

which is also equivalent to

$$A_{nU}V = A_{nV}U, \tag{21}$$

for any $U, V \in \Gamma(TN_\perp)$. Whereas, taking into account of (3) and the shape operator A is self-adjoint, we get

$$\begin{aligned} g(A_{nU}V, Z) &= g(h(V,Z), nU) = g(\bar{\nabla}_Z V, FU) = g(\bar{\nabla}_Z FV, U) \\ &= -g(A_{nV}Z, U) = -g(A_{nV}U, Z), \end{aligned}$$

which gives

$$A_{nU}V = -A_{nV}U, \tag{22}$$

for any $U, V \in \Gamma(TN_\perp)$ and $Z \in \Gamma(TN)$. Thus (21) and (22) give us

$$A_{nU}V = 0 \quad \text{and} \quad Bh(U,V) = 0, \tag{23}$$

for any $U, V \in \Gamma(TN_\perp)$. In the same way, taking account of (1), (2), (8), (9) and (17), we get

$$\begin{aligned} -A_{nU}X + \nabla^\perp_X nU &= F(\nabla_U X) + Ch(U,X) \\ &= U(\ln f)tX + Ch(U,X), \end{aligned}$$

for any $U \in \Gamma(TN_\perp)$ and $X \in \Gamma(TN_T)$. Thus we have

$$A_{nU}X = -U(\ln f)tX \quad \text{and} \quad \nabla^\perp_X nU = Ch(U,X). \tag{24}$$

Furthermore, from (1), (2), (8), (9) and consider Lemma 2.1, we have

$$\begin{aligned} h(Y,tX) + \nabla_Y tX &= F(\nabla_Y X) + Fh(X,Y) \\ &= F\left(\nabla^{N_2}_Y X - g(X,Y)\frac{\mathrm{grad} f}{f}\right) + Bh(X,Y) + Ch(X,Y) \\ &= t(\nabla^{N_2}_Y X) - g(X,Y)n\left(\frac{\mathrm{grad} f}{f}\right) + Bh(X,Y) + Ch(X,Y), \end{aligned}$$

for any $X, Y \in \Gamma(TN_T)$. Thus we arrive

$$h(Y,tX) = -g(X,Y)n\left(\frac{\mathrm{grad} f}{f}\right) + Ch(X,Y), \tag{25}$$

and

$$\nabla^{N_2}_Y tX - g(tX,Y)\frac{\mathrm{grad}f}{f} = t(\nabla^{N_2}_Y X) + Bh(X,Y). \tag{26}$$

The equation (25) implies

$$h(Y,tX) = h(X,tY), \tag{27}$$

for any $X, Y \in \Gamma(TN_T)$. By using (15) and (27), we have

$$\begin{aligned} n([X,Y]) &= n(\nabla_X Y - \nabla_Y X) = \nabla^{\perp}_X nY - (\nabla_X n)Y - \nabla^{\perp}_Y nX + (\nabla_Y n)X \\ &= (\nabla_Y n)X - (\nabla_X n)Y = Ch(Y,X) - h(Y,tX) - Ch(X,Y) \\ &+ h(X,tY) = 0, \end{aligned}$$

for any $X, Y \in \Gamma(TN_T)$, that is, $[X,Y] \in \Gamma(TN_T)$. In the same way, by using (14) and (23) for any $U, V \in \Gamma(TN_\perp)$, we get

$$\begin{aligned} t([U,V]) &= t(\nabla_U V - \nabla_V U) \\ &= \nabla_U tV - (\nabla_U t)V - \nabla_V tU + (\nabla_V t)U \\ &= (\nabla_U t)V - (\nabla_U t)V = A_{nV}U - A_{nU}V = 0, \end{aligned}$$

that is, $[U,V] \in \Gamma(TN_\perp)$. This completes the proof. $\square$

Theorem 3.3. *Let $N = N_\perp \times_f N_T$ be a warped product of a locally Riemannian product manifold M such that $N_\perp$ is an anti-invariant submanifold and N_T is an invariant submanifold of M. Then N is a warped product semi-invariant submanifold if and only $nt = 0$.*

Proof. Let us assume that N is a warped product semi-invariant submanifold of a locally Riemannian product manifold M and by Q and P, we denote the projection operators on subspaces $\Gamma(TN_\perp)$ and $\Gamma(TN_T)$, respectively, then we have

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0.$$

By using (8), we get

$$QtP = 0, \quad nP = 0, \quad tP = t,$$

from which, consider (10) and (11), we can derive

$$nt = 0, \tag{28}$$

which is also equivalent to

$$Cn = 0. \tag{29}$$

Conversely, for a warped product submanifold N of a locally Riemannian product manifold M, we suppose that $nt = 0$. For any vector fields tangent X to N and V normal to N, by using (8), (9) and (29), we have

$$\begin{aligned} g(X, BV) &= g(nX, V) \\ g(X, FBV) &= g(FnX, V) \\ g(X, tBV) &= g(CnX, V) = 0, \end{aligned}$$

which gives $tB = 0$ which is equivalent to $BC = 0$ from (11). Then from (10), we conclude that

$$t^3 = t \text{ and } C^3 = C. \tag{30}$$

Now, if we put

$$P = t^2 \text{ and } Q = I - P, \tag{31}$$

then we can derive that $P + Q = I$, $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$, which show that Q and P are orthogonal complementary projection operators and define orthogonal complementary distributions such as $D^{\perp}$ and D, respectively, where D and $D^{\perp}$ denote the distributions which are belong to TN_T and $TN_{\perp}$, respectively. From the equations (28), (30) and (31) we can derive

$$tP = t, \quad tQ = 0, \quad QtP = 0 \text{ and } nP = 0.$$

These equations show that the distribution D is an invariant and the distribution $D^{\perp}$ is also an anti-invariant. This completes the proof. □

Theorem 3.4. *Let N be a semi-invariant submanifold of a locally Riemannian product manifold M. Then N is a locally warped product semi-invariant submanifold if and only if the shape operator of N satisfies*

$$A_{FU}Z = F(U(\mu))Z, \quad U \in \Gamma(TN_{\perp}), \quad Z \in \Gamma(TN_T), \tag{32}$$

for some function μ on N satisfying $W(\mu) = 0$, for any $W \in \Gamma(TN_T)$.

Proof. We suppose that $N = N_\perp \times_f N_T$ is a warped product semi-invariant submanifold in a locally Riemannian product manifold M. Then from (19), we have

$$A_{FU}X = -F(U(\ln f))X,$$

for any $U \in \Gamma(TN_\perp)$ and $X \in \Gamma(TN_T)$. Because f is a function on $N_\perp$, we can easily to see that $W(\ln f) = 0$, for all $W \in \Gamma(TN_T)$.

Conversely, let us assume that N is a semi-invariant submanifold in a locally Riemannian product manifold M satisfying

$$A_{FU}X = F(U(\mu))X, \quad U \in \Gamma(TN_\perp) \text{ and } X \in \Gamma(TN_T),$$

for some function μ with $W(\mu) = 0$, for all $W \in \Gamma(TN_T)$. Then from (1) and (23), we arrive

$$g(\nabla_U V, X) = g(\bar{\nabla}_U V, X) = g(\bar{\nabla}_U FV, FX) = -g(A_{FV}U, FX) = 0,$$

for any $U, V \in \Gamma(TN_\perp)$ and $X \in (TN_T)$. Thus the anti-invariant submanifold $N_\perp$ is totally geodesic in N. In the same way;

$$\begin{aligned} g(\nabla_X Y, U) &= g(\bar{\nabla}_X Y, U) = -g(\bar{\nabla}_X U, Y) = -g(\bar{\nabla}_X FU, FY) \\ &= g(A_{FU}X, FY) = U(\mu)g(X, Y), \end{aligned}$$

for any $X, Y \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\perp)$. Since the invariant submanifold N_T of semi-invariant submanifold N is always integrable and $W(\mu) = 0$, for each $W \in \Gamma(TN_T)$, which implies that N_T is an extrinsic sphere in N, that is, it is a totally umbilical submanifold with the mean curvature is parallel in N. Thus we know that N is a locally Riemannian warped product $N_\perp \times_f N_T$, where $N_\perp$ and N_T are anti-invariant and invariant submanifolds of M, respectively, and f is the warping function. The proof is complete. □

Lemma 3.1. *Let $N = N_\perp \times_f N_T$ be a warped product semi-invariant submanifold of a locally Riemannian product manifold M. Then we have*

1.) $g(h(TN_\perp, TN_\perp), FTN_\perp) = 0$

2.) $g(h(FX, U), FY) = U(\ln f)g(X, Y)$

3.) $g(h(TN_\perp, TN_T), FTN_\perp) = 0$

4.) *$g(h(TN_T, F(TN_T)), FTN_\perp) = 0$ if and only if $N = N_\perp \times_f N_T$ is a trivial Riemannian product in M, for each $U \in \Gamma(TN_\perp)$, $X, Y \in \Gamma(TN_T)$.*

Proof. 1.) The proof is obvious from (3) and (23).

$$2.)\ g(h(FX,U),FY) = g(\bar{\nabla}_U FX, FY) = g(\bar{\nabla}_U X, Y) = g(\nabla_U X, Y) = U(\ln f)g(X,Y),$$

for any $U \in \Gamma(TN_\perp)$ and $X, Y \in \Gamma(TN_T)$.
3.) The proof is obvious from (3) and (23).

$$4.)\ g(h(X,FY),FU) = g(\bar{\nabla}_X FY, FU) = g(\bar{\nabla}_X Y, U) = -g(\bar{\nabla}_X U, Y) = -g(\nabla_X U, Y) = U(\ln f)g(X,Y) = 0$$

for any $X, Y \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\perp)$, if and only if f is a constant function on $N_\perp$ if and only if $N = N_\perp \times_f N_T$ is a locally Riemannian product. □

4. Conclusion. The geometry of warped products in the Riemannian product manifolds is totally different from the geometry of warped products in Complex manifolds. Namely, In Kaehler manifolds, if $N = N_\perp \times_f N_T$ is a warped product CR-submanifold such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold, then it has to be a CR-product(see [3]), whereas, in the Riemannian product manifolds, if $N = N_T \times_f N_\perp$ is a warped product semi-invariant submanifold such that N_T is an invariant submanifold and $N_\perp$ is an anti-invariant submanifold, then N has to be a Riemannian product(see Theorem 3.1). Moreover, In complex manifolds, the dimension of holomorphic distribution is even, whereas, in the Riemannian product manifolds, the dimension of invariant distribution may be even or odd(see Example 3.1 and 3.2).

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