Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **35** (2009), 301–310

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

COMPOUND COMPOUND POISSON RISK MODEL*

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Communicated by S. T. Rachev

Abstract. The compound Poisson risk models are widely used in practice. In this paper the counting process in the insurance risk model is a compound Poisson process. The model is called Compound Compound Poisson Risk Model. Some basic properties and ruin probability are given. We analyze the model under the proportional reinsurance. The optimal retention level and the corresponding adjustment coefficient are obtained. The particular case of the Pólya-Aeppli risk model is discussed.

1. Introduction. Assume that the standard model of an insurance company, called risk process $\{X(t), t \geq 0\}$ is given by

*This paper is partially supported by Sofia University grant 221/2008.

2000 *Mathematics Subject Classification:* 60K10, 62P05.

Key words: Compound Poisson process, Pólya-Aeppli risk model, ruin probability, Cramér-Lundberg approximation.

$$X(t) = ct - S(t). \tag{1}$$

Here c is a premium income per unit time and $S(t)$ represents the aggregate amount of claims up to time t. The process $S(t)$ is a compound process, given by

$$S(t) = \sum_{i=1}^{N(t)} Z_i, \quad \left(\sum_1^0 = 0\right), \tag{2}$$

where $N(t)$ is the counting process, $\{Z_i\}_{i=1}^{\infty}$ is a sequence of independent identically distributed, positive random variables, independent of $N(t)$ with Z_i representing the ith claim amount. We assume that the individual claim amounts have a continuous distribution with distribution function F, $F(0) = 0$, and mean value $\mu = EZ_1 < \infty$. In the classical risk model the process $N(t)$ is a homogeneous Poisson process, see for instance [1] and [7]. In this paper we suppose that the counting process $N(t)$ is a compound Poisson process with discrete compounding distribution, i.e. $N(t) = \sum_{i=1}^{N_1(t)} Y_i$, where $Y_1, Y_2, \ldots$ are independent identically distributed random variables, independent of $N_1(t)$ and $N_1(t) \sim Po(\lambda t)$. Let Y denotes the compounding random variable with probability generating function (PGF) $\psi(s) = Es^Y$ and $EY = m_1$ and $EY^2 = m_2$. Then the PGF of the counting process is given by

$$P_{N(t)}(s) = e^{-\lambda t[1-\psi(s)]}$$

and the process (1) is called a Compound Compound Poisson risk model (CC Poisson risk model). It is easy to find that $EN(t) = \lambda m_1 t$, $ES(t) = \lambda\mu m_1 t$ and the safety loading coefficient

$$\theta = \frac{(c - \lambda\mu m_1)t}{\lambda\mu m_1 t} = \frac{c}{\lambda\mu m_1} - 1 > 0.$$

Denote by

$$\tau(u) = \inf\{t > 0, u + X(t) \le 0\}$$

the time to ruin of a company having initial capital u. We let $\tau = \infty$, if for all $t > 0$ $u + X(t) > 0$.

The probability of ruin in the infinite horizon time is

$$\Psi(u) = P(\tau(u) < \infty)$$

and in the finite horizon case

$$\Psi(u,t) = P(\tau(u) \leq t).$$

In Section 2 and Section 3 the CC Poisson risk model and martingale approach are discussed. In Section 4, we analyze the model under the proportional reinsurance in the case of small claims. The retention level maximizing the adjustment coefficient is obtained. The effect of reinsurance on the ruin probability is discussed. The particular case of Pólya-Aeppli risk model is given.

2. Martingales for the CC Poisson risk model. Let us denote by $(\mathcal{F}_t^X)$ the natural filtration generated by any stochastic process $X(t)$. $(\mathcal{F}_t^X)$ is the smallest complete filtration to which $X(t)$ is adapted.

Let us denote by $LS_Z(r) = \int_0^\infty e^{-rx} dF(x)$ the Laplace-Stieltjes transform of any random variable Z with distribution function $F(x)$ and $M_Z(r) = LS_Z(-r)$ the moment generating function (MGF) of Z.

Lemma 1. *For the CC Poisson risk model*

$$Ee^{-rX(t)} = e^{g(r)t},$$

where

$$g(r) = \lambda[\psi(M_Z(r)) - 1] - cr. \tag{3}$$

P r o o f. Let us consider the random sum from the right hand side of (1)

$$S_t = \sum_{k=1}^{N(t)} Z_k,$$

where $N(t)$ is a compound Poisson process, independent of $Z_k, k = 1, 2, \ldots$. S_t is a CC Poisson process and the MGF is given by

$$M_{S_t}(r) = P_{N(t)}(M_Z(r)) = e^{-\lambda t[1-\psi(M_Z(r))]}.$$

For the Laplace-Stieltjes transform of $X(t)$ we have the following

$$LS_{X(t)}(r) = Ee^{-rX(t)} = Ee^{-r[ct-S(t)]} = e^{-rct}Ee^{rS(t)} =$$

$$= e^{-rct}P_{N(t)}(M_Z(r)) = e^{-rct}e^{-\lambda t[1-\psi(M_Z(r))]} = e^{g(r)t},$$

where $g(r)$ is given by (3). $\square$

From the martingale theory we get the following

Lemma 2. *For all $r \in \mathbb{R}$ the process*

$$M(t) = e^{-rX(t)-g(r)t}, \ t \geq 0$$

is an $\mathcal{F}_t^X$-martingale, provided that $M_Z(r) < \infty$.

3. Martingale approach to the CC Poisson risk model. Using the martingale properties of $M(t)$, we will give some useful inequalities for the ruin probability.

Proposition 1. *Let $r > 0$. For the ruin probabilities of the CC Poisson risk model we have the following results*

i) $\Psi(u,t) \leq e^{-ru} \sup\limits_{0 \leq s \leq t} e^{g(r)s}, 0 \leq t < \infty$

ii) $\Psi(u) \leq e^{-ru} \sup\limits_{s \geq 0} e^{g(r)s}.$

iii) *If the Lundberg exponent R exists, then R is the unique strictly positive solution of*

$$\lambda[\psi(M_Z(r)) - 1] - cr = 0 \tag{4}$$

and

$$\Psi(u) \leq e^{-Ru}. \tag{5}$$

Proof. i) Since at the time of ruin $X(\tau) \leq -u$, for any $t < \infty$, the martingale stopping time theorem yields the following

$$
\begin{aligned}
1 &= M(0) = EM(t \wedge \tau) \\
&= E[M(t \wedge \tau)|\tau \leq t]P(\tau \leq t) + E[M(t \wedge \tau)|\tau > t]P(\tau > t) \\
&\geq E[M(t \wedge \tau)|\tau \leq t]P(\tau \leq t) = E[e^{-rX(\tau)-g(r)\tau}|\tau \leq t]P(\tau \leq t) \\
&\geq e^{ru}E[e^{-g(r)\tau}|\tau \leq t]P(\tau \leq t),
\end{aligned}
$$

from which

$$P(\tau \leq t) \leq \frac{e^{-ru}}{E[e^{-g(r)\tau}|\tau \leq t]}. \tag{6}$$

The statement i) follows from the above relation.

ii) follows immediately from i) when $t \to \infty$.

iii) Under the condition (4) (g(r) = 0), (5) follows from (6). Let R be a positive solution of $g(r) = 0$. Because $g(0) = 0, g'(0) = \lambda\psi'(1)\mu - c = \lambda m_1\mu - c < 0, g''(r) = \lambda\psi''(M_Z(r))(M_Z'(r))^2 + \lambda\psi'(M_Z(r))M_Z''(r)$ and $g''(0) = \lambda[m_2\mu^2 + m_1 Var(Z)] > 0$, there is at most one strictly positive solution of the equation $g(r) = 0$. □

Remark 1. The condition (4) is known as Cramér condition and (5) as Lundberg inequality.

4. Reinsurance.

Suppose the insurer has the possibility to choose proportional reinsurance with retention level $b \in [0, 1]$. This means that the insurer pays bZ of a claim. The premium rate for the reinsurance is

$$(1+\eta)(1-b)\lambda\mu m_1,$$

where $\eta > 0$ is the relative safety loading, defined by the reinsurance company. We consider the case $\eta > \theta$, i.e. the reinsurance is more expensive than first insurance. The premium rate for the insurer is

$$[(1+\theta)-(1+\eta)(1-b)]\lambda\mu m_1 = [b(1+\eta)-(\eta-\theta)]\lambda\mu m_1,$$

and the surplus process becomes

$$U(t,b) = u + [b(1+\eta)-(\eta-\theta)]\lambda\mu m_1 t - \sum_{k=1}^{N(t)} bZ_k, \tag{7}$$

In order that the net profit condition is fulfilled we need

$$\frac{[b(1+\eta)-(\eta-\theta)]\lambda\mu m_1}{b\lambda\mu m_1} > 1,$$

i.e.

$$b > 1 - \frac{\theta}{\eta}.$$

The adjustment coefficient $R(b)$ under proportional reinsurance solves the equation:

$$\lambda[\psi(M_Z(br)) - 1] - [b(1+\eta)-(\eta-\theta)]\lambda\mu m_1 r = 0. \tag{8}$$

Let $\Psi(u,b)$ denote the probability of ultimate ruin when the proportional reinsurance is chosen. Then

$$\Psi(u,b) = P(U(t,b) < 0 \text{ for some } \ t > 0).$$

Our objective is to find the retention level that minimizes $\Psi(u,b)$. The problem related to classical risk model is given in [8]. According the Lundberg inequality (7), the retention level will be optimal, if the corresponding Lundberg exponent R is maximal. We know that there is a unique $b \in [0,1]$ where the maximum is attained. If the maximiser $b > 1$ we know from the uni-modality that the optimal b is 1, i.e. no reinsurance is chosen.

The next result gives the optimal retention level b and maximal adjustment coefficient $R(b)$. Similar result is obtained by Hald and Schmidli [2] for the classical risk model.

Lemma 3. *The solution of* (8) *is given by*

$$R(b(r)) = \frac{1-\psi(M_Z(r))+(1+\eta)\mu m_1 r}{(\eta-\theta)\mu m_1}, \tag{9}$$

where $r(b)$ *is invertible function.*

Proof. Assume that $r(b) = bR((b))$, where $R(b)$ will be the maximal value of the adjustment coefficient and $r(b)$ is invertible. If we consider the function $r \to b(r)$, it follows that

$$b(r) = \frac{(\eta-\theta)\mu m_1 r}{1-\psi(M_Z(r))+(1+\eta)\mu m_1 r}. \tag{10}$$

Now $R(b(r)) = \dfrac{r}{b(r)}$ in details is given by (9). □

The next result gives a way to calculate b and $R(b)$.

Theorem 1. *Assume that* $M_Z(r) < \infty$. *Suppose there is a (unique) solution* r *to*

$$\psi'(M_Z(r))M_Z'(r) - \mu m_1(1+\eta) = 0. \tag{11}$$

Then $r > 0$, *the maximal value of* $R(b(r))$ *and the retention level* $b(r)$ *are given by* (9) *and* (10).

Proof. The necessary condition for maximizing the value of the adjustment coefficient is given by equation (11).

Because $R'(b(0)) = \dfrac{\eta}{\eta-\theta} > 0$, the function $R(b(r))$ is strictly increasing in 0. The second derivative in zero $R''(b(0)) = -\dfrac{(m_2-m_1)\mu^2+m_1EZ^2}{(\eta-\theta)\mu m_1} < 0$ shows that $R(b(r))$ is strictly concave. Consequently, the function $R(b(r))$ has an unique maximum in r, which is the solution of (11). The retention level is given by (10). □

Remark 2. Note that the value R does not depend on c but on the relative safety loadings only.

5. Pólya-Aeppli risk model. The Pólya-Aeppli risk model is defined in [4]. In this case the counting process $N(t)$ has a Pólya-Aeppli distribution (see [3] and [5]). The random variables Y_i are geometrically distributed with parameter $1-\rho$ and $m_1 = \dfrac{1}{1-\rho}$. The Pólya-Aeppli counting process is the only compound Poisson process that is also a renewal process. The problem of reinsurance for the Pólya-Aeppli risk model is discussed in [6].

The Lundberg exponent R of the Pólya-Aeppli risk model is given by the positive solution of the equation

$$\rho cr M_Z(r) + \lambda(M_Z(r)-1) - cr = 0.$$

From the equation one can see that the solution will not be explicit as in the case of exponentially distributed claims.

5.1. Exponentially distributed claims. Here we suppose that the claim sizes are exponentially distributed with parameter $\mu > 0$, i.e.

$$F'(z) = \frac{1}{\mu} e^{-\frac{z}{\mu}}, \quad z \geq 0.$$

The moment generating function is given by

$$M_Z(r) = \frac{1}{1-\mu r}, \quad r < \frac{1}{\mu}$$

and

$$M'_Z(r) = \frac{\mu}{(1-\mu r)^2}.$$

The Lundberg exponent r in this case is obtained as the positive solution of the equation

$$(1-\rho)^2 \frac{\mu}{(1-\mu r)^2} = (1+\eta)\mu \left[1 - \rho\frac{1}{1-\mu r}\right]^2 = 0,$$

i.e.

$$r = \frac{1}{\mu}\left(1 - (1+\eta)^{-\frac{1}{2}}\right).$$

From (10) we find

$$b = \frac{(\eta - \theta)}{\sqrt{1+\eta}(1+\sqrt{1+\eta})}$$

yielding

$$R(b) = \frac{(1-\rho)[1-\sqrt{1+\eta}]^2}{(\eta-\theta)\mu}.$$

In this example there are closed form expressions for b and $R(b)$, if $b \leq 1$ and the retention level does not depend from ρ.

Analyzing the optimal retention levels for the Pólya-Aeppli risk model and the classical model gives the possibility to compare the ruin probabilities. In the case of proportional reinsurance and $\rho = 0$, the retention level $b(r)$, given by (10) coincides with the retention level for the classical risk model, say b_0, obtained by Hald and Schmidli [2]. It is easy to verify that $b(r) < b_0$. For the maximal values $R(b_0)$ and $R(b(r))$ is fulfilled

$$R(b_0) \geq R(b(r)),$$

which means that the Pólya-Aeppli risk model with proportional reinsurance is more dangerous than the classical risk model.

Acknowledgment. I thank both the associate editor and the referee for the valuable suggestions and comments.

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Received August 6, 20009