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Number Sequences in an Integral Form with a Generalized Convolution Property and Somos-4 Hankel Determinants

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This paper is dealing with the Hankel determinants of the special number sequences given in an integral form. We show that these sequences satisfy a generalized convolution property and the Hankel determinants have the generalized Somos-4 property. Here, we recognize well known number sequences such as: the Fibonacci, Catalan, Motzkin and Schröder sequences, like special cases.

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1. Introduction

The closed-form computation of the Hankel determinants is of great combinatorial interest related to partitions and permutations. A long list of known determinant evaluations and methods can be seen, for example, in [6]. The most useful methods are based on: LU-decompositions, continued fractions, lattice paths and orthogonal polynomials.

Definition 1. A sequence $\{g_n\}$ is of the *B–integral form* if its members can be written as

$$g_n^{(p)} = \frac{1}{2\pi}\int_a^b x^{n-\delta_{1,p}}(x-a)^{\mu-1}(b-x)^{\nu-1}\, dx, \tag{1}$$

$$(a<b;\ p\in\{0,1\};\ \mu,\nu>0;\ n\in\mathbb{N}).$$

The letter B is here to remind on the beta function $B(\mu,\nu)$ because of the fact that those integrals often can be reduced to it by a simple change $u=(x-$

$a)/(b-a)$. Here, the Euler integral representation for the Gauss hypergeometric function will be very useful (see [3], p.33 or [4], p.201):

$$\int_0^1 x^{a-1}(1-x)^{b-1}(1-sx)^{-d}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\ {}_2F_1\Big(\begin{matrix} d, a \\ a+b \end{matrix}\Big| s\Big), \tag{2}$$

$$(\mathrm{Re}(a) > 0;\ \mathrm{Re}(b) > 0),$$

where

$${}_2F_1\Big(\begin{matrix} a, b \\ c \end{matrix}\Big| z\Big) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\ \frac{z^n}{n!}\ ,$$

with $(a)_0 = 1, \quad (a)_n = a(a+1)\ldots(a+n-1),\ n \in \mathbb{N}$.

Example 1. The *Catalan numbers* $\{C_n\}_{n\geq 0} = \{1, 1, 2, 5, 14, 42, \ldots\}$ can be represented in the integral form

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2\pi}\int_0^4 x^{n-1}\sqrt{(4-x)x}\ dx \qquad (n \in \mathbb{N}_0). \tag{3}$$

The *large Schröder numbers* $\{S_n\}_{n\geq 0} = \{1, 2, 6, 22, 90, 394, \ldots\}$ are the numbers of Schröder paths of semi-length n from $(0,0)$ to $(2n,0)$ and can be written in the following form

$$S_n = \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{2n-2k}{n-k}\binom{2n-k}{k} = \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} x^{n-1}\sqrt{1-6x+x^2}\ dx.$$

The *Motzkin numbers* $\{M_n\}_{n\geq 0} = \{1, 1, 2, 4, 9, 21, 51, \ldots\}$ are given by

$$M_n = \int_{-1}^{3} x^n \sqrt{(3-x)(x+1)}\ dx \qquad (n \in \mathbb{N}_0)\ .$$

Definition 2. A sequence $\{g_n\}$ satisfies the *convolution property of order* r, if it is valid

$$g_n = \sum_{k=1}^{r} \alpha_k g_{n-k} + \beta \sum_{k=0}^{n-r} g_k g_{n-r-k}. \tag{4}$$

For $r = 1$ and $\alpha_1 = 0$, we say that the *regular convolution property* is satisfied.

Definition 3. A sequence $\{g_n\}$ has the *generalized Somos-4 property*, if there exists a pair (r, s) such that

$$g_n g_{n-4} = r\ g_{n-1} g_{n-3} + s\ g_{n-2}^2 \quad (n = 4, 5, \ldots). \tag{5}$$

The Somos-4 sequences are associated with the abscissae of rational points on an elliptic curve (see [8]). If the initial values and the parameters are integers, usually the complete sequences are integers. Also, taking concrete initial values and holding parameters as variables, the sequence members are Laurent polynomials in variables r and s.

Definition 4. The *Hankel transform* of a number sequence $G = \{g_n\}$ is the sequence of Hankel determinants $H = \{h_n\}$ given by

$$G = \{g_n\}_{n\in\mathbb{N}_0} \quad \to \quad H = \{h_n\}_{n\in\mathbb{N}}: \quad h_n = |g_{i+j-2}|_{i,j=1}^{n}. \tag{6}$$

Some authors start with a generating function $G(x) = \sum_{n\geq 0} g_n x^n$ and its sequence of coefficients.

Definition 5. For a given function $v = f(u)$ with the property $f(0) = 0$, the *series reversion* is the sequence $\{s_k\}$ such that

$$u = f^{-1}(v) = s_0 + s_1 v + \cdots + s_n v^n + \cdots, \tag{7}$$

where $u = f^{-1}(v)$ is the inverse function of $v = f(u)$.

2. The special numbers in an integral form

Let $a < b$ and $c \neq 0$. Consider a sequence $\{g_n^{(0)}\}$ defined by

$$g_n^{(0)} = g_n^{(0)}(a,b,c,d) = \frac{1}{2\pi}\int_a^b (cx+d)^n \sqrt{(b-x)(x-a)}\; dx \quad (n \in \mathbb{N}_0). \tag{8}$$

Their generating function $G_0(t) = G_0(t; a, b; c, d)$ is

$$G_0(t) = \sum_{n=0}^{\infty} g_n^{(0)} t^n = \frac{1}{2\pi}\int_a^b \frac{\sqrt{(b-x)(x-a)}}{1-(cx+d)t}\, dx,$$

wherefrom

$$G_0(t) = \frac{2 - ((a+b)c + 2d)t - 2\sqrt{(1-(ac+d)t)(1-(bc+d)t)}}{(2ct)^2}\,.$$

Since

$$G_0^2(t) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} g_k^{(0)} g_{n-k}^{(0)}\Big)t^n,$$

the generating function satisfies the following functional equation

$$(4ct)^2 G_0^2(t) + 8(((a+b)c + 2d)t - 2)G_0(t) + (b-a)^2 = 0.$$

Hence

$$g_n^{(0)} = \frac{(a+b)c+2d}{2}\ g_{n-1}^{(0)} + c^2 \sum_{k=0}^{n-2} g_k^{(0)} g_{n-2-k}^{(0)} \qquad (n=2,3,\ldots),$$

with initial values

$$g_0^{(0)} = \Big(\frac{b-a}{4}\Big)^2, \qquad g_1^{(0)} = \frac{(a+b)c+2d}{2}\ g_0^{(0)}\ .$$

Remark 1. In the special case, when $a > 0$ or $B \leq 0$, this sequence can be written in the hypergeometric form

$$g_n^{(0)} = \frac{(a-b)^2 b^n}{16}\ {}_2F_1\Big(\begin{matrix}3/2, -n\\ 3\end{matrix}\Big| 1 - a/b\Big) \qquad (a > 0 \vee b \leq 0).$$

3. The shifted case

Let

$$g_0^{(1)} = \frac{1}{2\pi}\int_a^b \frac{\sqrt{(b-x)(x-a)}}{cx+d}\,dx, \qquad g_n^{(1)} = g_{n-1}^{(0)} \quad (n \in \mathbb{N}_0).$$

The generating function of this sequence is $G_1(t) = g_0^{(1)} + tG_0(t)$, i.e.

$$G_1(t) = \frac{1 - \sqrt{(ac+d)(bc+d)}\ t - \sqrt{(1-(ac+d)t)(1-(bc+d)t)}}{2c^2 t}\ .$$

It satisfies the following equation

$$c^2 t G_1^2(t) - (1 - \sqrt{(ac+d)(bc+d)}\ t)G_1(t) + \Big(\frac{\sqrt{bc+d} - \sqrt{ac+d}}{2c}\Big)^2 = 0.$$

Hence $\{g_k^{(1)}\}$ has the following convolution property

$$g_n^{(1)} = \sqrt{(ac+d)(bc+d)}\ g_{n-1}^{(1)} + c^2 \sum_{k=0}^{n-1} g_k^{(1)} g_{n-1-k}^{(1)} \qquad (n=2,3,\ldots),$$

with initial values

$$g_0^{(1)} = \Big(\frac{\sqrt{bc+d} - \sqrt{ac+d}}{2c}\Big)^2, \quad g_1^{(1)} = (g_0^{(1)})^2 + \sqrt{(ac+d)(bc+d)}\ g_0^{(1)}.$$

In another form, it can be written like

$$g_n^{(1)} = \frac{a+b}{2}\, g_{n-1}^{(1)} + \sum_{k=1}^{n-2} g_k^{(1)} g_{n-1-k}^{(1)}.$$

Example 2. If $a = 0,\ b = 4$, then $g_n^{(1)} = C_n$ is the Catalan number. The generating function is often denoted by

$$c(t) = \sum_{n=0}^{\infty} C_n t^n = \frac{1-\sqrt{1-4t}}{2t}.$$

It satisfies the following functional equation and the convolution property

$$tc^2(t) - c(t) + 1 = 0, \qquad C_0 = 1, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} \qquad (n \in \mathbb{N}_0).$$

Example 3. Also, we mentioned that for $a = 3 - 2\sqrt{2},\ b = 3 + 2\sqrt{2}$, we get $g_n^{(1)} = S_n$, the large Schröder numbers. They satisfy the convolution property

$$S_0 = 1, \quad S_1 = 2, \quad S_n = 3S_{n-1} + \sum_{k=0}^{n-2} C_k C_{n-1-k} \qquad (n \in \mathbb{N}_0), \tag{9}$$

4. Computing Hankel determinants via orthogonal polynomials

A few methods are known for evaluating the Hankel determinants. Especially, we have published our considerations about the method based on the theory of distributions and orthogonal polynomials in the papers [2], [7] and [1], in chronological order.

Namely, the Hankel determinant h_n of the sequence $\{a_n\}$ equals

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^2 \beta_{n-1} \qquad (n = 1, 2, \ldots)\ , \tag{10}$$

where $\{\beta_n\}$ is the sequence of the coefficients in the recurrence relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x)\ , \qquad P_{-1} \equiv 0, \quad P_0 \equiv 1. \tag{11}$$

Here, $\{P_n(x)\}$ is the monic polynomial sequence orthogonal with respect to the inner product

$$(f,\ g) = \mathcal{U}[f(x)g(x)], \tag{12}$$

where $\mathcal{U}$ is a functional determined by

$$a_n = \mathcal{U}[x^n] \quad (n = 0, 1, 2, \ldots) \ . \tag{13}$$

In some cases, there exists a weight function $w(x)$ such that the functional $\mathcal{U}$ can be expressed by

$$\mathcal{U}[f] = \int_{\mathbb{R}} f(x)\ w(x)\ dx \qquad (f(x) \in C(\mathbb{R});\ w(x) \geq 0) \ . \tag{14}$$

So, we can join to every weight $w(x)$ two sequences of coefficients, i.e.

$$w(x) \ \mapsto \ \{\alpha_n,\ \beta_n\}_{n \in \mathbb{N}_0}.$$

The statements of the next lemma will be very useful (see proofs in [5]).

Lemma 1. *Let $w(x)$ be a weight function with the support* $\text{supp}(w) = (a, b)$ *and* $\{\alpha_n,\ \beta_n\}_{n\in\mathbb{N}_0}$ *the corresponding sequences of coefficients in monic three-term recurrence relation. Also, let $\tilde{w}(x)$ be a modified weight of $w(x)$ and* $\{\tilde{\alpha}_n,\ \tilde{\beta}_n\}_{n\in\mathbb{N}_0}$ *its sequences of coefficients. Then*
(i) If $\tilde{w}(x) = Cw(x) \quad \Rightarrow \quad \{\tilde{\alpha}_n = \alpha_n, \quad \tilde{\beta}_0 = C\beta_0, \quad \tilde{\beta}_n = \beta_n \ (n \in \mathbb{N})\}$.
(ii) If $\tilde{w}(x) = w(cx + d) \quad \Rightarrow \quad \{\tilde{\alpha}_n = \frac{\alpha_n - d}{c}, \quad \tilde{\beta}_0 = \frac{\beta_0}{|c|}, \quad \tilde{\beta}_n = \frac{\beta_n}{c^2}\ (n \in \mathbb{N})\},$

$$\text{supp}(\tilde{w}) = \Big(\frac{a-d}{c},\ \frac{b-d}{c}\Big).$$

(iii) If $\tilde{w}_c(x) = (x - c)\ w(x)\ (c < a < b)$, *then, for every* $n \in \mathbb{N}$, *it is valid*

$$\tilde{\alpha}_{c,0} = \alpha_0 + r_1 - r_0 \ , \ \tilde{\alpha}_{c,n} = \alpha_{n+1} + r_{n+1} - r_n, \ \tilde{\beta}_{c,0} = -r_0\beta_0, \ \tilde{\beta}_{c,n} = \beta_n\ \frac{r_n}{r_{n-1}},$$

where $r_0 = c - \alpha_0, \quad r_1 = c - \alpha_1 - \frac{\beta_1}{r_0}, \quad r_{n+1} = c - \alpha_{n+1} - \frac{\beta_{n+1}}{r_n} \quad (n \in \mathbb{N}_0)$.
(iv) If $\tilde{w}_d(x) = (d - x)\ w(x)\ (a < b < d)$, *then*

$$\tilde{\alpha}_{d,n} = d + q_{n+1} + e_{n+1}\ (n \in \mathbb{N}_0),\ \tilde{\beta}_{d,0} = \int_{\mathbb{R}} \tilde{w}_d(x)\ dx,\ \tilde{\beta}_{d,n} = q_{n+1}e_n\ (n \in \mathbb{N}),$$

where $e_0 = 0, \quad q_n = \alpha_{n-1} - e_{n-1} - d, \quad e_n = \frac{\beta_n}{q_n} \quad (n \in \mathbb{N})$.
(v) If $\tilde{w}_c(x) = \frac{w(x)}{x-c} \qquad (c < a < b)$, *then*

$$\tilde{\alpha}_{c,0} = \alpha_0 + r_0 \ , \ \tilde{\alpha}_{c,n} = \alpha_n + r_n - r_{n-1},\ \tilde{\beta}_{c,0} = -r_{-1},\ \tilde{\beta}_{c,n} = \beta_{n-1}\frac{r_{n-1}}{r_{n-2}} \quad (n \in \mathbb{N}),$$

where $r_{-1} = -\int_{\mathbb{R}} \tilde{w}_c(x)\ dx, \qquad r_n = c - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0)$.

(vi) If $\tilde{w}_d(x) = \frac{w(x)}{d-x}$ $\quad (a < b < d)$ *, then*

$$\tilde{\alpha}_{d,0} = \alpha_0 + r_0 \ , \ \tilde{\alpha}_{d,n} = \alpha_n + r_n - r_{n-1}, \ \tilde{\beta}_{d,0} = r_{-1}, \ \tilde{\beta}_{d,n} = \beta_{n-1} \frac{r_{n-1}}{r_{n-2}} \quad (n \in \mathbb{N}),$$

where $r_{-1} = \int_{\mathbb{R}} \tilde{w}_d(x)\ dx, \qquad r_n = d - \alpha_n - \frac{\beta_n}{r_{n-1}} \quad (n \in \mathbb{N}_0).$

5. Orthogonal polynomials

The monic Chebyshev polynomials of the second kind

$$Q_n^{(1)}(x) = S_n(x) = \frac{\sin((n+1)\arccos x)}{2^n \cdot \sqrt{1-x^2}}$$

are orthogonal with respect to the weight $w^{(1)}(x) = \sqrt{1-x^2}$ on $(-1,1)$. The corresponding coefficients in the three-term recurrence relation are

$$\beta_0^{(1)} = \frac{\pi}{2}, \quad \beta_n^{(1)} = \frac{1}{4} \quad (n \geq 1), \qquad \alpha_n^{(1)} = 0 \quad (n \geq 0)\ .$$

Let us introduce a new weight function

$$w^{(2)}(x) = w^{(1)}\Big(\frac{2}{b-a}x - \frac{b+a}{b-a}\Big), \qquad x \in (a,b).$$

By usage of the previous lemma, we get

$$\beta_0^{(2)} = \pi\frac{b-a}{4}, \quad \beta_n^{(2)} = \Big(\frac{b-a}{4}\Big)^2 \quad (n \in \mathbb{N}), \qquad \alpha_n^{(2)} = \frac{a+b}{2} \qquad (n \in \mathbb{N}_0)\ .$$

Also, considering $w^{(3)}(x) = \frac{b-a}{4\pi}\ w^{(2)}(x), \quad x \in (a,b)$, we get

$$\beta_n^{(3)} = \Big(\frac{b-a}{4}\Big)^2, \qquad \alpha_n^{(3)} = \frac{a+b}{2} \qquad (n \in \mathbb{N}_0)\ .$$

Finally, by introducing a new weight function

$$w^{(4)}(x) = \frac{w^{(3)}(x)}{x} = \frac{1}{2\pi}\ \frac{\sqrt{(b-x)(x-a)}}{x}, \qquad x \in (a,b),$$

we obtain $r_n = \frac{-1}{4}(a+b-2\sqrt{ab}) \quad (n = -1, 0, 1 \ldots)$, and

$$\beta_0^{(4)} = \frac{1}{4}(a+b-2\sqrt{ab}), \quad \beta_n^{(4)} = \Big(\frac{b-a}{4}\Big)^2,$$

$$\alpha_0^{(4)} = \frac{a+b+\sqrt{ab}}{4}\ , \quad \alpha_n^{(4)} = \frac{a+b}{2}\ (n \in \mathbb{N}).$$

5. The Hankel determinants

Notice that $g_n^{(0)}$ can be written in the form

$$g_n^{(0)} = \frac{1}{2c^2\pi}\int_{ac+d}^{bc+d} z^n\sqrt{(bc+d-z)(z-ac-d)}\,dz \quad (n\in\mathbb{N}_0).$$

For computing the Hankel determinants $h_n^{(0)} = |g_{i+j-2}^{(0)}|$, it is enough to consider

$$w^{(3)}(x) = \frac{b-a}{4c\pi}w^{(1)}\Big(\frac{2}{(b-a)c}x - \frac{(a+b)c+2d}{(b-a)c}\Big).$$

Since

$$\beta_0^{(3)} = \Big(\frac{b-a}{4}\Big)^2, \quad \beta_n^{(3)} = \Big(\frac{(b-a)c}{4}\Big)^2, \quad (n\in\mathbb{N}) \ ,$$

we have

$$h_n^{(0)} = \Big(\frac{b-a}{4}\Big)^{n(n+1)} c^{n(n-1)} \ .$$

Hence we conclude that $\{h_n^{(0)}\}$ is a (r,s) Somos-4 sequence for every pair (r,s) such that

$$\Big(\frac{b-a}{4}c\Big)^2 r + s = \Big(\frac{b-a}{4}c\Big)^8 \ . \tag{15}$$

Similarly,

$$h_n^{(1)} = \Big(\frac{\sqrt{b}-\sqrt{a}}{2}\Big)^{2n}\Big(\frac{b-a}{4}\Big)^{(n-1)n} \qquad (n\in\mathbb{N}) \ .$$

Hence we conclude that $\{h_n^{(1)}\}$ has the same (r,s) Somos-4 property (15). Especially, we can say that both sequences of determinants are $(((b-a)/4)^6, 0)$ Somos-4 or $(0, ((b-a)/4)^8)$ Somos-4 sequences.

Example 4. Here a few known special number sequences are presented:

(1) In the Motzkin case ($a=-1;\ b=3$) and the Catalan case ($a=0;\ b=4$), we have that $h_n^{(0)}=1$ is $(1,0)$ Somos-4 sequence;

(2) In the Schröder case ($a=3-2\sqrt{2};\ b=3+2\sqrt{2}$), we have that $h_n^{(1)} = 2^{n(n-1)/2}$ and it is $(8,0)$ Somos-4 sequence.

6. An analogue case with complex numbers

Let

$$g_n^{(2)} = \frac{1}{2\pi}\int_{-v}^{v}(iz+u)^n\sqrt{(v-z)(z+v)}\,dz \qquad (i=\sqrt{-1})\ (n\in\mathbb{N}_0). \tag{16}$$

Hence we get the first sequence $g_n^{(2)} = g_n^{(0)}(a,b,c,d)$ with $a=-v,\ b=v$ and $c=i,\ d=u$.

Remark 2. This case can be written in the form (8) with $a = u - iv$ and $b = u + iv$. But, it can cause a lot of troubles with two valued square root complex functions. That is why we choose the form (16).

Example 5. Taking $u = -1$ and $v = 2$, in (16) we get the sequence $\{g_n\} = \{1, -1, 0, 2, -3, -1, 11, -15, -13, 77, -86, \ldots\}$, whose Hankel determinants are $h_n^{(2)} = (-1)^{n(n-1)/2}$, which is an (r, s) Somos-4 sequence for any r and s satisfying $s - r = 1$.

The sequence $\{g_n\}$ has the generating function

$$G(x) = \frac{-1 - x - \sqrt{1 + 2x + 5x^2}}{2x^2} = \frac{y(x)}{x}, \quad \text{where} \quad \frac{y}{1 - y - y^2} = x.$$

Here, $y(x)$ is the series reversion of the generating function of the Fibonacci numbers, i.e.

$$\frac{y}{1 - y - y^2} = \sum_{n=0}^{\infty} F_n y^n,$$

where $F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad (n \in \mathbb{N})$.

7. A conjecture

Let

$$a_1 = -\frac{1 + \sqrt{5}}{2}, \quad b_1 = -\frac{\sqrt{13} - 3}{2}, \quad a_2 = \frac{\sqrt{5} - 1}{2}, \quad b_2 = \frac{3 + \sqrt{13}}{2},$$

and

$$w(x) = \frac{1}{|x|} \cdot \sqrt{-(b_1 - x)(x - a_1)(b_2 - x)(x - a_2)}, \quad x \in D = (a_1, b_1) \cup (a_2, b_2).$$

Consider the sequence $\{g_n\}$ defined by

$$g_n = \frac{1}{2\pi} \int_D x^n w(x)\, dx \qquad (n \in \mathbb{N}_0).$$

Here is $g = \{1, 1, 3, 6, 16, 40, 109, 297, \ldots\}$, i.e. it is the sequence $A128720$. The following convolution property is valid:

$$g_n = g_{n-1} + g_{n-2} + \sum_{k=0}^{n-2} g_k g_{n-2-k} \qquad (n = 2, 3, \ldots).$$

We conjecture that their Hankel transform $h = \{1, 2, 5, 17, 109, \ldots\}$, is the sequence known as $A174168$. This is a $(1, 3)$ Somos-4 sequence because of the property

$$h_n h_{n-4} = h_{n-1} h_{n-3} + 3h_{n-2}^2.$$

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