Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **35** (2009), 311–316

**Serdica**
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

# A NOTE ABOUT THE NOWICKI CONJECTURE ON WEITZENBÖCK DERIVATIONS

Leonid Bedratyuk

*Communicated by V. Drensky*

Abstract. We reduce the Nowicki conjecture on Weitzenböck derivations of polynomial algebras to a well known problem of classical invariant theory.

**1.** Let $\mathbb{K}$ be a field of characteristic 0. A linear locally nilpotent derivation $\mathcal{D}$ of the polynomial algebra $\mathbb{K}[Z] = \mathbb{K}[z_1, z_2, \ldots, z_m]$ is called a Weitzenböck derivation. It is well known that the kernel

$$\ker \mathcal{D} := \{f \in \mathbb{K}[Z] \mid \mathcal{D}(f) = 0\}$$

of the linear locally nilpotent derivation $\mathcal{D}$ is a finitely generated algebra, see [13, 11, 12].

Let $\mathbb{K}[X, Y] = \mathbb{K}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ be the polynomial $\mathbb{K}$-algebra in $2\,n$ variables. Consider the following Weitzenböck derivation $\mathcal{D}_1$ of $\mathbb{K}[X, Y]$:

$$\mathcal{D}_1(x_i) = 0, \mathcal{D}_1(y_i) = x_i, \quad i = 1, 2, \ldots, n.$$

2000 *Mathematics Subject Classification:* 13N15, 13A50, 16W25.
*Key words:* Classical invariant theory, covariants of binary form, derivations.

In [8] Nowicki conjectured that $\ker \mathcal{D}_1$ is generated by the elements $x_1, x_2, \ldots, x_n$ and the determinants

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \quad 1 \leq i < j \leq n.$$

The conjecture was confirmed by several authors, see [6, 1, 7].

In this note we show that the Nowicki conjecture is equivalent to a well known problem of classical invariant theory, namely, to the problem to describe the algebra of joint covariants of $n$ linear binary forms. Using the same idea we present an explicit set of generators of the kernel of the derivation $\mathcal{D}_2$ of

$$\mathbb{K}[X, Y, Z] = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n]$$

defined by

$$\mathcal{D}_2(x_i) = 0, \mathcal{D}_2(y_i) = x_i, \mathcal{D}_2(z_i) = y_i, \quad i = 1, \ldots, n.$$

**2.** It is well known that there is a one-to-one correspondence between the $\mathbb{G}_a$-actions on an affine algebraic variety $V$ and the locally nilpotent $\mathbb{K}$-derivations on its algebra of polynomial functions. Let us identify the algebra $\mathbb{K}[X, Y]$ with the algebra $\mathcal{O}[\mathbb{K}^{2\,n}]$ of polynomial functions of the algebraic variety $\mathbb{K}^{2n}$. Then, the kernel of the derivation $\mathcal{D}_1$ coincides with the invariant ring of the induced via $\exp(t\,\mathcal{D}_1)$ action:

$$\ker \mathcal{D}_1 = \mathcal{O}[\mathbb{K}^{2n}]^{\mathbb{G}_a} = \mathbb{K}[X, Y]^{\mathbb{G}_a}.$$

Now, let

$$V_1 := \{\alpha \mathcal{X} + \beta \mathcal{Y} \mid \alpha, \beta \in \mathbb{K}\}$$

be the vector $\mathbb{K}$-space of linear binary forms endowed with the natural action of the group $SL_2$. Consider the induced action of the group $SL_2$ on the algebra of polynomial functions $\mathcal{O}[n\,V_1 \oplus \mathbb{K}^2]$ on the vector space $n\,V_1 \oplus \mathbb{K}^2$, where

$$n\,V_1 := \underbrace{V_1 \oplus V_1 \oplus \ldots \oplus V_1}_{n \text{ times}}.$$

Let

$$U_2 = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \;\middle|\; \lambda \in \mathbb{K} \right\}$$

be the maximal unipotent subgroup of the group $SL_2$. The application of the Grosshans principle, see [5, 10], gives

$$\mathcal{O}[n\,V_1 \oplus \mathbb{K}^2]^{SL_2} \cong \mathcal{O}[n\,V_1]^{U_2}.$$

Thus,

$$\mathcal{O}[n\,V_1 \oplus \mathbb{K}^2]^{\,\mathfrak{sl}_2} \cong \mathcal{O}[n\,V_1]^{\,\mathfrak{u}_2}.$$

Since $U_2 \cong (\mathbb{K}, +)$, it follows that

$$\ker \mathcal{D}_1 \cong \mathcal{O}[nV_1 \oplus \mathbb{K}^2]^{\,\mathfrak{sl}_2} \cong \mathcal{O}[n\,V_1]^{\,\mathfrak{u}_2}.$$

In the language of classical invariant theory the algebra $C_1 := \mathcal{O}[nV_1 \oplus \mathbb{K}^2]^{\,\mathfrak{sl}_2}$ is called the algebra of joint covariants of $n$ linear binary forms and the algebra $S_1 := \mathcal{O}[nV_1]^{\,\mathfrak{u}_1}$ is called the algebra of joint semi-invariants of $n$ linear binary forms. Algebras of joint covariants of binary forms were an object of research in the classical invariant theory of the 19th century.

**3.** Let us consider the set of $n$ linear binary forms $f_i = x_i\mathcal{X} + y_i\mathcal{Y}$, $i = 1, \ldots, n$. Then any element of $\mathcal{O}[nV_1 \oplus \mathbb{K}^2]$ can be considered as a polynomial from $\mathbb{K}[X, Y, \mathcal{X}, \mathcal{Y}]$. Gordan's famous theorem, see [3, 2], implies:

**Theorem 1** (A weak form of Gordan's theorem)**.** *If $T$ is a subalgebra of $C_1$ with the property that $(f_i, z)^r \in T$ whenever $r \in \mathbb{N}$, $z \in T$, then $T = C_1$.*

Here $(u, v)^r$ denotes the $r$-transvectants of the binary forms $u, v \in \mathbb{K}[X, Y, \mathcal{X}, \mathcal{Y}]$:

$$(u, v)^r := \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r u}{\partial \mathcal{X}^{r-i} \partial \mathcal{Y}^i} \frac{\partial^r v}{\partial \mathcal{X}^i \partial \mathcal{Y}^{r-i}}.$$

Observe, that $(u, v)^0 = u\,v$ and $(u, v)^1$ is exactly the Jacobian $J(u, v)$ of the forms $u, v$. The above theorem yields:

**Theorem 2.** *The algebra of joint covariants $C_1$ of $n$ linear binary forms $f_i$, $i = 1, \ldots, n$, is generated by the forms $f_1, f_2, \ldots, f_n$ and their jacobians $J(f_i, f_j)$, $1 \le i < j \le n$.*

P r o o f. All forms $f_i$, $i = 1, \ldots, n$, belong to the algebra of covariants $C_1$. By direct calculations we get

$$(f_i, f_j)^1 = J(f_i, f_j) = \begin{vmatrix} \dfrac{\partial f_i}{\partial \mathcal{X}} & \dfrac{\partial f_i}{\partial \mathcal{Y}} \\ \\ \dfrac{\partial f_j}{\partial \mathcal{X}} & \dfrac{\partial f_i}{\partial \mathcal{Y}} \end{vmatrix} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix},$$

and

$$(f_i, f_j)^r = 0 \quad \text{for} \quad r > 1.$$

Let us consider the subalgebra $T$ of $C_1$ generated by the linear forms $f_1$, $f_2, \ldots,$ $f_n$ and theirs jacobians $J(f_i, f_j), i < j$. Since $(f_i, J(f_j, f_k))^r = 0$ for all $r \geq 1$, it follows that $T = C_1$. $\square$

Let us show that the result is equivalent to the Nowicki conjecture:

Identify the algebra of semi-invariants $S_1$ with $\ker \mathcal{D}_1$. The isomorphism $\tau : C_1 \to S_1$ takes each homogeneous covariant of degree $m$ (with respect to the variables $\mathcal{X}, \mathcal{Y}$) to its coefficient of $\mathcal{X}^m$. The proof proceeds in the same manner as the proof in the case of a unique binary form, see [9, Proposition 9.45].

Thus, the following statement holds:

**Theorem 3.** *The algebra of joint semi-invariants $S_1 = \ker \mathcal{D}_1$ of $n$ linear binary forms $f_i$, $i = 1, \ldots, n$, is generated by the elements $x_1, x_2, \ldots, x_n$ and the determinants*

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \quad 1 \leq i < j \leq n.$$

P r o o f. The algebra $S_1$ is generated by the images of the generating elements of the algebra $C_1$ under the homomorphism $\tau$. We have $\tau(f_i) = x_i$ and

$$\tau(J(f_i, f_j)) = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}. \qquad \square$$

Theorem 3 is exactly the Nowicki conjecture.

**4.** Other ways to prove Theorem 3 were suggested by Dersken and Panyushev, see the comments in [1]. Taking into account $\mathbb{K}^2 \cong_{\mathfrak{sl}_2} V_1$ we get

$$\ker \mathcal{D}_1 \cong \mathcal{O}[nV_1 \oplus \mathbb{K}^2]^{\,\mathfrak{sl}_2} \cong \mathcal{O}[(n+1)V_1]^{\,\mathfrak{sl}_2}.$$

But then the invariant algebra $\mathcal{O}[(n+1)V_1]^{\,\mathfrak{sl}_2}$ is well known because of the First fundamental theorem of invariant theory for $SL_2$, see [14].

**5.** A natural generalization of the above problem looks as follows.

Let

$$\mathbb{K}[X, Y, Z] = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n]$$

be the polynomial $\mathbb{K}$-algebra in $3\,n$ variables. Consider the following derivation $\mathcal{D}_2$ of the algebra $\mathbb{K}[X, Y, Z]$:

$$\mathcal{D}_2(x_i) = 0,\ \mathcal{D}_2(y_i) = x_i,\ \mathcal{D}_2(z_i) = y_i, \quad i = 1, 2, \ldots, n.$$

The following theorem holds.

**Theorem 4.** *The kernel of the derivation $\mathcal{D}_2$ is generated by the elements of the following types:*

1. $x_1, x_2, \ldots, x_n,$
2. $J_{1,2}, J_{1,3}, \ldots, J_{n-1,n},$
3. $H_{1,2}, H_{1,3}, \ldots, H_{n-1,n},$
4. $\Delta_{1,2,3}, \Delta_{1,2,4}, \ldots, \Delta_{n-2,n-1,n},$

*where*

$$J_{i,j} := \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \quad 1 \leq i < j \leq n,$$

$$H_{i,j} = x_i z_j - y_i y_j + z_i x_j, \quad 1 \leq i \leq j \leq n,$$

*and*

$$\Delta_{i,j,k} := \begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{vmatrix}, \quad 1 \leq i < j < k \leq n.$$

The proof follows from the description of the generating elements of the algebra of covariants for $n$ quadratic binary forms, see [4, page 162].

**6.** Any Weitzenböck derivation of the polynomial algebra is completely determined by its Jordan normal form. Denote by $\mathcal{D}_k$ the Weitzenböck derivation with Jordan normal form which consists of $n$ Jordan blocks of size $k+1$.

**Problem.** Find a generating set of $\ker \mathcal{D}_k$.

## REFERENCES

[1] V. Drensky, L. Makar-Limanov. The conjecture of Nowicki on Weitzenböck derivations of polynomial algebras. *J. Algebra Appl.* **8**, *1* (2009) 41–51.

[2] O. E. Glenn. Treatise on Theory of Invariants. Boston, 1915.

[3] P. Gordan. Vorlesungen über Invariantentheorie. Teubner, Leipzig, 1885. Reprinted by Chelsea Publishing Company, New York, 1987.

[4] J. Grace, A. Young. The Algebra of Invariants. Cambrige Univ. Press, 1903.

[5] F. Grosshans. Observable groups and Hilbert's fourteenth problem. *Amer. J. Math.* **95** (1973), 229–253.

[6] J. Khoury. Locally Nilpotent Derivations and Their Rings of Constants, Ph.D. Thesis, Univ. Ottawa, 2004.

[7] S. Kuroda. A simple proof of Nowicki's conjecture on the kernel of an elementary derivation. *Tokyo J. Math.* **32**, *1* (2009), 247–251.

[8] A. Nowicki. Polynomial Derivations and their Rings of Constants. Uniwersytet Mikolaja Kopernika, Torun, 1994 [Available at: `http://www-users.mat.uni.torun.pl/~anow/polder.html`.]

[9] P. Olver. Classical Invariant Theory. Cambridge University Press, 1999.

[10] K. Pommerening. Invariants of unipotent groups. – A survey. In: Invariant Theory, Symp. West Chester/Pa, 1985, Lect. Notes Math. vol. **1278**, 1987, 8–17.

[11] C. S. Seshadri. On a theorem of Weitzenböck in invariant theory. *J. Math. Kyoto Univ.* **1** (1962), 403–409.

[12] A. Tyc. An elementary proof of the Weitzenböck theorem. *Colloq. Math.* **78** (1998), 123–132.

[13] R. Weitzenböck. Über die Invarianten von linearen Gruppen. *Acta Math.* **58** (1932), 231–293.

[14] H. Weyl. The Classical Groups. Their Invariants and Representations. Princeton University Press, 1997.

*Khmelnytsky National University*
*11, Instytuts'ka st.*
*29016 Khmelnytsky, Ukraine*
*e-mail:* `leonid.uk@gmail.com`

*Received October 3, 2009*